Vectors

A **vector** has direction and magnitude both but **scalar** has only magnitude. e.g. Vector quantities are displacement, velocity, acceleration, etc. and scalar quantities are length, mass, time, etc.

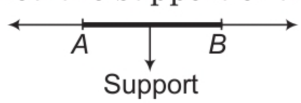
Characteristics of a Vector

(i) **Magnitude** The length of the vector $\mathbf{AB}$ or $\mathbf{a}$ is called the magnitude of $\mathbf{AB}$ or $\mathbf{a}$ and it is represented as $|\mathbf{AB}|$ or $|\mathbf{a}|$.

(ii) **Sense** The direction of a line segment from its initial point to its terminal point is called its sense.

e.g. The sense of $\mathbf{AB}$ is from A to B and that of $\mathbf{BA}$ is from B to A.

(iii) **Support** The line of infinite length of which the line segment AB is a part, is called the support of the vector $\mathbf{AB}$.



Types of Vectors

(i) **Zero or Null Vector** A vector whose initial and terminal points are coincident is called zero or null vector. It is denoted by $\mathbf{0}$.

(ii) **Unit Vector** A vector whose magnitude is unity i.e., 1 unit is called a unit vector. The unit vector in the direction of $\mathbf{n}$ is given by $\frac{\mathbf{n}}{|\mathbf{n}|}$ and it is denoted by $\hat{\mathbf{n}}$.

(iii) **Free Vector** If the initial point of a vector is not specified, then it is said to be a free vector.

(iv) **Like and Unlike Vectors** Vectors are said to be like when they have the same direction and unlike when they have opposite direction.

(v) **Collinear or Parallel Vectors** Vectors having the same or parallel supports are called collinear vectors.

(vi) **Equal Vectors** Two vectors **a** and **b** are said to be equal, written as **a** = **b**, if they have same length and same direction.

(vii) **Negative Vector** A vector having the same magnitude as that of a given vector **a** and the direction opposite to that of **a** is called the negative vector **a** and it is denoted by – **a**.

(viii) **Coinitial Vectors** Vectors having same initial point are called coinitial vectors.

(ix) **Coterminus Vectors** Vectors having the same terminal point are called coterminus vectors.

(x) **Localised Vectors** A vector which is drawn parallel to a given vector through a specified point in space is called localised vector.

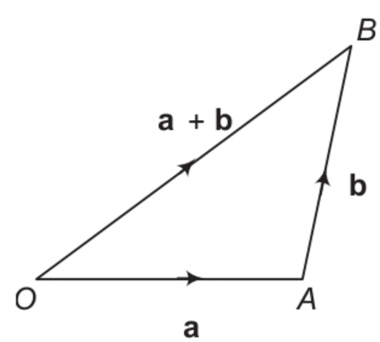
(xi) **Coplanar Vectors** A system of vectors is said to be coplanar, if their supports are parallel to the same plane. Otherwise they are called non-coplanar vectors.

(xii) **Reciprocal of a Vector** A vector having the same direction as that of a given vector but magnitude equal to the reciprocal of the given vector is known as the reciprocal of **a** and it is denoted by $\mathbf{a}^{-1}$, i.e. if $|\mathbf{a}| = a$, then $|\mathbf{a}^{-1}| = \frac{1}{a}$.

Addition of Vectors

Triangle Law of Vector Addition

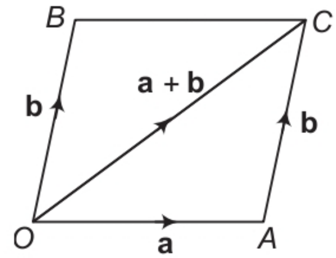
Let **a** and **b** be any two vectors. From the terminal point of **a**, vector **b** is drawn. Then, the vector from the initial point O of **a** to the terminal point B of **b** is called the sum of vectors **a** and **b** and is denoted by **a** + **b**. This is called the triangle law of addition of vectors.



Note When the sides of a triangle are taken in order, then the resultant will be $\mathbf{AB} + \mathbf{BC} + \mathbf{CA} = \mathbf{0}$

Parallelogram Law of Vector Addition

Let **a** and **b** be any two vectors. From the initial point of **a**, vector **b** is drawn and parallelogram *OACB* is completed with *OA* and *OB* as adjacent sides. The diagonal of the parallelogram through the common vertex represents the vector **OC** and it is defined as the sum of **a** and **b**. This is called the parallelogram law of vector addition.



The sum of two vectors is also called their resultant and the process of addition as **composition**.

Properties of Vector Addition

Let **a, b** and **c** are three vectors.

(i) $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$ (commutative)

(ii) $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$ (associative)

(iii) $\mathbf{a}+\mathbf{0}=\mathbf{a}$ (additive identity)

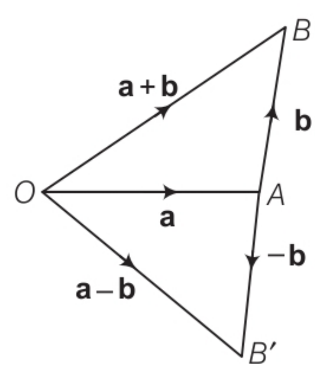
(iv) $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$ (additive inverse)

Note The bisector of the angle between two non-collinear vectors **a** and **b** is given by

$$\lambda(\hat{\mathbf{a}}+\hat{\mathbf{b}}) \text{ or } \lambda\left(\frac{\mathbf{a}}{|\mathbf{a}|}\pm\frac{\mathbf{b}}{|\mathbf{b}|}\right).$$

Difference (Subtraction) of Vectors

If **a** and **b** are any two vectors, then their difference $\mathbf{a}-\mathbf{b}$ is defined as $\mathbf{a}+(-\mathbf{b})$. In the given figure the vector **AB′** is said to represent the difference of **a** and **b**.



Multiplication of a Vector by a Scalar

Let **a** be a given vector and λ be a scalar. Then, the product of the vector **a** by the scalar λ is $\lambda\,\mathbf{a}$ and is called the multiplication of vector by the scalar.

Important Properties

(i) $|\lambda\,\mathbf{a}| = |\lambda||\mathbf{a}|$, where λ be a scalar.

(ii) $\lambda\,\mathbf{0} = \mathbf{0}$

(iii) $m(-\mathbf{a}) = -m\,\mathbf{a} = -(m\,\mathbf{a})$

(iv) $(-m)(-\mathbf{a}) = m\,\mathbf{a}$

(v) $m(n\,\mathbf{a}) = mn\,\mathbf{a} = n(m\,\mathbf{a})$

(vi) $(m+n)\mathbf{a} = m\,\mathbf{a} + n\,\mathbf{a}$

(vii) $m(\mathbf{a}+\mathbf{b}) = m\,\mathbf{a} + m\,\mathbf{b}$

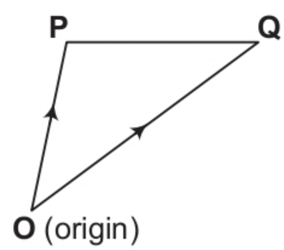
Position Vector of a Point

The position vector of a point P with respect to a fixed point say O, is the vector **OP**. The fixed point is called the **origin**.

Let **PQ** be any vector. We have,

$$\mathbf{PQ} = \mathbf{PO} + \mathbf{OQ} = -\mathbf{OP} + \mathbf{OQ} = \mathbf{OQ} - \mathbf{OP}$$

$$= \text{Position vector of } Q - \text{Position vector of } P.$$



i.e. $\mathbf{PQ} = \text{PV of } \mathbf{Q} - \text{PV of } \mathbf{P}$

Collinear Points

Let A, B and C be any three points.

Points A, B, C are collinear $\Leftrightarrow$ **AB, BC** are collinear vectors

$\Leftrightarrow \mathbf{AB} = \lambda\mathbf{BC}$ for some non-zero scalar λ.

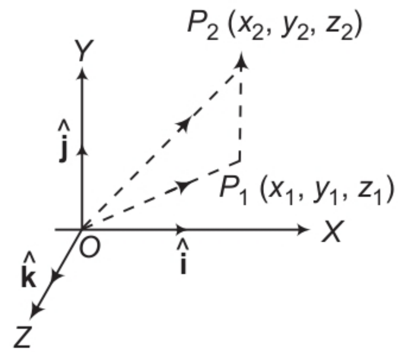
Components of a Vector

1. **In Two-dimension** Let $P(x, y)$ be any point in a plane and O be the origin. Let $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ be the unit vectors along X and Y-axes, then the component of vector P is $\mathbf{OP} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$.

2. **In Three-dimension** Let $P(x, y, z)$ be any point is a space and O be the origin. Let $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ be the unit vectors along X, Y and Z-axes, then the component of vector P is $\mathbf{OP} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

Vector Joining Two Points

Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points, then the vector joining P_1 and P_2 is the vector $\mathbf{P_1P_2}$.



The position vectors of P_1 and P_2 with respect to the origin O are

$$\mathbf{OP_1} = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}} \text{ and } \mathbf{OP_2} = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_2\hat{\mathbf{k}}$$

Then, the component form of $\mathbf{P_1P_2}$ is

$$\mathbf{P_1P_2} = (x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_2\hat{\mathbf{k}}) - (x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}})$$

$$= (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} + (z_2 - z_1)\hat{\mathbf{k}}$$

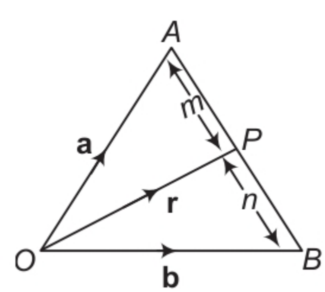
Here, vector component of $\mathbf{P_1P_2}$ are $(x_2 - x_1)\hat{\mathbf{i}}$, $(y_2 - y_1)\hat{\mathbf{j}}$ and $(z_2 - z_1)\hat{\mathbf{k}}$ along X-axis, Y-axis and Z-axis respectively.

Its magnitude is $|\mathbf{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Section Formulae

Let A and B be two points with position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively and $\mathbf{OP} = \mathbf{r}$.

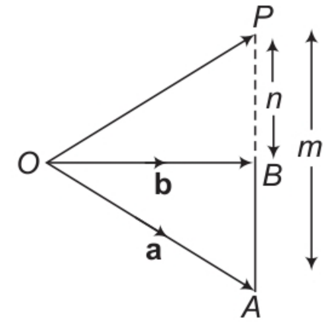
(i) **Internal division** Let P be a point dividing AB internally in the ratio $m : n$. Then, position vector of P is



$$\mathbf{OP} = \frac{m\,\mathbf{OB} + n\,\mathbf{OA}}{(m + n)}$$

i.e. $$\mathbf{r} = \frac{m\,\mathbf{b} + n\,\mathbf{a}}{m + n}$$

(ii) The position vector of the mid-point of $\mathbf{a}$ and $\mathbf{b}$ is $\dfrac{\mathbf{a}+\mathbf{b}}{2}$.



(iii) **External division** Let P be a point dividing AB externally in the ratio $m:n$. Then, position vector of P is

$$\mathbf{OP} = \frac{m\mathbf{OB} - n\mathbf{OA}}{m-n}$$

i.e.

$$\mathbf{r} = \frac{m\mathbf{b} - n\mathbf{a}}{m-n}.$$

Position Vector of Different Centre of a Triangle

(i) If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be PV's of the vertices A, B, C of a ΔABC respectively, then the PV of the centroid G of the triangle is $\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$.

(ii) The PV of incentre of ΔABC is $\dfrac{(BC)\,\mathbf{a} + (CA)\mathbf{b} + (AB)\mathbf{c}}{BC + CA + AB}$

(iii) The PV of orthocentre of ΔABC is

$$\frac{\mathbf{a}(\tan A) + \mathbf{b}(\tan B) + \mathbf{c}(\tan C)}{\tan A + \tan B + \tan C}$$

Linear Combination of Vectors

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ be vectors and $x, y, z, \ldots$ be scalars, then the expression $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + \ldots$ is called a linear combination of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$.

Collinearity of Three Points

The necessary and sufficient condition that three points with PV's $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are collinear, if there exist three scalars x, y, z not all zero such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0} \Rightarrow x + y + z = 0$.

Coplanarity of Four Points

The necessary and sufficient condition that four points with PV's $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ are coplanar, if there exist scalar x, y, z and t not all zero, such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + t\mathbf{d} = 0 \Leftrightarrow x + y + z + t = 0$.

If

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}\ldots$$

then, the vector $\mathbf{r}$ is said to be a linear combination of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$.

Linearly and Dependent and Independent System of Vectors

(i) The system of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ is said to be **linearly dependent**, if there exists some scalars x, y, z, … not all zero, such that $x\,\mathbf{a} + y\,\mathbf{b} + z\,\mathbf{c} + \ldots = \mathbf{0}$.

(ii) The system of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ is said to be **linearly independent**, if $x\mathbf{a} + y\,\mathbf{b} + z\,\mathbf{c} + t\,\mathbf{d} = \mathbf{0} \Rightarrow x = y = z = t \ldots = 0$.

Important Points to be Remembered

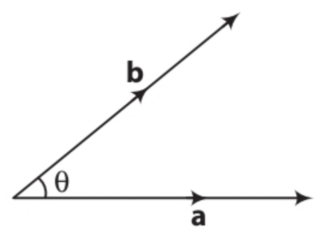
(i) Two non-zero, non-collinear vectors **a** and **b** are linearly independent.

(ii) Three non-zero, non-coplanar vectors **a**, **b** and **c** are linearly independent.

(iii) More than three vectors are always linearly dependent.

Scalar or Dot Product of Two Vectors

If **a** and **b** are two non-zero vectors, then the scalar or dot product of **a** and **b** is denoted by $\mathbf{a} \cdot \mathbf{b}$ and is defined as $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$, where θ is the angle between the two vectors and $0 \le \theta \le \pi$.



(i) Angle between two like vectors is 0 and angle between two unlike vectors is π.

(ii) If either **a** or **b** is the null vector, then scalar product of the vector is zero.

(iii) If **a** and **b** are two unit vectors, then $\mathbf{a} \cdot \mathbf{b} = \cos\theta$.

(iv) The scalar product is commutative

i.e. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(v) If $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are mutually perpendicular unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, then

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$

and $$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$$

(vi) The scalar product of vectors is distributive over vector addition.

(a) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (left distributive)

(b) $(\mathbf{b} + \mathbf{c}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{a}$ (right distributive)

(vii) $(m\mathbf{a}) \cdot (\mathbf{b}) = m(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (m\mathbf{b})$, where m is any scalar.

(viii) If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, then $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2$

or $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

(ix) **Angle between Two Vectors** If θ is angle between two non-zero vectors, **a, b**, then we have

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

or
$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$

Then, the angle θ between **a** and **b** is given by

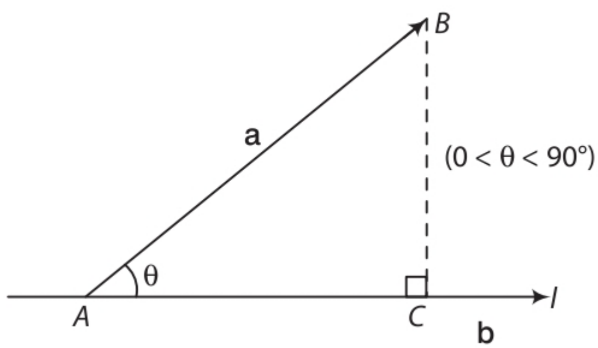
$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

Condition of perpendicularity $\mathbf{a} \cdot \mathbf{b} = 0 \Leftrightarrow \mathbf{a} \perp \mathbf{b}, \mathbf{a}$ and **b** being non-zero vectors.

(x) **Projection and Component of a Vector on a Line**

The projection of **a** on $\mathbf{b} = \mathbf{a} \cdot \hat{\mathbf{b}} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$

The projection of **b** on $\mathbf{a} = \mathbf{b} \cdot \hat{\mathbf{a}} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$,



Components of **a** along and perpendicular to **b** are

$$\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \cdot \mathbf{b} \text{ and } \mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} \cdot \mathbf{b}$$

(xi) **Work done by a Force** The work done by a force is a scalar quantity equal to the product of the magnitude of the force and the resolved part of the displacement.

$\therefore$ $\mathbf{F} \cdot \mathbf{S}$ = dot products of force and displacement.

Suppose $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ are n forces acted on a particle, then during the displacement **S** of the particle, the separate forces to quantities of work $\mathbf{F}_1 \cdot \mathbf{S}, \mathbf{F}_2 \cdot \mathbf{S}, \ldots, \mathbf{F}_n \cdot \mathbf{S}$.

The total work done is $\sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{S} = \sum_{i=1}^{n} \mathbf{S} \cdot \mathbf{F}_i = \mathbf{S} \cdot \mathbf{R}$

Here, system of forces were replaced by its resultant **R**.

Important Results of Dot Product

(i) $(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2$

(ii) $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2(\mathbf{a} \cdot \mathbf{b})$

(iii) $|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2(\mathbf{a} \cdot \mathbf{b})$

(iv) $|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2)$

and $|\mathbf{a}+\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2 = 4(\mathbf{a} \cdot \mathbf{b})$

or $\mathbf{a} \cdot \mathbf{b} = \frac{1}{4}[|\mathbf{a}+\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2]$

(v) If $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$, then **a** is parallel to **b**.

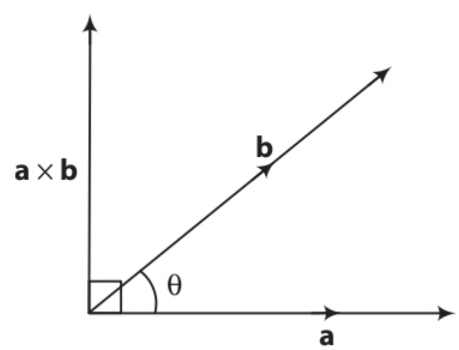
(vi) If $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$, then **a** is perpendicular to **b**.

(vii) $(\mathbf{a} \cdot \mathbf{b})^2 \le |\mathbf{a}|^2 |\mathbf{b}|^2$

Vector or Cross Product of Two Vectors

The vector product of the vectors **a** and **b** is denoted by $\mathbf{a} \times \mathbf{b}$ and it is defined as

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}| \sin\theta)\, \hat{\mathbf{n}} = ab \sin\theta\, \hat{\mathbf{n}} \quad \ldots\text{(i)}$$



where, $a = |\mathbf{a}|$, $b = |\mathbf{b}|$, θ is the angle between the vectors **a** and **b** and $\hat{\mathbf{n}}$ is a unit vector which is perpendicular to both **a** and **b**.

Important Results of Cross Product

(i) Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$

Then, $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(ii) If $\mathbf{a} = \mathbf{b}$ or if $\mathbf{a}$ is parallel to $\mathbf{b}$, then $\sin\theta = 0$ and so $\mathbf{a}\times\mathbf{b} = \mathbf{0}$.

(iii) The direction of $\mathbf{a}\times\mathbf{b}$ is regarded positive, if the rotation from $\mathbf{a}$ to $\mathbf{b}$ appears to be anti-clockwise.

(iv) $\mathbf{a}\times\mathbf{b}$ is perpendicular to the plane, which contains both $\mathbf{a}$ and $\mathbf{b}$. Thus, the unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ or to the plane containing is given by $\hat{\mathbf{n}} = \dfrac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|} = \dfrac{\mathbf{a}\times\mathbf{b}}{ab\sin\theta}$.

(v) Vector product of two parallel or collinear vectors is zero.

(vi) If $\mathbf{a}\times\mathbf{b} = \mathbf{0}$, then $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ and $\mathbf{b}$ are parallel or collinear.

(vii) **Vector Product of Two Perpendicular Vectors**
If $\theta = 90°$, then $\sin\theta = 1$, i.e. $\mathbf{a}\times\mathbf{b} = (ab)\,\hat{\mathbf{n}}$ or $|\mathbf{a}\times\mathbf{b}| = |ab\hat{\mathbf{n}}| = ab$
$[\because |\mathbf{a}| = a$ and $|\mathbf{b}| = b]$

(viii) **Vector Product of Two Unit Vectors** If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then

$$a = |\mathbf{a}| = 1, b = |\mathbf{b}| = 1$$

$\therefore$
$$\mathbf{a}\times\mathbf{b} = ab\sin\theta\cdot\hat{\mathbf{n}} = (\sin\theta)\cdot\hat{\mathbf{n}}$$

(ix) **Vector Product is not Commutative** The two vector products $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{a}$ are equal in magnitude but opposite in direction.

i.e.
$$\mathbf{b}\times\mathbf{a} = -\,\mathbf{a}\times\mathbf{b} \quad \ldots\text{(i)}$$

(x) **Distributive Law** For any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c})$$

(xi) **Area of a Triangle and Parallelogram**

(a) The area of a ΔABC is equal to $\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}|$ or $\frac{1}{2}|\mathbf{BC}\times\mathbf{BA}|$ or $\frac{1}{2}|\mathbf{CB}\times\mathbf{CA}|$.

(b) The area of a ΔABC with vertices having PV's $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively, is $1/2|\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}|$.

(c) The points whose PV's $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are collinear, if and only if $\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a} = \mathbf{0}$.

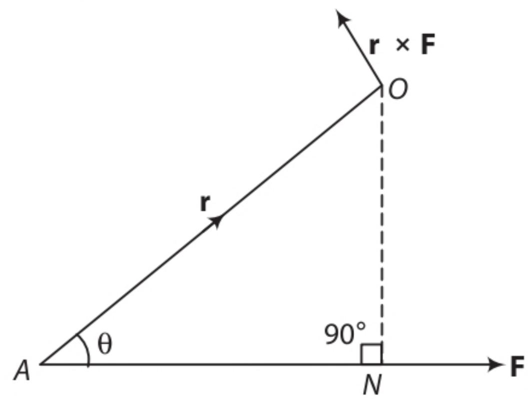
(d) The area of a parallelogram with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a}\times\mathbf{b}|$.

Contd. ...

(e) The area of a parallelogram with diagonals **a** and **b** is $\frac{1}{2}\ |\mathbf{a}\times\mathbf{b}|$.

(f) The area of a quadrilateral *ABCD* is equal to $\frac{1}{2}|\mathbf{AC}\times\mathbf{BD}|$.

(xii) **Vector Moment of a Force about a Point** The vector moment of torque **M** of a force **F** about the point *O* is the vector whose magnitude is equal to the product of **F** and the perpendicular distance of the point *O* from the line of action of **F**.



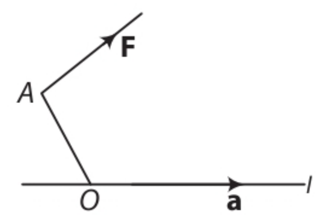
$$\therefore \qquad \mathbf{M}=\mathbf{r}\times\mathbf{F}$$

where, **r** is the position vector of *A* referred to *O*.

(a) The moment of force **F** about *O* is independent of the choice of point *A* on the line of action of **F**.

(b) If several forces are acting through the same point *A*, then the vector sum of the moments of the separate forces about a point *O* is equal to the moment of their resultant force about *O*.

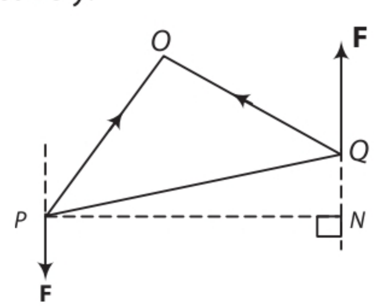
(xiii) **The Moment of a Force about a Line** Let **F** be a force acting at a point *A, O* be any point on the given line *l* and **a** be the unit vector along the line, then moment of **F** about the line *l* is a scalar given by $(\mathbf{OA}\times\mathbf{F})\cdot\mathbf{a}$.



(xiv) **Moment of a Couple**

(a) Two equal and unlike parallel forces whose lines of action are different is said to constitute a couple.

(b) Let *P* and *Q* be any two points on the lines of action of the forces $-\mathbf{F}$ and **F**, respectively.



The moment of the couple $=\mathbf{PQ}\times\mathbf{F}$

Scalar Triple Product

If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors, then $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is called scalar triple product and is denoted by $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

$\therefore$ $$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

Geometrical Interpretation of Scalar Triple Product

The scalar triple product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ represents the volume of a parallelopiped whose coterminus edges are represented by $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ which form a right handed system of vectors.

Expression of the scalar triple product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ in terms of components

$$\mathbf{a} = a_1\hat{\mathbf{i}} + b_1\hat{\mathbf{j}} + c_1\hat{\mathbf{k}}, \mathbf{b} = a_2\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + c_2\hat{\mathbf{k}}$$

and $$\mathbf{c} = a_3\hat{\mathbf{i}} + b_3\hat{\mathbf{j}} + c_3\hat{\mathbf{k}} \text{ is}$$

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & b_3 \end{vmatrix}$$

Properties of Scalar Triple Product

(i) The scalar triple product is independent of the positions of dot and cross i.e. $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

(ii) The scalar triple product of three vectors is unaltered so long as the cyclic order of the vectors remains unchanged.

i.e. $$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$$

or $$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] = [\mathbf{c}\ \mathbf{a}\ \mathbf{b}].$$

(iii) The scalar triple product changes in sign but not in magnitude, when the cyclic order is changed.

i.e. $$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = -[\mathbf{a}\ \mathbf{c}\ \mathbf{b}]$$

(iv) The scalar triple product vanishes, if any two of its vectors are equal.

i.e. $$[\mathbf{a}\ \mathbf{a}\ \mathbf{b}] = 0, [\mathbf{a}\ \mathbf{b}\ \mathbf{a}] = 0 \text{ and } [\mathbf{b}\ \mathbf{a}\ \mathbf{a}] = 0.$$

(v) The scalar triple product vanishes, if any two of its vectors are parallel or collinear.

(vi) For any scalar x, $[x\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = x\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

Also, $[x\mathbf{a}\ y\mathbf{b}\ z\mathbf{c}] = xyz\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

(vii) For any vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$; $[\mathbf{a} + \mathbf{b}\ \mathbf{c}\ \mathbf{d}] = [\mathbf{a}\ \mathbf{c}\ \mathbf{d}] + [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]$

(viii) The scalar triple product of cyclic components $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ is 1, i.e. $[\mathbf{i}\ \mathbf{j}\ \mathbf{k}] = 1$.

(ix) $(\mathbf{a}\times \mathbf{b})\cdot(\mathbf{c}\times \mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$

(x) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\ [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \begin{vmatrix} \mathbf{a}\cdot\mathbf{u} & \mathbf{b}\cdot\mathbf{u} & \mathbf{c}\cdot\mathbf{u} \\ \mathbf{a}\cdot\mathbf{v} & \mathbf{b}\cdot\mathbf{v} & \mathbf{c}\cdot\mathbf{v} \\ \mathbf{a}\cdot\mathbf{w} & \mathbf{b}\cdot\mathbf{w} & \mathbf{c}\cdot\mathbf{w} \end{vmatrix}$

(xi) Three non-zero vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are coplanar, if and only if $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$.

(xii) Four points A, B, C, D with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ respectively are coplanar, if and only if $[\mathbf{AB}\ \mathbf{AC}\ \mathbf{AD}] = 0$.

i.e. if and only if $[\mathbf{b}-\mathbf{a}\ \ \mathbf{c}-\mathbf{a}\ \ \mathbf{d}-\mathbf{a}] = 0$.

(xiii) Volume of parallelopiped with three coterminus edges $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ is $|[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]|$.

(xiv) Volume of prism on a triangular base with three coterminus edges $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ is $\frac{1}{2}|[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]|$.

(xv) Volume of a tetrahedron with three coterminus edges $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ is $\frac{1}{6}|[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]|$.

(xvi) If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ are position vectors of vertices of a tetrahedron, then

$$\text{Volume} = \frac{1}{6}|[\mathbf{b}-\mathbf{a}\ \ \mathbf{c}-\mathbf{a}\ \ \mathbf{d}-\mathbf{a}]|.$$

Vector Triple Product

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be any three vectors, then $(\mathbf{a}\times \mathbf{b})\times\mathbf{c}$ and $\mathbf{a}\times(\mathbf{b}\times \mathbf{c})$ are known as vector triple product.

$\therefore$ $$\mathbf{a}\times(\mathbf{b}\times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot \mathbf{b})\,\mathbf{c}$$

and $$(\mathbf{a}\times \mathbf{b})\times \mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot \mathbf{c})\,\mathbf{a}$$

Important Properties

(i) The vector $\mathbf{r} = \mathbf{a}\times(\mathbf{b}\times \mathbf{c})$ is perpendicular to $\mathbf{a}$ and lies in the plane $\mathbf{b}$ and $\mathbf{c}$.

(ii) $\mathbf{a}\times(\mathbf{b}\times \mathbf{c}) \neq (\mathbf{a}\times \mathbf{b})\times \mathbf{c}$, the cross product of vectors is not associative.

(iii) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, if and only if

$$(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a}, \text{ i.e. } \mathbf{c} = \frac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\mathbf{a}$$

or vectors **a** and **c** are collinear.

Reciprocal System of Vectors

Let **a**, **b** and **c** be three non-coplanar vectors and let

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

Then, $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ are said to form a reciprocal system of a, b and c.

Properties of Reciprocal System

(i) $\mathbf{a} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{b}' = \mathbf{c} \cdot \mathbf{c}' = 1$

(ii) $\mathbf{a} \cdot \mathbf{b}' = \mathbf{a} \cdot \mathbf{c}' = \mathbf{0},\ \mathbf{b} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{c}' = \mathbf{0},\ \mathbf{c} \cdot \mathbf{a}' = \mathbf{c} \cdot \mathbf{b}' = 0$

(iii) $[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'][\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 1 \Rightarrow [\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] = \dfrac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

(iv) $\mathbf{a} = \dfrac{\mathbf{b}' \times \mathbf{c}'}{[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']},\ \mathbf{b} = \dfrac{\mathbf{c}' \times \mathbf{a}'}{[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']},\ \mathbf{c} = \dfrac{\mathbf{a}' \times \mathbf{b}'}{[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']}$

Thus, **a**, **b**, **c** is reciprocal to the system $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$.

(v) The orthonormal vector triad **i**, **j**, **k** form self reciprocal system.

(vi) If **a**, **b**, **c** are a system of non-coplanar vectors and $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ are the reciprocal system of vectors, then any vector **r** can be expressed as $\mathbf{r} = (\mathbf{r} \cdot \mathbf{a}')\mathbf{a} + (\mathbf{r} \cdot \mathbf{b}')\mathbf{b} + (\mathbf{r} \cdot \mathbf{c}')\,\mathbf{c}$.