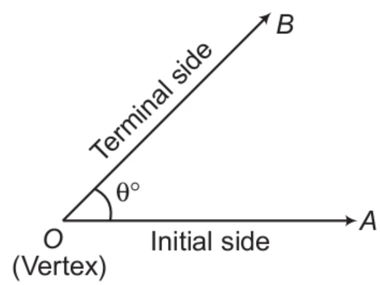
Trigonometric Functions, Identities and Equations

Angle

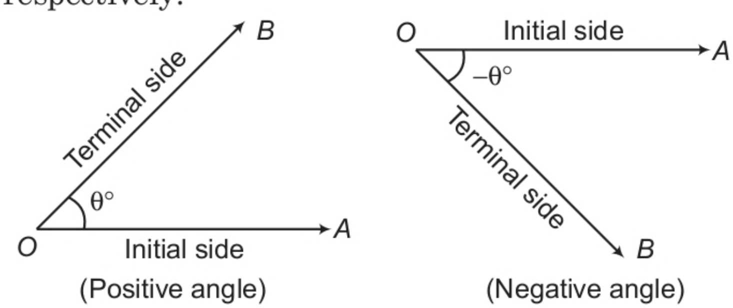
When a ray OA starting from its initial position OA rotates about its end point O and takes the final position OB, we say that angle AOB (written as $\angle AOB$) has been formed.



The amount of rotation from the initial side to the terminal side is called the measure of the angle.

Positive and Negative Angles

An angle formed by a rotating ray is said to be positive or negative depending on whether it moves in an anti-clockwise or a clockwise direction, respectively.



Measurement of Angles

There are three system for measuring the angles, which are given below

1. **Sexagesimal System** (Degree Measure)

In this system, a right angle is divided into 90 equal parts, called the degrees. The symbol 1° is used to denote one degree. Each degree is divided into 60 equal parts, called the minutes and one minute is divided into 60 equal parts, called the seconds. Symbols $1'$ and $1''$ are used to denote one minute and one second, respectively.

i.e. 1 right angle $= 90°$, $1° = 60'$, $1' = 60''$

2. **Circular System** (Radian Measure)

In this system, angle is measured in radian. A radian is the angle subtended at the centre of a circle by an arc, whose length is equal to the radius of the circle. The number of radians in an angle subtended by an arc of circle at the centre is equal to $\frac{\text{arc}}{\text{radius}}$.

3. **Centesimal System** (French System)

In this system, a right angle is divided into 100 equal parts, called the grades. Each grade is subdivided into 100 min and each minute is divided into 100 s.

i.e. 1 right angle $= 100$ grades $= 100^g$, $1^g = 100'$, $1' = 100''$

Relation between Degree and Radian

(i) π radian $= 180°$

or 1 radian $= \left(\frac{180°}{\pi}\right) = 57°16'22''$ where, $\pi = \frac{22}{7} = 3.14159$

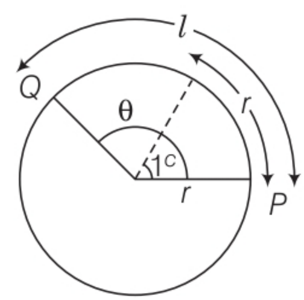
(ii) $1° = \left(\frac{\pi}{180}\right)$ rad $= 0.01746$ rad

(iii) If D is the number of degrees, R is the number of radians and G is the number of grades in an angle θ, then

$$\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$$

Length of an Arc of a Circle

If in a circle of radius r, an arc of length l subtend an angle θ radian at the centre, then

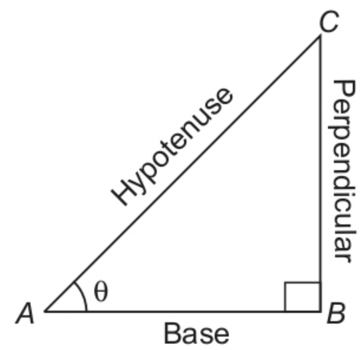


$$\theta = \frac{l}{r} = \frac{\text{Length of arc}}{\text{Radius}} \text{ or } l = r\theta$$

Trigonometric Ratios For acute Angle

Relation between different sides and angles of a right angled triangle are called trigonometric ratios or T-ratios.

Trigonometric ratios can be represented as



$$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{BC}{AC},$$

$$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{AB}{AC},$$

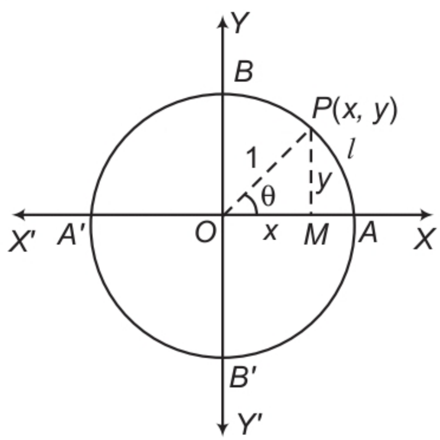
$$\tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{BC}{AB},$$

$$\text{cosec}\,\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}, \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

Trigonometric (or Circular) **Functions**

Let $X'OX$ and YOY' be the coordinate axes. Taking O as the centre and a unit radius, draw a circle, cutting the coordinate axes at A, B, A' and B', as shown in the figure.



$$\left[\because \angle AOP = \frac{\text{arc } AP}{\text{radius } OP} = \frac{\theta}{1} = \theta°, \text{ using } \theta = \frac{l}{r}\right]$$

Now, six circular functions may be defined as

(i) $\cos\theta = x$ (ii) $\sin\theta = y$

(iii) $\sec\theta = \frac{1}{x}, x \neq 0$ (iv) $\text{cosec}\,\theta = \frac{1}{y}, y \neq 0$

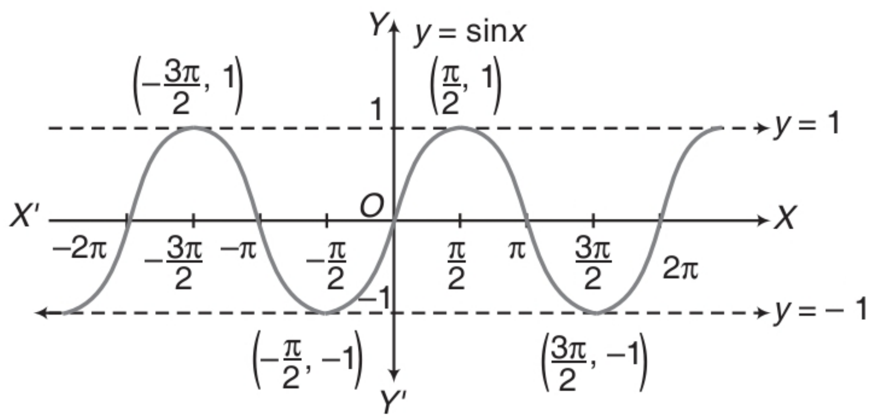
(v) $\tan\theta = \frac{y}{x}, x \neq 0$ (vi) $\cot\theta = \frac{x}{y}, y \neq 0$

Trigonometric Function of Some Standard Angles

Angle	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	$-\infty$
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

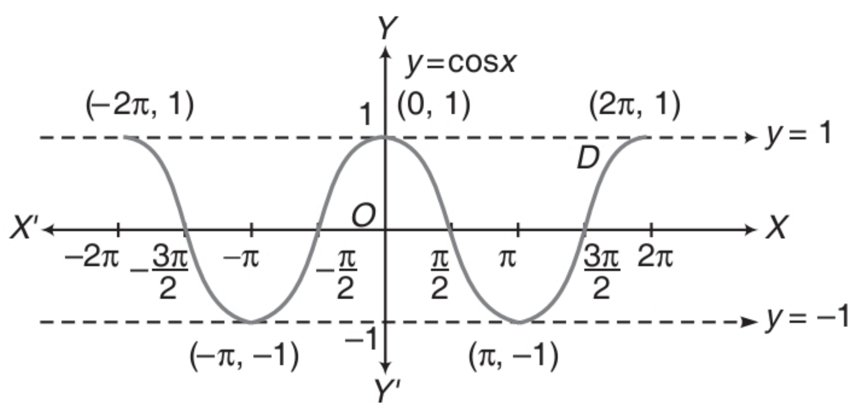
Graph of Trigonometric Functions

1. **Graph of sin *x***



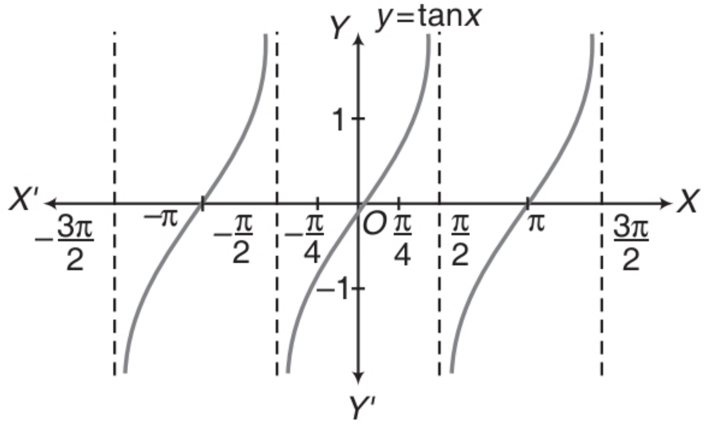
(i) Domain = R (ii) Range = $[-1, 1]$ (iii) Period = 2π

2. **Graph of cos *x***



(i) Domain = R (ii) Range = $[-1, 1]$ (iii) Period = 2π

3. **Graph of tan *x***

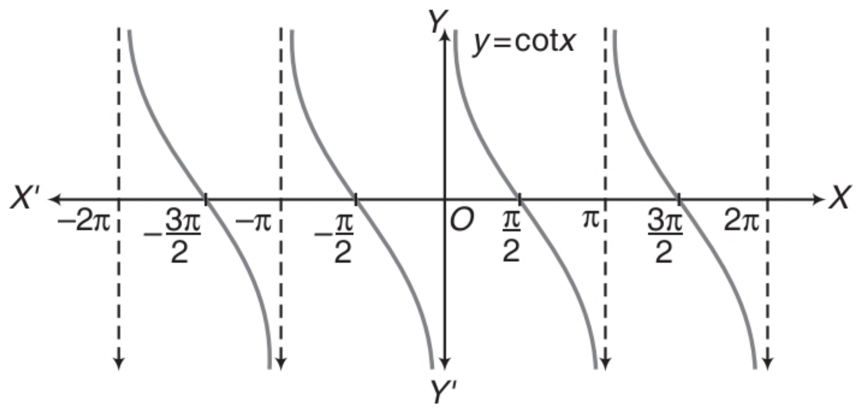


(i) Domain $= R \sim (2n+1)\dfrac{\pi}{2}, n \in I$

(ii) Range $= (-\infty, \infty)$

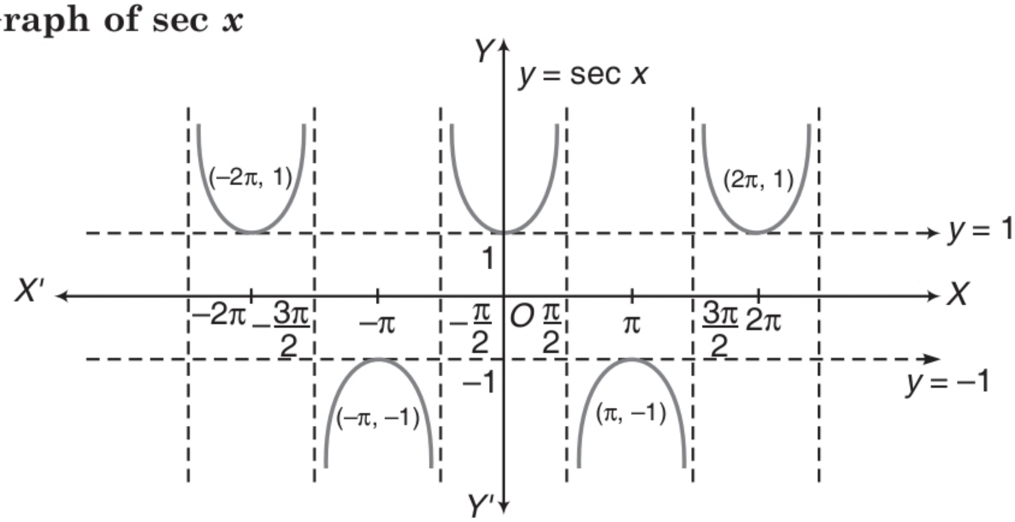
(iii) Period $= \pi$

4. **Graph of cot *x***



(i) Domain $= R \sim n\pi, n \in I$ (ii) Range $= (-\infty, \infty)$ (iii) Period $= \pi$

5. **Graph of sec *x***

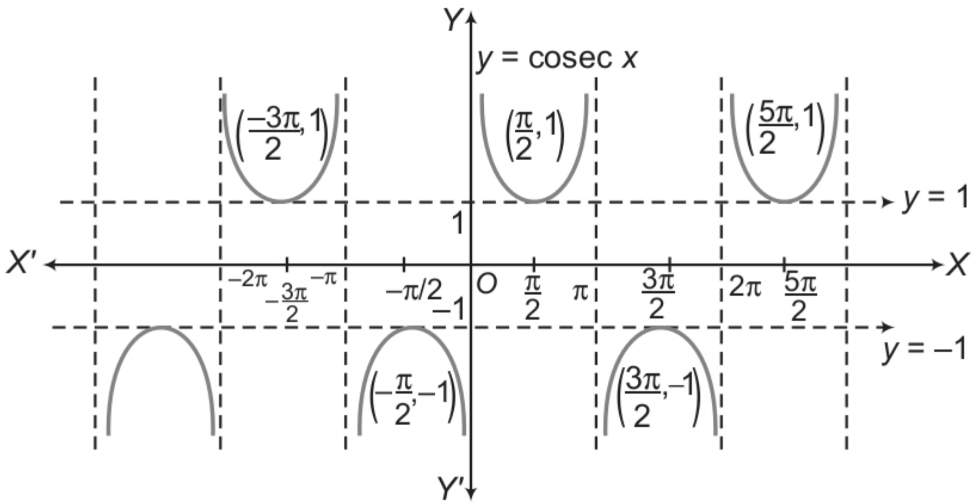


(i) Domain $= R \sim (2n+1)\dfrac{\pi}{2}, n \in I$

(ii) Range $= (-\infty, -1] \cup [1, \infty)$

(iii) Period $= 2\pi$

6. **Graph of cosec x**



(i) Domain $= R \sim n\pi, n \in I$

(ii) Range $= (-\infty, -1] \cup [1, \infty)$

(iii) Period $= 2\pi$

Note $|\sin\theta| \le 1, |\cos\theta| \le 1, |\sec\theta| \ge 1, |\text{cosec}\,\theta| \ge 1$ for all values of θ, for which the functions are defined.

Trigonometric Functions in Terms of sine and cosine Functions

Given below are trigonometric functions defined in terms of sine and cosine functions

(i) $\sin\theta = \dfrac{1}{\text{cosec}\,\theta}$ or $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$

(ii) $\cos\theta = \dfrac{1}{\sec\theta}$ or $\sec\theta = \dfrac{1}{\cos\theta}$

(iii) $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$ or $\tan\theta = \dfrac{1}{\cot\theta} = \dfrac{\sin\theta}{\cos\theta}$

Fundamental Trigonometric Identities

An equation involving trigonometric functions which is true for all those angles for which the functions are defined is called trigonometrical identity.

(i) $\cos^2\theta + \sin^2\theta = 1$ or $1 - \cos^2\theta = \sin^2\theta$ or $1 - \sin^2\theta = \cos^2\theta$

(ii) $1 + \tan^2\theta = \sec^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$ or $\sec^2\theta - \tan^2\theta = 1$

(iii) $1 + \cot^2\theta = \text{cosec}^2\,\theta$ or $\cot^2\theta = \text{cosec}^2\theta - 1$ or $\text{cosec}^2\theta - \cot^2\theta = 1$

Transformation of One Trigonometric Function to Another Trigonometric Function

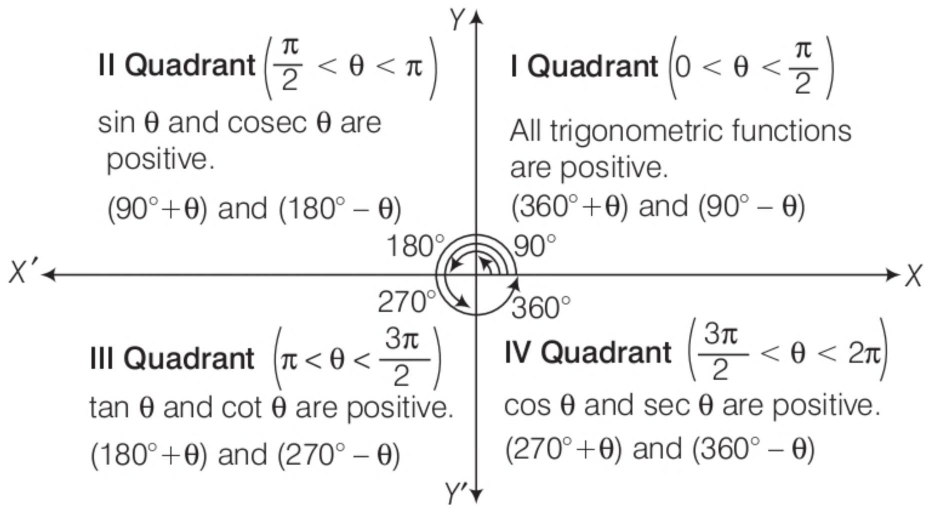
Trigonometric function	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{(1-\cos^2\theta)}$	$\dfrac{\tan\theta}{\sqrt{(1+\tan^2\theta)}}$	$\dfrac{1}{\sqrt{1+\cot^2\theta}}$	$\dfrac{\sqrt{(\sec^2\theta-1)}}{\sec\theta}$	$\dfrac{1}{\operatorname{cosec}\theta}$
$\cos\theta$	$\sqrt{(1-\sin^2\theta)}$	$\cos\theta$	$\dfrac{1}{\sqrt{(1+\tan^2\theta)}}$	$\dfrac{\cot\theta}{\sqrt{(1+\cot^2\theta)}}$	$\dfrac{1}{\sec\theta}$	$\dfrac{\sqrt{(\operatorname{cosec}^2\theta-1)}}{\operatorname{cosec}\theta}$
$\tan\theta$	$\dfrac{\sin\theta}{\sqrt{(1-\sin^2\theta)}}$	$\dfrac{\sqrt{(1-\cos^2\theta)}}{\cos\theta}$	$\tan\theta$	$\dfrac{1}{\cot\theta}$	$\sqrt{(\sec^2\theta-1)}$	$\dfrac{1}{\sqrt{(\operatorname{cosec}^2\theta-1)}}$
$\cot\theta$	$\dfrac{\sqrt{(1-\sin^2\theta)}}{\sin\theta}$	$\dfrac{\cos\theta}{\sqrt{(1-\cos^2\theta)}}$	$\dfrac{1}{\tan\theta}$	$\cot\theta$	$\dfrac{1}{\sqrt{(\sec^2\theta-1)}}$	$\sqrt{(\operatorname{cosec}^2\theta-1)}$
$\sec\theta$	$\dfrac{1}{\sqrt{(1-\sin^2\theta)}}$	$\dfrac{1}{\cos\theta}$	$\sqrt{(1+\tan^2\theta)}$	$\dfrac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\dfrac{\operatorname{cosec}\theta}{\sqrt{(\operatorname{cosec}^2\theta-1)}}$
$\operatorname{cosec}\theta$	$\dfrac{1}{\sin\theta}$	$\dfrac{1}{\sqrt{(1-\cos^2\theta)}}$	$\dfrac{\sqrt{(1+\tan^2\theta)}}{\tan\theta}$	$\sqrt{(1+\cot^2\theta)}$	$\dfrac{\sec\theta}{\sqrt{(\sec^2\theta-1)}}$	$\operatorname{cosec}\theta$

Note Above table is applicable only when $\theta \in (0^\circ, 90^\circ)$.

Sign of Trigonometric Functions in Different Quadrants

If we draw two mutually perpendicular (intersecting) lines in the plane of paper, then these lines divide the plane of paper into four parts, known as quadrants.

In anti-clockwise order, these quadrants are numbered as I, II, III and IV. All angles from 0° to 90° are taken in I quardant, 90° to 180° in II quardant, 180° to 270° in III quadrant and 270° to 360° in IV quadrant.



Trigonometric Ratios of Some Special Angles

Angle	$7\frac{1}{2}^\circ$	15°	18°	$22\frac{1}{2}^\circ$	36°
$\sin\theta$	$\frac{\sqrt{4-\sqrt{2}-\sqrt{6}}}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}\sqrt{2-\sqrt{2}}$	$\frac{1}{4}\sqrt{10-2\sqrt{5}}$
$\cos\theta$	$\frac{\sqrt{4+\sqrt{2}+\sqrt{6}}}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{1}{4}\sqrt{10+2\sqrt{5}}$	$\frac{1}{2}\sqrt{2+\sqrt{2}}$	$\frac{\sqrt{5}+1}{4}$
$\tan\theta$	$(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$	$2-\sqrt{3}$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$	$\sqrt{2}-1$	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$

Trigonometric Ratios (or Functions) of Allied Angles

Two angles are said to be allied when their sum or difference is either zero or a multiple of 90°. The angles $-\theta, 90^\circ \pm \theta, 180^\circ \pm \theta, 270^\circ \pm \theta$, $360^\circ - \theta$ etc., are angles allied to the angle θ, if θ is measured in degrees.

Allied Angles	$\sin\theta$	$\text{cosec}\,\theta$	$\cos\theta$	$\sec\theta$	$\tan\theta$	$\cot\theta$
$-\theta$	$-\sin\theta$	$-\text{cosec}\,\theta$	$\cos\theta$	$\sec\theta$	$-\tan\theta$	$-\cot\theta$
$90^\circ-\theta$	$\cos\theta$	$\sec\theta$	$\sin\theta$	$\text{cosec}\,\theta$	$\cot\theta$	$\tan\theta$
$90^\circ+\theta$	$\cos\theta$	$\sec\theta$	$-\sin\theta$	$-\text{cosec}\,\theta$	$-\cot\theta$	$-\tan\theta$
$180^\circ-\theta$	$\sin\theta$	$\text{cosec}\,\theta$	$-\cos\theta$	$-\sec\theta$	$-\tan\theta$	$-\cot\theta$
$180^\circ+\theta$	$-\sin\theta$	$-\text{cosec}\,\theta$	$-\cos\theta$	$-\sec\theta$	$\tan\theta$	$\cot\theta$
$270^\circ-\theta$	$-\cos\theta$	$-\sec\theta$	$-\sin\theta$	$-\text{cosec}\,\theta$	$\cot\theta$	$\tan\theta$
$270^\circ+\theta$	$-\cos\theta$	$-\sec\theta$	$\sin\theta$	$\text{cosec}\,\theta$	$-\cot\theta$	$-\tan\theta$
$360^\circ-\theta$	$-\sin\theta$	$-\text{cosec}\,\theta$	$\cos\theta$	$\sec\theta$	$-\tan\theta$	$-\cot\theta$

Trigonometric Functions of Compound Angles

The algebraic sum of two or more angles are generally called compound angles and the angles are known as the constituent angle. Some standard formulae of compound angles have been given below

(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(ii) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(iii) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(iv) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

(vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

(vii) $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot B + \cot A}$

(viii) $\cot(A-B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$

Some Important Results

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(iii) $\sin(A+B+C) = \cos A\cos B\sin C + \cos A\sin B\cos C + \sin A\cos B\cos C - \sin A\sin B\sin C$

or $\sin(A+B+C) = \cos A\cos B\cos C(\tan A + \tan B + \tan C - \tan A\tan B\tan C)$

(iv) $\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$

or $\cos(A+B+C) = \cos A \cos B \cos C(1 - \tan A \tan B - \tan B \tan C - \tan C \tan A)$

(v) $$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

If $A+B+C=0$, then $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vi) (a) $\sin(A_1 + A_2 + \ldots + A_n) = (\cos A_1 \cos A_2 \cos A_3 \ldots \cos A_n) \times (S_1 - S_3 + S_5 - S_7 + \ldots)$

(b) $\cos(A_1 + A_2 + \ldots + A_n) = (\cos A_1 \cos A_2 \cos A_3 \ldots \cos A_n) \times (1 - S_2 + S_4 - S_6 + \ldots)$

(c) $$\tan(A_1 + A_2 + \ldots + A_n) = \frac{S_1 - S_3 + S_5 - S_7 + \ldots}{1 - S_2 + S_4 - S_6 + \ldots}$$

where, $S_1 = \tan A_1 + \tan A_2 + \ldots + \tan A_n$

[sum of the tangents of the separate angles]

$S_2 = \tan A_1 \tan A_2 + \tan A_2 \tan A_3 + \ldots$

[sum of the tangents taken two at a time]

$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \ldots$

[sum of the tangents taken three at a time]

Note If $A_1 = A_2 = \cdots A_n = A$, then we have

$S_1 = n \tan A, S_2 = {}^nC_2 \tan^2 A, S_3 = {}^nC_3 \tan^3 A, \ldots$ so on.

Transformation Formulae

(i) $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

(ii) $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$

(iii) $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

(iv) $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

(v) $$\sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$$

(vi) $$\sin C - \sin D = 2 \cos\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$$

(vii) $$\cos C + \cos D = 2 \cos\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$$

(viii) $$\cos C - \cos D = -2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$$

$$= 2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{D-C}{2}\right)$$

Trigonometric Functions of Multiple Angles

(i) $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$

$= 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

(iv) $\sin 3A = 3\sin A - 4\sin^3 A$

(v) $\cos 3A = 4\cos^3 A - 3\cos A$

(vi) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

Trigonometric Functions of Sub-multiple Angles

(i) $\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2} = \dfrac{2\tan\dfrac{A}{2}}{1+\tan^2\dfrac{A}{2}}$

(ii) $\cos A = \cos^2\dfrac{A}{2} - \sin^2\dfrac{A}{2} = 2\cos^2\dfrac{A}{2} - 1 = 1 - 2\sin^2\dfrac{A}{2} = \dfrac{1-\tan^2\dfrac{A}{2}}{1+\tan^2\dfrac{A}{2}}$

(iii) $\tan A = \dfrac{2\tan\dfrac{A}{2}}{1-\tan^2\dfrac{A}{2}}$

(iv) $1 - \cos A = 2\sin^2\dfrac{A}{2}$

(v) $1 + \cos A = 2\cos^2\dfrac{A}{2}$

(vi) $\dfrac{1-\cos A}{1+\cos A} = \tan^2\dfrac{A}{2}$

(vii) $\sin\left(\dfrac{A}{2}\right) + \cos\left(\dfrac{A}{2}\right) = \pm\sqrt{1+\sin A}$

(viii) $\sin\left(\dfrac{A}{2}\right) - \cos\left(\dfrac{A}{2}\right) = \pm\sqrt{1-\sin A}$

Some Important Results

1. Product of Trigonometric Ratio

(i) $\sin\theta \sin(60° - \theta) \sin(60° + \theta) = \frac{1}{4}\sin 3\theta$

(ii) $\cos\theta \cos(60° - \theta) \cos(60° + \theta) = \frac{1}{4}\cos 3\theta$

(iii) $\tan\theta \tan(60° - \theta) \tan(60° + \theta) = \tan 3\theta$

(iv) $\cos 36° \cos 72° = \frac{1}{4}$

(v) $\cos A \cos 2A \cos 4A \ldots \cos 2^{n-1} A = \frac{1}{2^n \sin A} \sin(2^n A)$

2. Sum of Trigonometric Ratios

(i) $\sin A + \sin(A + B) + \sin(A + 2B) + \ldots + \sin(A + (n-1)B)$

$$= \frac{\sin\left\{A + (n-1)\frac{B}{2}\right\} \sin\frac{nB}{2}}{\sin\frac{B}{2}}$$

(ii) $\cos A + \cos(A + B) + \cos(A + 2B) + \ldots + \cos(A + (n-1)B)$

$$= \frac{\sin\frac{nB}{2}}{\sin\frac{B}{2}} \cos\left\{A + \frac{(n-1)B}{2}\right\}$$

3. Identities for Angles of a Triangle

If A, B and C are angles of a triangle (or $A + B + C = \pi$), then

(i) (a) $\sin(B + C) = \sin A$ (b) $\cos(B + C) = -\cos A$

(c) $\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$ (d) $\cos\left(\frac{B+C}{2}\right) = \sin\frac{A}{2}$

(ii) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

(iii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

(iv) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(v) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(vi) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vii) $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$

(viii) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(ix) $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$

Trigonometric Periodic Functions

A function $f(x)$ is said to be periodic, if there exists a real number $T>0$ such that $f(x+T)=f(x)$ for all x. T is called the period of the function, all trigonometric functions are periodic.

Important Points to be Remembered

(i) $\sin\theta$, $\cos\theta$, $\text{cosec}\,\theta$ and $\sec\theta$ have a period of 2π.
(ii) $\tan\theta$, $\cot\theta$ have a period of π.
(iii) Period of $\sin k\theta$ is $2\pi/k$.
(iv) Period of $\tan k\theta$ is π/k.
(v) Period of $\sin^n\theta$, $\cos^n\theta$, $\sec^n\theta$ and $\text{cosec}^n\theta$ is 2π, if n is odd and, π if n is even.
(vi) Period of $\tan^n\theta$, $\cot^n\theta$ is π, if n is even or odd.
(vii) Period of $|\sin\theta|, |\cos\theta|, |\tan\theta|, |\cot\theta|, |\sec\theta|$ and $|\text{cosec}\,\theta|$ is π.
(viii) Period of $|\sin\theta|+|\cos\theta|$, $|\tan\theta|+|\cot\theta|$ and $|\sec\theta|+|\text{cosec}\,\theta|$ is $\pi/2$.

Maximum and Minimum Values of a Trigonometric Expression

(i) Maximum value of $a\cos\theta\pm b\sin\theta=\sqrt{a^2+b^2}$

Minimum value of $a\cos\theta\pm b\sin\theta=-\sqrt{a^2+b^2}$

(ii) Maximum value of $a\cos\theta\pm b\sin\theta+c=c+\sqrt{a^2+b^2}$

Minimum value of $a\cos\theta\pm b\sin\theta+c=c-\sqrt{a^2+b^2}$

Hyperbolic Functions

The hyperbolic functions $\sinh z, \cosh z, \tanh z, \text{cosech}\ z, \text{sec h}\ z, \coth z$ are angles of the circular functions, defined by removing is appearing in the complex exponentials.

(i) $\sinh x=\frac{e^x-e^{-x}}{2}$

(ii) $\cosh x = \dfrac{e^x + e^{-x}}{2}$

(iii) $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

(iv) $\text{cosech}\, x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

(v) $\text{sech}\, x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

(vi) $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

Domain and Range of Hyperbolic Function

Hyperbolic function	Domain	Range
$\sinh x$	R	R
$\cosh x$	R	$[1, \infty)$
$\tanh x$	R	$(-1, 1)$
$\text{cosech}\, x$	$R - \{0\}$	$R - \{0\}$
$\text{sech}\, x$	R	$(0, 1]$
$\coth x$	$R - \{0\}$	$R - [-1, 1]$

Identities

(i) $\cosh^2 x - \sinh^2 x = 1$

(ii) $\text{sech}^2 x + \tanh^2 x = 1$

(iii) $\coth^2 x - \text{cosech}^2 x = 1$

(iv) $\cosh^2 x + \sinh^2 x = \cosh 2x$

Formulae for the Sum and Difference

(i) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

(ii) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(iii) $\tanh\ (x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

Formulae to Transform the Product into Sum or Difference

(i) $\sinh x + \sinh y = 2 \sinh\left(\dfrac{x+y}{2}\right) \cosh\left(\dfrac{x-y}{2}\right)$

(ii) $\sinh x - \sinh y = 2 \cosh\left(\dfrac{x+y}{2}\right) \sinh\left(\dfrac{x-y}{2}\right)$

(iii) $\cosh x + \cosh y = 2\cosh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$

(iv) $\cosh x - \cosh y = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$

(v) $2\sinh x\cosh y = \sinh(x+y) + \sinh(x-y)$

(vi) $2\cosh x\sinh y = \sinh(x+y) - \sinh(x-y)$

(vii) $2\cosh x\cosh y = \cosh(x+y) + \cosh(x-y)$

(viii) $2\sinh x\sinh y = \cosh(x+y) - \cosh(x-y)$

Formulae for Multiples of *x*

(i) $\sinh 2x = 2\sinh x\cosh x = \frac{2\tanh x}{1-\tanh^2 x}$

(ii) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

$$= \frac{1+\tanh^2 x}{1-\tanh^2 x}$$

(iii) $\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$

(iv) $\sinh 3x = 3\sinh x + 4\sinh^3 x$

(v) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

(vi) $\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$

Important Formulae

1. (i) $\sinh^2 x - \sinh^2 y = \sinh(x+y)\sinh(x-y)$
   (ii) $\cosh^2 x + \sinh^2 y = \cosh(x+y)\cosh(x-y)$
   (iii) $\cosh^2 x - \cosh^2 y = \sinh(x+y)\sinh(x-y)$
2. (i) $\sin ix = i\sinh x$ (ii) $\cos(ix) = \cosh x$
   (iii) $\tan(ix) = i\tanh x$ (iv) $\cot(ix) = -i\coth x$
   (v) $\sec(ix) = \operatorname{sech} x$ (vi) $\operatorname{cosec}(ix) = -i\operatorname{cosech} x$
3. (i) $\sinh x = -i\sin(ix)$ (ii) $\cosh x = \cos(ix)$
   (iii) $\tanh x = -i\tan(ix)$ (iv) $\coth x = i\cot(ix)$
   (v) $\operatorname{sech} x = \sec(ix)$ (vi) $\operatorname{cosech} x = i\operatorname{cosec}(ix)$

Trigonometric Equations

An equation involving one or more trigonometrical ratios of unknown angle is called a trigonometric equation.

Solution/Roots of a Trigonometric Equation

A value of the unknown angle which satisfies the given equation, is called a solution or root of the equation.

The trigonometric equation may have infinite number of solutions.

(i) **Principal Solution** The least value of unknown angle which satisfies the given equation, is called a principal solution of trigonometric equation.

(ii) **General Solution** We know that trigonometric function are periodic and solution of trigonometric equations can be generalised with the help of the periodicity of the trigonometric functions. The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

Some Important Results

(i) $\sin\theta = 0 \Rightarrow \theta = n\pi$, where $n \in z$

(ii) $\cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}$, where $n \in z$

(iii) $\tan\theta = 0 \Rightarrow \theta = n\pi$, where $n \in z$

(iv) $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$, where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $n \in z$

(v) $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$, where $\alpha \in [0, \pi]$ and $n \in z$

(vi) $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$, where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $n \in z$

(vii) $\sin^2\theta = \sin^2\alpha$, $\cos^2\theta = \cos^2\alpha$, $\tan^2\theta = \tan^2\alpha$

$\Rightarrow \quad \theta = n\pi \pm \alpha$, where $n \in z$

(viii) $\sin\theta = 1 \Rightarrow \theta = (4n+1)\frac{\pi}{2}$, where $n \in z$

(ix) $\cos\theta = 1 \Rightarrow \theta = 2n\pi$, where $n \in z$

(x) $\cos\theta = -1 \Rightarrow \theta = (2n+1)\pi$, where $n \in z$

(xi) $$\left.\begin{array}{l}\sin\theta = \sin\alpha \text{ and } \cos\theta = \cos\alpha \\ \sin\theta = \sin\alpha \text{ and } \tan\theta = \tan\alpha \\ \tan\theta = \tan\alpha \text{ and } \cos\theta = \cos\alpha\end{array}\right\} \Rightarrow \theta = 2n\pi + \alpha, \text{ where } n \in z$$

(xii) Equation of the form $a \cos\theta + b\sin\theta = c$

Put $a = r\cos\alpha$ and $b = r\sin\alpha$, where

$$r = \sqrt{a^2 + b^2} \text{ and } |c| \le \sqrt{a^2 + b^2}$$

$$\therefore \quad \theta = 2n\pi \pm \alpha + \phi,\ n \in I$$

where, $\alpha = \cos^{-1}\dfrac{|c|}{\sqrt{a^2+b^2}}$ and $\phi = \tan^{-1}\dfrac{b}{a}$

(a) If $|c| > \sqrt{a^2 + b^2}$, equation has no solution.

(b) If $|c| \le \sqrt{a^2 + b^2}$, equation is solvable.

(xiii) $\sin\left(\dfrac{n\pi}{2} + \theta\right) = (-1)^{\frac{n}{2}}\cos\theta$, if n is odd.

$= (-1)^{\frac{n}{2}}\sin\theta$, if n is even.

(xiv) $\cos\left(\dfrac{n\pi}{2} + \theta\right) = (-1)^{\frac{n-1}{2}}\sin\theta$, if n is odd.

$= (-1)^{\frac{n}{2}}\cos\theta$, if n is even.

(xv) $\sin\theta_1 + \sin\theta_2 + \ldots + \sin\theta_n = n \Rightarrow \sin\theta_1 = \sin\theta_2 = \ldots = \sin\theta_n = 1$

(xvi) $\cos\theta_1 + \cos\theta_2 + \ldots + \cos\theta_n = n \Rightarrow \cos\theta_1 = \cos\theta_2 = \ldots = \cos\theta_n = 1$

(xvii) $\sin\theta + \operatorname{cosec}\theta = 2 \Rightarrow \sin\theta = 1$

(xviii) $\cos\theta + \sec\theta = 2 \Rightarrow \cos\theta = 1$

(xix) $\sin\theta + \operatorname{cosec}\theta = -2 \Rightarrow \sin\theta = -1$

(xx) $\cos\theta + \sec\theta = -2 \Rightarrow \cos\theta = -1$

Important Points to be Remembered

(i) While solving an equation, we have to square it, sometimes the resulting roots does not satisfy the original equation.

(ii) Do not cancel common factors involving the unknown angle on LHS and RHS. Because it may be the solution of given equation.

(iii) (a) Equation involving $\sec\theta$ or $\tan\theta$ can never be a solution of the form $(2n+1)\dfrac{\pi}{2}$.

(b) Equation involving $\operatorname{cosec}\theta$ or $\cot\theta$ can never be a solution of the form $\theta = n\pi$.