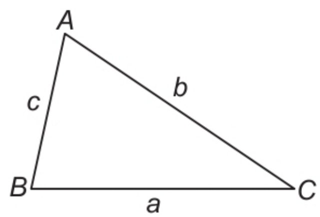
Solution of Triangles

Basic Rules of Triangle

In a ΔABC, the angles are denoted by capital letters A, B and C and the lengths of the sides opposite to these angles are denoted by small letters a, b and c, respectively. Area and perimeter of a triangle are denoted by Δ and 2s respectively.



Semi-perimeter of the triangle is written as $s = \dfrac{a+b+c}{2}$.

(i) **Sine Rule** $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = \dfrac{1}{2R}$, where R is radius of the circumcircle of ΔABC.

(ii) **Cosine Rule** $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

and $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

(iii) **Projection Rule** $a = b\cos C + c\cos B$, $b = c\cos A + a\cos C$ and $c = a\cos B + b\cos A$

(iv) **Napier's Analogy** $\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$,

$$\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2} \text{ and } \tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$$

Trigonometrical Ratios of Half of the Angles of Triangle

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ac}}$,

$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$, $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Area of a Triangle

Consider a triangle of side a, b and c.

(i) $\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \frac{1}{2} ab \sin C$

(ii) $\Delta = \frac{c^2 \sin A \sin B}{2 \sin C} = \frac{a^2 \sin B \sin C}{2 \sin A} = \frac{b^2 \sin C \sin A}{2 \sin B}$

(iii) $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, its known as **Heron's formula**.

where, $s = \frac{a+b+c}{2}$ [semi-perimeter of triangle]

(iv) $\Delta = \frac{abc}{4R} = rs$, where R and r are radii of the circumcircle and the incircle of ΔABC, respectively.

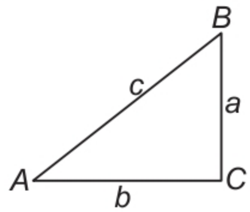
Solutions of a Triangle

Elements of a Triangle

There are six elements of a triangle, in which three are its sides and other three are its angle. If three elements of a triangle are given, atleast one of which is its side, then other elements can be uniquely calculated. This is called **solving the triangle**.

1. Solutions of a Right Angled Triangle

Let ΔABC be a given triangle with right angle at C, then



(i) the solution when two sides are given

Given	To be calculated
a, b	$\tan A = \frac{a}{b}$; $B = 90° - A$, $c = \frac{a}{\sin A}$
a, c	$\sin A = \frac{a}{c}$; $B = 90° - A$ $b = c \cos A$ or $b = \sqrt{c^2 - a^2}$

(ii) the solution when one side and one acute angle are given

Given	To be calculated
a, A	$B = 90° - A$, $b = a \cot A$, $c = \frac{a}{\sin A}$
c, A	$B = 90° - A$, $a = c \sin A$, $b = c \cot A$

2. Solutions of a Triangle in General

(i) When three sides a, b and c are given, then

$$\sin A = \frac{2\Delta}{bc}, \sin B = \frac{2\Delta}{ac}, \sin C = \frac{2\Delta}{ab}$$

where, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ in which $s = \frac{a+b+c}{2}$

and $A + B + C = 180°$.

(ii) When two sides and the included angle are given, then

(a) $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b} \cot \frac{C}{2}, \frac{A+B}{2} = 90° - \frac{C}{2}, c = \frac{a \sin C}{\sin A}$

(b) $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c} \cot \frac{A}{2}, \frac{B+C}{2} = 90° - \frac{A}{2}, a = \frac{b \sin A}{\sin B}$

(c) $\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a} \cot \frac{B}{2}, \frac{C+A}{2} = 90° - \frac{B}{2}, b = \frac{c \sin B}{\sin C}$

This is called as **Napier's analogy**.

(iii) When one side a and two angles A and B are given, then

$$C = 180° - (A + B) \Rightarrow b = \frac{c \sin B}{\sin C} \text{ and } c = \frac{a \sin C}{\sin A}$$

(iv) When two sides a, b and the opposite $\angle A$ is given, then

$$\sin B = \frac{b}{a} \sin A, C = 180° - (A + B), \ c = \frac{a \sin C}{\sin A}$$

Now, different cases arise here

(a) If A is an acute angle and $a < b \sin A$, then $\sin B = \frac{b}{a} \sin A$ gives $\sin B > 1$, which is not possible, so no such triangle is possible.

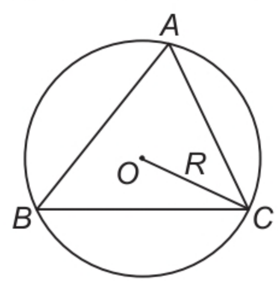
(b) When A is an acute angle and $a = b \sin A$. In this case, only one triangle is possible, which is right angled at B.

(c) If A is an acute angle and $a > b \sin A$. In this case, there are two values of B given by $\sin B = \frac{b \sin A}{a}$, say B_1 and B_2 such that $B_1 + B_2 = 180°$, side c can be calculated from $c = \frac{a \sin C}{\sin A}$.

Circles Connected with Triangle

1. Circumcircle

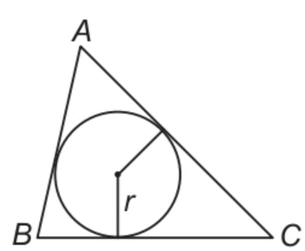
The circle passing through the vertices of the ΔABC is called the circumcircle and its radius R is called the circumradius.



$$\therefore \qquad R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} = \frac{abc}{4\Delta}$$

2. Incircle

The circle touches the three sides of the triangle internally is called the inscribed or the incircle of the triangle and its radius r is called the inradius of circle.



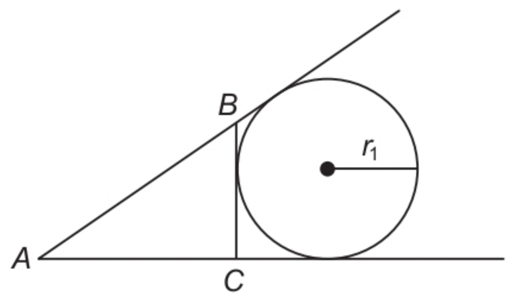
$\therefore \qquad r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2}$

$$r = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

and $$r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \frac{b\sin\frac{C}{2}\sin\frac{A}{2}}{\cos\frac{B}{2}} = \frac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$$

3. **Escribed Circle**

The circle touches BC and the two sides AB and AC produced of ΔABC externally is called the escribed circle opposite to A. Its radius is denoted by r_1.



Similarly, r_2 and r_3 denote the radii of the escribed circles opposite to angles B and C, respectively. Hence, r_1, r_2 and r_3 are called the exradius of ΔABC. Here,

(i) $$r_1 = \frac{\Delta}{s-a} = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

(ii) $$r_2 = \frac{\Delta}{s-b} = s\tan\frac{B}{2} = 4R\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2} = \frac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}$$

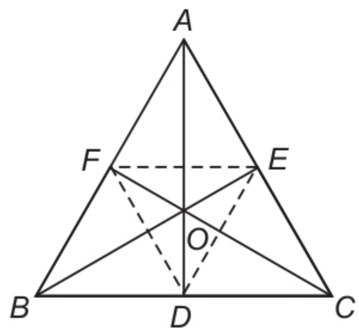
(iii) $$r_3 = \frac{\Delta}{s-c} = s\tan\frac{C}{2} = 4R\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2} = \frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$$

(iv) $r_1 + r_2 + r_3 = 4R + r$

(v) $r_1 r_2 + r_2 r_3 + r_3 r_1 = \frac{r_1 r_2 r_3}{r}$

4. Orthocentre and Pedal Triangle

The point of intersection of perpendicular drawn from the vertices on the opposite sides of a triangle is called orthocentre.



The ΔDEF formed by joining the feet of the altitudes is called the pedal triangle.

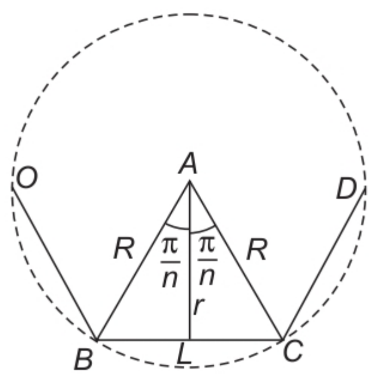
(i) Distance of the orthocentre of the triangle from the angular points are $2R\cos A$, $2R\cos B$, $2R\cos C$ and its distances from the sides are $2R\cos B\cos C$, $2R\cos C\cos A$, $2R\cos A\cos B$.

(ii) The length of medians AD, BE and CF of a ΔABC are

$$AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2},\ BE = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$$

and $$CF = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$$

Radii of the Inscribed and Circumscribed Circles of Regular Polygon



(i) Radius of circumcircle $(R) = \dfrac{a}{2}\operatorname{cosec}\dfrac{\pi}{n}$

(ii) Radius of incircle $(r) = \dfrac{a}{2}\cot\dfrac{\pi}{n}$, where a is the length of a side of polygon.

(iii) The area of the polygon = n (Area of ΔABC)

$$= \frac{1}{4} n a^2 \cot\left(\frac{\pi}{n}\right)$$

$$= n r^2 \tan\frac{\pi}{n} = \frac{n}{2} R^2 \sin\left(\frac{2\pi}{n}\right)$$

Important Points to be Remembered

(i) Distance between circumcentre and orthocentre

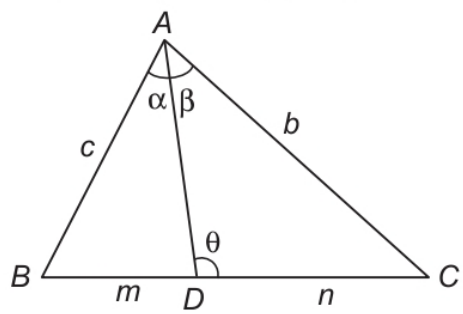
$$= R^2[1 - 8 \cos A \cos B \cos C]$$

(ii) Distance between circumcentre and incentre

$$= R^2\left[1 - 8 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}\right] = R^2 - 2Rr$$

(iii) Distance between circumcentre and centroid $= R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$

(iv) ***m-n* Theorem** In a ΔABC, D is a point on the line BC such that $BD : DC = m : n$ and $\angle ADC = \theta$, $\angle BAD = \alpha$, $\angle DAC = \beta$, then



(a) $(m + n) \cot\theta = m \cot\alpha - n \cot\beta$

(b) $(m + n) \cot\theta = n \cot B - m \cot C$