HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

2. $m^3 - 3m^2 - 4m + 12 = 0 \quad \Rightarrow \quad m = \pm 2, 3$

$m \in N$ hence $m \in \{2, 3\} \quad \Rightarrow \quad$ **(C)**

4. $y \sin 2x - \cos x + (1 + \sin^2 x)\frac{dy}{dx} = 0$ where $y = f(x)$

$$\frac{dy}{dx} + \left(\frac{\sin 2x}{1+\sin^2 x}\right)y = \frac{\cos x}{1+\sin^2 x}$$

I.F. $= e^{\int \frac{\sin 2x}{1+\sin^2 x}dx} = e^{\int \frac{dt}{t}} = e^{\ln(1+\sin^2 x)} = 1 + \sin^2 x$

(by putting $1 + \sin^2 x = t$)

$y(1 + \sin^2 x) = \int \cos x \, dx$

$y(1 + \sin^2 x) = \sin x + C \quad ; \quad (y(0) = 0) \quad \Rightarrow \quad C = 0$

hence, $\quad y = \frac{\sin x}{1+\sin^2 x}$

$y\left(\frac{\pi}{6}\right) = \frac{2}{5}$

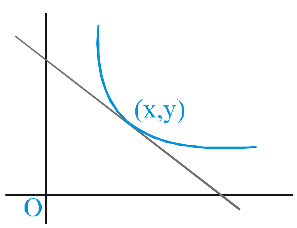
5. $Y - y = m(X - x)$

for X-intercept $Y = 0$

$X = x - \frac{y}{m}$

hence $\quad x - \frac{y}{m} = y$

or $\quad \frac{dy}{dx} = \frac{y}{x-y}$



put $y = Vx$

$V + x\frac{dV}{dx} = \frac{V}{1-V}$

$x\frac{dV}{dx} = \frac{V}{1-V} - V = \frac{V - V + V^2}{1-V}$

$\int \frac{1-V}{V^2}dV = \int \frac{dx}{x}$

$-\frac{1}{V} - \ln V = \ln x + c$

$-\frac{x}{y} - \ln\frac{y}{x} = \ln x + c$

$-\frac{x}{y} = \ln y + c$

$x = 1, y = 1 \quad \Rightarrow \quad c = -1$

$1 - \frac{x}{y} = \ln y$

$y = e \cdot e^{-x/y}$

$e^{-x/y} = \frac{e}{y}$

$y\, e^{x/y} = e \quad \Rightarrow \quad$ **(A)**

6. put $xe^x = t$

$(e^x + xe^x)\frac{dx}{dy} = \frac{dt}{dy}$

$\therefore \quad \frac{dt}{dy} + (ye^y - t) = 0 \quad \Rightarrow \quad \frac{dt}{dy} - t + ye^y = 0$

I.F. $\quad e^{-\int dy} = e^{-y}$

$t \cdot e^{-y} = -\int ye^y e^{-y} dy$

$x\, e^x e^{-y} = -\frac{y^2}{2} + C$

$f(0) = 0 \quad \Rightarrow \quad C = 0; \quad 2x\, e^x e^{-y} + y^2 = 0$

8. $\frac{dy}{dt} = -k\sqrt{y}\,; \quad$ when $t = 0; \quad y = 4$

$\int_4^0 \frac{dy}{\sqrt{y}} = -k\int_0^t dt$

$2\sqrt{y}\Big|_4^0 = -kt = -\frac{t}{15}$

$0 - 4 = -\frac{t}{15} \quad \Rightarrow \quad t = 60$ minutes $\quad \Rightarrow \quad$ **(C)**

10. $y \cdot e^{-2x} = Ax\, e^{-2x} + B$

$e^{-2x} \cdot y_1 - 2y\, e^{-2x} = A(e^{-2x} - 2x\, e^{-2x})$

Cancelling e^{-2x} throughout

$y_1 - 2y = A(1 - 2x) \qquad$ ....(i)

differentiating again

$y_2 - 2y_1 = -2A \quad \Rightarrow \quad A = \frac{2y_1 - y_2}{2}$

hence substituting A in (1)

$2(y_1 - 2y) = (2y_1 - y_2)(1 - 2x)$

$$2y_1 - 4y = 2y_1(1-2x) - (1-2x)y_2$$

$$(1-2x)\frac{d}{dx}\left(\frac{dy}{dx} - 2y\right) + 2\left(\frac{dy}{dx} - 2y\right) = 0$$

hence $k = 2$ and $l = -2$

$\Rightarrow$ ordered pair $(k, l) \equiv (2, -2)$

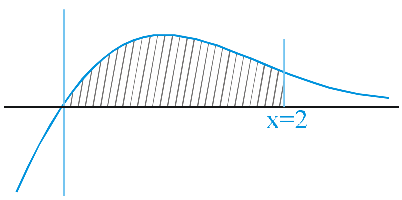
11. $y = xe^{-x}$

$y' = e^{-x} - x\,e^{-x} = (1-x)e^{-x} \uparrow$ for $x < 1$

$y'' = -e^{-x} - [e^{-x} - xe^{-x}] = e^{-x}[-1-1+x]$

$= (x-2)e^{-x}$

for point of inflection $y'' = 0 \quad \Rightarrow \quad x = 2$

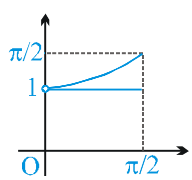


$$A = \int_0^2 xe^{-x}\,dx = -\,xe^{-x}\Big|_0^2 + \int_0^2 e^{-x}$$

$$= (-2\,e^{-2}) - \left(e^{-x}\right)_0^2$$

$$= -2\,e^{-2} - (e^{-2} - 1) = 1 - e^{-2} - 2e^{-2} = 1 - 3e^{-2}$$

13. $\sin x \dfrac{dy}{dx} + y\cos x = 1$



$$\frac{dy}{dx} + y\cot x = \operatorname{cosec} x$$

I.F. $= e^{\int \cot x\,dx} = e^{\ln(\sin x)} = \sin x$

$y\sin x = \int \operatorname{cosec} x \cdot \sin x\,dx$

$y\sin x = x + C$

if $x = 0$, y is finite

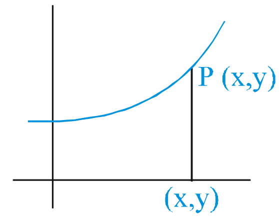
$\therefore \quad C = 0$

$$y = x(\operatorname{cosec} x) = \frac{x}{\sin x}$$

Now $\quad I < \dfrac{\pi^2}{4} \quad$ and $\quad I > \dfrac{\pi}{2}$

Hence $\quad \dfrac{\pi}{2} < I < \dfrac{\pi^2}{4} \Rightarrow$ **(A)**

14. $\displaystyle\int_0^x f(x)\,dx = y^3$



Differentiating

$$f(x) = 3y^2.\frac{dy}{dx}$$

$$y = 3y^2\frac{dy}{dx} \quad \Rightarrow \quad y = 0 \ \text{(rejected)}$$

or $\quad 3y\,dy = dx$

$$\frac{3y^2}{2} = x + c \Rightarrow \text{parabola} \Rightarrow C$$

15. $(x-h)^2 + (y-k)^2 = a^2$ ....**(i)**

$2(x-h) + 2(y-k)\dfrac{dy}{dx} = 0$ ....**(ii)**

$1 + \left(\dfrac{dy}{dx}\right)^2 + (y-k)\dfrac{d^2y}{dx^2} = 0$ ....**(iii)**

From (3) we have $(y-k)$, use in (2) to get $(x-h)$ and put $(x-h)$ and $(y-k)$ in (1)

16. $\dfrac{dy}{dx} - y = y^2(\sin x + \cos x)$

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = \sin x + \cos x \ ;$$

Let $-\dfrac{1}{y} = t \Rightarrow \dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$$\frac{dt}{dx} + t = \sin x + \cos x$$

I.F. e^x

$\therefore \quad t \cdot e^x = \int e^x(\sin x + \cos x)\,dx \ ; -\dfrac{1}{y}e^x = e^x\sin x + C$

if $x = 0$, $y = 1 \quad \Rightarrow \quad C = -1$

$$-\frac{e^x}{y} = e^x\sin x - 1$$

if $x = \pi$ then $-\dfrac{e^x}{y} = -1 \quad \Rightarrow \quad y = e^{\pi}$

17. $\frac{dy}{dx} = 100 - y$

$\Rightarrow \int \frac{dy}{100-y} = \int dx \quad \Rightarrow -\ell n\,(100-y) = x + c$

since $y(0) = 50 \quad \Rightarrow -\ell n\ 50 = c$

$\therefore -\ell n(100-y) = x - \ell n\ 50$

$\Rightarrow \ell n\left(\frac{50}{100-y}\right) = x \Rightarrow \frac{50}{100-y} = e^x$

18. $y = mx + c;\quad \frac{dy}{dx} = m;\ \frac{d^2y}{dx^2} = 0$

substituting in $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = -4x$

$0 - 3m - 4(mx + c) = -4x$

$-3m - 4c - 4mx = -4x$

$-(3m + 4c) = 4x(m-1)$ ....**(i)**

(i) is true for all real x if

$m = +1$ and $c = -3/4$ $\Rightarrow$ **(B)**

19. $\frac{dy}{dx} - \frac{y}{x^2}\tan\frac{1}{x} = -\sec\frac{1}{x}\cdot\frac{1}{x^2}$.

$IF = e^{-\int \frac{1}{x^2}\tan\frac{1}{x}\,dx} = \sec\frac{1}{x}$

$\Rightarrow y\,.\sec\frac{1}{x} = -\int \sec^2\left(\frac{1}{x}\right)\frac{1}{x^2}\,dx = \tan\frac{1}{x} + c$

if $y \to -1$ then $x \to \infty \Rightarrow c = -1$

$\Rightarrow y = \sin\frac{1}{x} - \cos\frac{1}{x}$

20. S_1 **:** $y^2 = 4ax$

$2yy' = 4a$

$y^2 = 2yy'x \Rightarrow \frac{dy}{dx} = \frac{y}{2x} \Rightarrow \frac{dy}{y} = \frac{dx}{2x}$

S_2 **:** $x\cos\alpha + y\sin\alpha = p$ ...**(1)**

$\cos\alpha + \sin\alpha y' = 0$

$y' = -\cot\alpha$ ...**(2)**

solving (1) & (2) we get we get 2 degree equation

S_3 **:** $ax^2 + by^2 + c = 0$

$x^2 + k_1y^2 + k_2 = 0$ two arbitary constant

$\therefore$ order = 2

21. $Y - y = -\frac{1}{m}(X - x)$

when $m = \frac{dy}{dx}$

take, let $Y = 0$

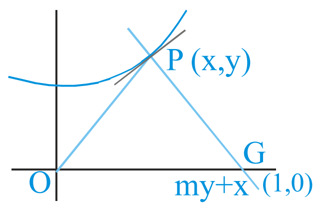
$X = my + x$

hence $x(my + x) = 2(x^2 + y^2)$

$xy\frac{dy}{dx} = x^2 + 2y^2$ **(Step-A)**

Now put $y = vx$

$v + x\frac{dv}{dx} = \frac{x^2(1+2v^2)}{x^2v} = \frac{1+2v^2}{v}$



$x\frac{dv}{dx} = \frac{1+2v^2}{v} - v = \frac{1+v^2}{v}$

$\int \frac{v\,dv}{1+v^2} = \int \frac{dx}{x}$

$\frac{1}{2}\,ln\,(1+v^2) = ln\,x + c$

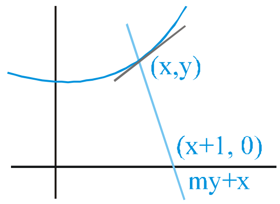
$ln\left(\frac{1+v^2}{x^2}\right) = c$

$x^2 + y^2 = cx^4 \quad \Rightarrow$ A

22. Given $y\frac{dy}{dx} + x = x + 1$

$\frac{y^2}{2} = x + c$

$x = 0, y = 0 \quad \Rightarrow c = 0$



$\therefore\ y^2 = 2x \quad \Rightarrow$ latus rectum= 2 $\Rightarrow$ B

23. $\phi(x) = \phi'(x) \quad \phi(1) = 2$

$\frac{d\varphi}{dx} = \phi$

$\ell n\,\phi(x) = x + c$

$\ell n\,2 = 1 + c \quad \Rightarrow c = \ell n2 - 1$

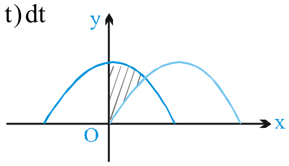
$\ell n\phi(3) = 3 + c = 2 + \ell n2 \quad \Rightarrow \phi(3) = 2e^2$

27. I.F. $= e^{-x}$

$\therefore \quad ye^{-x} = \int e^{-x}(\cos x - \sin x)\,dx$ put $-x = t$

$= -\int e^{t}(\cos t + \sin t)\,dt$

$= -e^{t}\sin t + c$



$y\,e^{-x} = e^{-x}\sin x + c$

since y is bounded when $x \to \infty \Rightarrow c = 0$

$\therefore \quad y = \sin x$

$\text{Area} = \int_0^{\pi/4}(\cos x - \sin x)\,dx = \sqrt{2} - 1 \Rightarrow$ **(A)**

29. $\ln c + \ln|x| = \dfrac{x}{y}$

diff. w.r.t. x, $\dfrac{1}{x} = \dfrac{y - x y_1}{y^2}$

$\dfrac{y^2}{x} = y - x\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{y}{x} - \dfrac{y^2}{x^2} \Rightarrow \varphi\left(\dfrac{x}{y}\right) = -\dfrac{y^2}{x^2} \Rightarrow$ **(D)**

31. $\dfrac{dy}{dx} = 2ax = 2x\left(\dfrac{y}{x^2}\right); \quad \dfrac{dy}{dx} = \dfrac{2y}{x};$

now $m = \dfrac{dy}{dx} = -1$

$\Rightarrow \quad m = -\dfrac{x}{2y} \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{2y}$

33. $+\sin y\dfrac{dy}{dx} = +ae^{-x}$

$\sin y\dfrac{dy}{dx} = \cos y \Rightarrow \dfrac{dy}{dx} = \cot y$

equation to the orthogonal trajectory will be obtained by solving the differential equation

$-\dfrac{dy}{dx} = \cot y$

$c - x = \ln(\sin y) \Rightarrow \sin y = c\,e^{-x}$

$y^2 = -\dfrac{x^2}{2} + c$

34. $f'(x) - \dfrac{2x(x+1)}{x+1}f(x) = \dfrac{e^{x^2}}{(x+1)^2}$

I. F. $= e^{\int -2x\,dx} = e^{-x^2}$

$\therefore \quad f(x).\,e^{-x^2} = \int \dfrac{dx}{(x+1)^2} \Rightarrow f(x).\,e^{-x^2} = -\dfrac{1}{x+1} + C$

at $x = 0, f(0) = 5 \Rightarrow C = 6$

$\therefore \quad f(x) = \left(\dfrac{6x+5}{x+1}\right).e^{x^2}$

36. differentiate $x\,y(x) = x^2 y'(x) + 2xy(x)$

or $\quad x\,y(x) + x^2\,y'(x) = 0$

$x\dfrac{dy}{dx} + y = 0$

$\ln y + \ln x = \ln c$

$xy = c \Rightarrow$ **(D)**

37. Let equation of St. Line

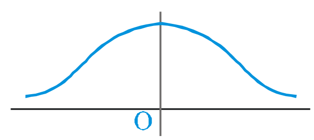
$Y - y = m(X - x)$

Distance from origin $\Rightarrow \left|\dfrac{mx - y}{\sqrt{1+m^2}}\right| = 1$

$\therefore \quad (mx - y)^2 = 1 + m^2$

$\left(y - \dfrac{dy}{dx}x\right)^2 = 1 + \left(\dfrac{dy}{dx}\right)^2$

38. $y = f(x) = e^{-\frac{x^2}{2}}$. The graph is as shown



39. $\dfrac{x}{x^2+y^2}dy = \left(\dfrac{y}{x^2+y^2} - 1\right)dx$

$\Rightarrow \dfrac{xdy - ydx}{x^2+y^2} = -dx$

$\Rightarrow \tan^{-1}\dfrac{y}{x} = -x + c \Rightarrow y = x\tan(c - x)$

40. $\int_0^1 f(tx)\,dt = n \cdot f(x)$

put $tx = y \Rightarrow dt = \frac{1}{x}\,dy$

$\therefore \frac{1}{x}\int_0^x f(y)\,dy = n\,f(x)$

$\therefore \int_0^x f(y)\,dy = x \cdot n \cdot f(x)$

Differentiating

$f(x) = n[f(x) + x f'(x)]$

$f(x)(1-n) = n x f'(x)$

$\therefore \frac{f'(x)}{f(x)} = \frac{1-n}{nx}$

Integrating $\ln f(x) = \left(\frac{1-n}{n}\right)\ln cx = \ln (cx)^{\frac{1-n}{n}}$

$\therefore f(x) = c.x^{\frac{1-n}{n}}$

EXERCISE - 2

Part # I : Multiple Choice

3. $y = e^{-x}\cos x$

$y_1 = -e^{-x}\cos x - e^{-x}\sin x = -e^{-x}(\sin x + \cos x)$

$= -\sqrt{2}\,e^{-x}(\cos x \cos\frac{\pi}{4} + \sin x \sin\frac{\pi}{4})$

$y_1 = -\sqrt{2}\,e^{-x}\cos\left(x - \frac{\pi}{4}\right)$

$y_2 = +\sqrt{2}\,e^{-x}\cos\left(x - \frac{\pi}{4}\right) + \sqrt{2}\,e^{-x}\sin\left(x - \frac{\pi}{4}\right)$

$= \sqrt{2}\,e^{-x}\left(\cos\left(x - \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right)\right)$

$= \sqrt{2}\,.\,\sqrt{2}\,e^{-x}\cos\left(x - \frac{\pi}{4} - \frac{\pi}{4}\right)$

similarly

$= \left(\sqrt{2}\right)^2 e^{-x}\cos\left(x - \frac{2\pi}{4}\right)$

$y_3 = -\left(\sqrt{2}\right)^3 e^{-x}\cos\left(x - \frac{3\pi}{4}\right)$

$y_4 = +\left(\sqrt{2}\right)^4 e^{-x}\cos\left(x - \frac{4\pi}{4}\right)$

$= -\left(\sqrt{2}\right)^4 e^{-x}\cos x$

$y_4 + 4y = 0$

Similarly

$y_8 = \left(\sqrt{2}\right)^8 e^{-x}\cos\left(x - \frac{8\pi}{4}\right) = \left(\sqrt{2}\right)^8 e^{-x}\cos(x - 2\pi)$

$= 16\,e^{-x}\cos x$

$y_8 - 16y = 0$

5. $\frac{dy}{1+y^2} + \frac{dx}{\sqrt{1-x^2}} = 0$

$\Rightarrow \tan^{-1} y + \sin^{-1} x + c = 0$

$\Rightarrow \cot^{-1}\frac{1}{y} + \cos^{-1}\sqrt{1-x^2} + c = 0$

9. $f(x) = \dfrac{\sin x}{x}$

13. $\dfrac{dy}{dx} + y = f(x)$

I.F. $= e^x$

$ye^x = \int e^x f(x)\,dx + C$

now if $0 \le x \le 2$ then $ye^x = \int e^x e^{-x}\,dx + C$

$\Rightarrow$ $ye^x = x + C$

$x = 0,\ y(0) = 1, \quad C = 1$

$\therefore$ $ye^x = x + 1$ **....(1)**

$y = \dfrac{x+1}{e^x}; \quad y(1) = \dfrac{2}{e}$ **Ans.** $\Rightarrow$ **(A)** is correct

$y' = \dfrac{e^x - (x+1)e^x}{e^{2x}}$

$y'(1) = \dfrac{e - 2e}{e^2} = \dfrac{-e}{e^2} = -\dfrac{1}{e}$ **Ans.**

$\Rightarrow$ **(B)** is correct

if $x > 2$

$ye^x = \int e^{x-2}\,dx$

$ye^x = e^{x-2} + C$

$y = e^{-2} + Ce^{-x}$

as y is continuous

$\therefore$ $\lim\limits_{x\to 2} \dfrac{x+1}{e^x} = \lim\limits_{x\to 2} \left(e^{-2} + Ce^{-x}\right)$

$3e^{-2} = e^{-2} + Ce^{-2} \Rightarrow C = 2$

$\therefore$ for $x > 2$

$y = e^{-2} + 2e^{-x}$ hence $y(3) = 2e^{-3} + e^{-2} = e^{-2}(2e^{-1} + 1)$

$y' = -2e^{-x}$

$y'(3) = -2e^{-3}$ **Ans.** $\Rightarrow$ **(D)** is correct

15. $x^2 y_1^2 + xy\,y_1 - 6y^2 = 0$

It is quadratic equation in y_1

$y_1 = \dfrac{-xy \pm \sqrt{x^2y^2 + 24y^2x^2}}{2x^2} = \dfrac{-xy \pm 5xy}{2x^2}$

$y_1 = -\dfrac{3y}{x} \quad | \quad y_1 = \dfrac{2y}{x}$

$\dfrac{dy}{dx} = \dfrac{-3y}{x} \quad | \quad \dfrac{dy}{dx} = \dfrac{2y}{x}$

$-\dfrac{dy}{y} = 3\dfrac{dx}{x} \quad | \quad \dfrac{dy}{dx} = \dfrac{2y}{x}$

$-\ell n\, y = 3\,\ell nx + \ell nc \quad | \quad \ln y = 2\,\ell nx + \ell nc$

$x^3y = C \quad | \quad y = cx^2$

Option **(C)**

$\dfrac{1}{2}\log y = c + \log x$

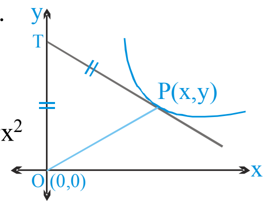
$\log y = \ell n\, c_1 + \log x^2$

$y = C_1x^2$

18. We have equation of tangent is $(Y - y) = m(X - x)$

Put $X = 0$, we get $Y = y - mx$.

Now $(OT)^2 = (PT)^2$ (given)



$\Rightarrow$ $(y - mx)^2 = x^2 + m^2x^2$

$\Rightarrow$ $y^2 + m^2x^2 - 2mxy = x^2 + m^2x^2$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy}$

Put $y^2 = t \Rightarrow 2y\dfrac{dy}{dx} = \dfrac{dt}{dx} \Rightarrow x\dfrac{dt}{dx} = t - x^2$

$\Rightarrow$ $\dfrac{dt}{dx} - \dfrac{t}{x} = -x$

$\therefore$ I.F. $= e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \dfrac{1}{x}$

The solution is given by $t\left(\dfrac{1}{x}\right) = \int -dx = -x + C$

$\Rightarrow$ $t = -x^2 + Cx$

Hence $y^2 + x^2 = Cx$.

If this is passing through (2, 2) $\Rightarrow C = 4 \Rightarrow x^2 + y^2 = 4x$

$\Rightarrow$ $(x - 2)^2 + y^2 = 4$

It's director circle will be $(x - 2)^2 + y^2 = 8$.

Put $x = 0,\ y^2 = 4 \Rightarrow y = 2$ or -2

Intercept on y-axis is 4.

This represents a family of circles with centre at y-axis and passing through origin.

Verify other alternatives.

22. Given that $\left|y\dfrac{dy}{dx}\right| = \left|\dfrac{4y}{\frac{dy}{dx}}\right|$ at (2, 1)

$\Rightarrow$ $\dfrac{dy}{dx} = \pm 2$ at (2, 1)

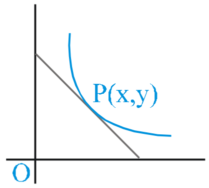
$\therefore$ Equation of tangent having positive slope is $y = 2x - 3$.

Now verify alternatives.

Part # II : Assertion & Reason

2. Equation of tangent $Y - y = m(X - x)$

put $X = 0$, $Y = y - mx$



hence initial ordinate is

$y - mx = x - 1 \Rightarrow mx - y = 1 - x$

$\frac{dy}{dx} - \frac{1}{x}y = \frac{1-x}{x}$ which is a linear differential equation

Hence statement-1 is correct and its degree is 1

$\Rightarrow$ statement-2 is also correct. Since every 1^{st} degree differential equation need not be linear hence statement-2 is not the correct explanation of statement-1.

3. Statement-2 $\ell n \frac{x}{y} - \frac{1}{xy} = c$

i.e $\ell nx - \ell ny - \frac{1}{xy} = c$

diff. $\frac{1}{x} - \frac{1}{y}\frac{dy}{dx} + \frac{y + x\frac{dy}{dx}}{(xy)^2} = 0$

$xy^2\,dx - x^2y\,dy + ydx + xdy = 0$

$(1 + xy)\,ydx + x(1 - xy)\,dy = 0$

$\therefore$ statement is true.

Statement -1 $x - \frac{y^2}{2} + \frac{1}{3}(x^2 + y^2)^{3/2} = c$

$1 - y\frac{dy}{dx} + \frac{1}{2}\sqrt{x^2+y^2}\left(2x + 2y\frac{dy}{dx}\right) = 0$

$dx - y\,dy + x\sqrt{x^2+y^2}\,dx + y\sqrt{x^2+y^2}\,dy = 0$

$(1 + x\sqrt{x^2+y^2})\,dx + y(-1 + \sqrt{x^2+y^2})\,dy = 0$

6. S-1: order is 2

7. $\frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{\left(x\frac{dy}{dx} - y\right)}{x^2} = mx + c \Rightarrow$

$\frac{x^2}{x^2+y^2} \cdot \frac{\left(x\frac{dy}{dx} - y\right)}{x^2} = mx + c \Rightarrow \frac{x\frac{dy}{dx} - y}{x^2+y^2} = mx + c$

statement-1 is false

statement-2 is linear form of differential equation which is true

8. Integral curves are

$y = cx - x^2$

The DE does not represent all the parabolas passing through origin but it represents all parabolas through origin with axis of symmetry parallel to y-axis and coefficient of x^2 as -1, hence statement-1 is false.

Statement-2 is universally true.]

10. Line touches the curve at (0, b) and $\left.\frac{dy}{dx}\right]_{x=0}$ also exists but even if $\frac{dy}{dx}$ fails to exist. tangents line can be drawn.

13. $C_1 : 2x - \frac{2y}{3}\frac{dy}{dx} = 0 \Rightarrow \left.\frac{dy}{dx}\right]_{x_1 y_1} = \frac{3x_1}{y_1} = m_1$

$C_2 : 3xy^2\frac{dy}{dx} + y^3 = 0 \Rightarrow \left.\frac{dy}{dx}\right]_{x_1 y_1} = -\frac{y_1}{3x_1} = m_2$

$\therefore m_1 \cdot m_2 = -1 \Rightarrow C_1$ and C_2 are orthogonal

15. We have $f(x) = \int_0^x f(t)\sin t\,dt - \underbrace{\int_0^x \sin(t - x)\,dt}_{\text{(King property)}}$

$\Rightarrow f(x) = \int_0^x f(t)\sin t\,dt + \int_0^x \sin t\,dt$

$\therefore f'(x) = f(x)\sin x + \sin x$

$\frac{dy}{dx} - y\sin x = \sin x$ which is linear

EXERCISE - 3

Part # I : Matrix Match Type

3. **(A)** $xdy = ydx + y^2 dy$

$\Rightarrow \dfrac{x\,dy - y\,dx}{y^2} = dy \quad \Rightarrow \; -d\left(\dfrac{x}{y}\right) = dy$

$-\dfrac{x}{y} = y + c$ put $x = 1 \quad y = 1 \quad \Rightarrow c = -2$

$-\dfrac{x}{y} = y - 2 \quad \Rightarrow \dfrac{x}{3} = -5$

(B) $\dfrac{dy}{dt} - \dfrac{t}{t+1}y = \dfrac{1}{t+1}$

$I.F = e^{-\int \frac{t+1-1}{t+1}dt} = e^{-t+\ell n(t+1)} = (t+1)e^{-t}$

solution is $(t+1)e^{-t}\, y = -e^{-t} + c$

put $t = 0$

and $\quad y = -1 \;\Rightarrow c = 0$

$\therefore \quad 2e^{-1}\, y = -e^{-1}$

$y = -\dfrac{1}{2}$

(C) $(x^2 + y^2)\, dy = xy\, dx$

$\dfrac{dy}{dx} = \dfrac{xy}{x^2 + y^2}$

put $\; y = vx \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\Rightarrow \; \ell n\, v - \dfrac{1}{2v^2} = -\ell n\, x + c$

$\therefore \quad c = -\dfrac{1}{2}$

$\therefore \quad \ell n\dfrac{y}{x} - \dfrac{1}{2}\dfrac{x^2}{y^2} = -\ell n\, x - \dfrac{1}{2}$ put $\; y = e$

$\therefore \quad x = \sqrt{3}\, e$

(D) $\dfrac{dy}{dx} + 2\dfrac{y}{x} = 0$

$x^2 y = C$ put $\; x = 1, y = 1$ and we get $C = 1$

put $\; x = 2$

$\Rightarrow \; y = \dfrac{1}{4}$

Part # II : Comprehension

Comprehension # 1

For reservoir A

$\dfrac{dV_A}{dt} \propto -V_A \quad \Rightarrow \quad \dfrac{dV_A}{dt} = -k_1 V_A$

$\Rightarrow \displaystyle\int_{V_{A_0}}^{V_A} \dfrac{dV_A}{V_A} = -k_1 \int_0^t dt \;\Rightarrow\; \log\dfrac{V_A}{V_{A_0}} = -k_1 t$

$\Rightarrow \; V_A = V_{A_0} e^{-k_1 t}$

Similarly $V_B = V_{B_0} e^{-k_2 t}$

so $\quad \dfrac{V_A}{V_B} = \dfrac{V_{A_0}}{V_{B_0}} e^{-(k_1 - k_2)t}$

At $\quad t = 0 \quad , \quad V_{A_0} = 2V_{B_0}$

and $\;$ at $t = 1$, $V_A = 1.5\, V_B$

so $\quad \dfrac{3}{2} = 2e^{-(k_1 - k_2)} \quad \therefore \quad e^{-(k_1 - k_2)} = \dfrac{3}{4}$

1. At $t = \dfrac{1}{2}$, $\quad V_A = kV_B$

so $\quad k = 2\left(\dfrac{3}{4}\right)^{1/2} \quad \Rightarrow \; k = \sqrt{3}$

2. Let at $t = t_0$ both the reservoirs have same quantity of water, then

$V_A = V_B \quad \Rightarrow \; 2e^{-(k_1 - k_2)t_0} = 1$

$\Rightarrow \; \left(\dfrac{3}{4}\right)^{t_0} = \dfrac{1}{2} \quad \therefore \; t_0 = \log_{3/4}\left(\dfrac{1}{2}\right)$

$\Rightarrow \; t_0 = \log_{4/3} 2$

and also $t_0 = \dfrac{1}{\log_2 \frac{4}{3}} = \dfrac{1}{2 - \log_2 3}$

3. Now $\dfrac{V_A}{V_B} = 2\, e^{-(k_1 - k_2)t} \;\Rightarrow\; f(t) = 2\, e^{-(k_1 - k_2)t}$

$f'(t) = -2(k_1 - k_2)\, e^{-(k_1 - k_2)t} = 2\ln\dfrac{3}{4} e^{-(k_1 - k_2)t}$

$\Rightarrow \;$ f(t) is decreasing.

Comprehension # 2

1-3

$$\frac{dy}{dx}+\left(\frac{2x}{1+x^2}\right)y=\frac{4x^2}{1+x^2}$$

I.F. $= e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = (1+x^2)$

$\therefore \quad y(1+x^2) = \int 4x^2\,dx = \frac{4x^3}{3}+C$

passing through (0, 0) $\Rightarrow \quad C=0$

$\therefore \quad y = \frac{4x^3}{3(1+x^2)}$

$$\frac{dy}{dx}=\frac{4}{3}\left[\frac{(1+x^2)3x^2-x^3\cdot 2x}{(1+x^2)^2}\right]=\frac{4}{3}\left[\frac{3x^2+x^4}{(1+x^2)^2}\right]$$

$$=\frac{4x^2(3+x^2)}{3(1+x^2)^2}$$

hence $\frac{dy}{dx} > 0 \;\; \forall\, x \neq 0;$

$\frac{dy}{dx} = 0$ at $x = 0$ and it does not change sign $\Rightarrow$ $x = 0$ is the point of inflection **Ans.**

$y = f(x)$ is increasing for all $x \in R$

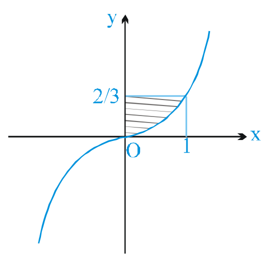
$x \to \infty; \;\; y \to \infty \quad ; \quad x \to -\infty; \;\; y \to -\infty$

Area enclosed by $y = f^{-1}(x)$, x-axis and ordinate at $x = \frac{2}{3}$

$$A = \frac{2}{3} - \frac{4}{3}\int_0^1 \frac{x^3}{1+x^2}dx$$

put $1+x^2 = t \quad \Rightarrow \quad 2x\,dx = dt$

$$A = \frac{2}{3}-\frac{2}{3}\int_1^2 \frac{(t-1)}{t}dt = \frac{2}{3}-\frac{2}{3}\int_1^2\left(1-\frac{1}{t}\right)dt$$



$$=\frac{2}{3}-\frac{2}{3}[t-\ln t]_1^2 = \frac{2}{3}-\frac{2}{3}[(2-\ln 2)-1]$$

$$=\frac{2}{3}-\frac{2}{3}[1-\ln 2)] = \frac{2}{3}\ln 2$$

Comprehension # 3

1. $2x^3dx + 2y^3dy - (xy^2\,dx + x^2y\,dy) = 0$

$$d\left(\frac{x^4}{2}\right)+d\left(\frac{y^4}{2}\right)-\frac{1}{2}\,d\,(x^2y^2)=0$$

$\Rightarrow \; d\,(x^4+y^4-x^2y^2)=0 \;\; \Rightarrow \;\; x^4+y^4-x^2y^2=c$

2. $\frac{xdy-ydx}{x^2+y^2}+dx=0 \qquad \Rightarrow \frac{\frac{xdy-ydx}{x^2}}{1+\left(\frac{y}{x}\right)^2}+dx=0$

$\Rightarrow \frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}+dx=0 \qquad \Rightarrow d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)+dx=0$

$\Rightarrow \; \tan^{-1}\left(\frac{y}{x}\right)+x=c$

3. $e^ydx + xe^ydy - 2ydy = 0$

$d\,(xe^y) - d(y^2) = 0$

Solution is $xe^y - y^2 = c$

Comprehension # 4

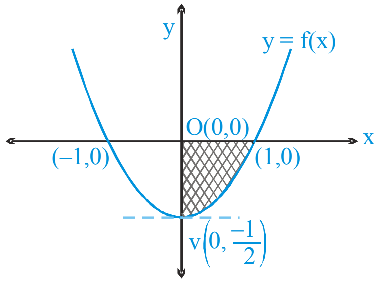
1-3

Equation of normal (N) : $Y-y=\frac{-1}{m}(X-x)$

Put $X = 0 \;\Rightarrow\; Y = y+\frac{x}{m}$

hence y-intercept of the normal is $y + \frac{x}{\frac{dy}{dx}}$.

So, $y+\frac{x}{\frac{dy}{dx}} = \sqrt{x^2+y^2} \Rightarrow \int\frac{ydy+xdx}{\sqrt{x^2+y^2}} = \int dy$



$$\Rightarrow \frac{1}{2}\int\frac{d\left(x^2+y^2\right)}{\sqrt{x^2+y^2}} = \int dy \Rightarrow \sqrt{x^2+y^2} = y+k$$

As $M\left(-\sqrt{3},1\right)$ satisfy it, so $k=1$

$\Rightarrow x^2+y^2=y^2+2y+1 \Rightarrow \quad x^2=2\left(y+\frac{1}{2}\right)$

1. From graph $\left.\frac{dy}{dx}\right]_{(1,0)}=1 \Rightarrow \tan\theta=1 \Rightarrow \theta=\tan^{-1}(1)$ **Ans.**

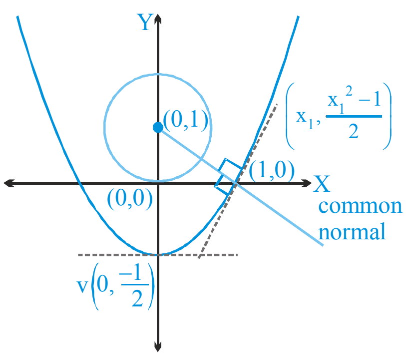
2. Clearly, required area

$=\left|\int_0^1 \frac{x^2-1}{2}dx\right| = \left|\frac{1}{2}\left(\frac{x^2}{3}-x\right)_0^1\right| = \left|\frac{1}{2}\left(\frac{1}{3}-1\right)\right| = \frac{1}{3}$

(square units).

3. Clearly, $\frac{-1}{x_1}=\frac{\left(\frac{x_1^2-1}{2}\right)-1}{x_1-0}$

$\Rightarrow \quad x_1^3-x_1=0 \quad \Rightarrow \quad x_1(x_1^2-1)=0$



$\therefore \quad x_1=-1, 0, 1$

Now, distance of $(1, 0)$ from $(0, 1) = \sqrt{2}$

So, shortest distance between the curve C and the circle $=\sqrt{2}-1$ (units).

Comprehension # 5

1. We have

$f(x)=2x^3+3\left(1-\frac{3a}{2}\right)x^2+3(a^2-a)x+b$ So,

$f'(x)=6x^2+6\left(1-\frac{3a}{2}\right)x+3(a^2-a)$

Now, for $f(x)$ to have negative point of local minimum, we must have both roots of the equation $f'(x)=0$ real (unequal) and negative.

$\therefore \quad$ Discriminant $>0 \Rightarrow 36\left[\left(1-\frac{3a}{2}\right)^2-2(a^2-a)\right]>0$

$\Rightarrow \quad \left(\frac{a}{2}-1\right)^2>0$

$\Rightarrow \quad a \in R-\{2\}$ ....(1)

Also, sum of roots <0

$\Rightarrow \quad \frac{-6\left(1-\frac{3a}{2}\right)}{6}<0 \Rightarrow \frac{3a}{2}-1<0 \Rightarrow a<\frac{2}{3}$ ....**(2)**

And product of roots >0

$\Rightarrow \quad \frac{3(a^2-a)}{6}>0$

$\Rightarrow \quad a(a-1)>0 \Rightarrow a \in (-\infty, 0)\cup(1,\infty)$ ....**(3)**

$\therefore \quad$ From (1) $\cap$ (2) $\cap$ (3), we get $a \in (-\infty, 0)$

2. We have

$g(x)=\frac{f'(x)}{6}=x^2+\left(1-\frac{3a}{2}\right)x+\left(\frac{a^2-a}{2}\right)$

As discriminant of $g(x)=\left(\frac{a}{2}-1\right)^2>0\ \forall\ a \in R-\{2\}$

$\Rightarrow \quad$ Ordinate of the vertex of parabola $y=g(x)$ lies below x-axis.

So, $a \in R-\{2\}$

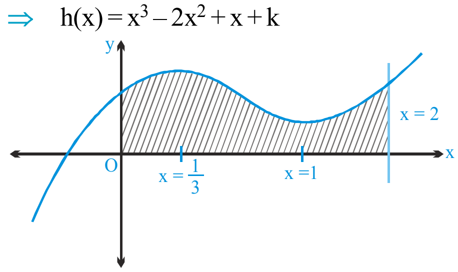
3. We have

$h''(x)=6x-4 \quad \Rightarrow \quad h'(x)=3x^2-4x+c$

As $h'(1)=0$, so $0=3-4+c \quad \Rightarrow \quad c=1$

$\therefore \quad h'(x)=3x^2-4x+1$

$\Rightarrow \quad h(x)=x^3-2x^2+x+k$



As $h(1)=5$, so $5=1-2+1+k \Rightarrow k=5$.

Hence, $h(x)=x^3-2x^2+x+5$

Clearly, required area

$=\int_0^2 h(x)dx=\int_0^2\left(x^3-2x^2+x+5\right)dx$

$=\left(\frac{x^4}{4}-\frac{2x^3}{3}+\frac{x^2}{2}+5x\right)_0^2=\frac{32}{3}$ (square units).

EXERCISE - 4
Subjective Type

1. **(i)** (2, 2) **(ii)** (3, 2) **(iii)** 1, 1
(iv) 1, 2
(v) 3, degree is not applicable **(vi)** 3, 2
(vii) 2, degree is not applicable

2.

(II) $x^2 + y^2 + 2gx + 2fy + c = 0$

Differentiation, we get,

$\Rightarrow 2x + 2yy' + 2g + 2fy' = 0$

Again differentiating,

$\Rightarrow 1 + (y')^2 + yy'' + fy'' = 0$ ....(i)

Again differentiating,

$\Rightarrow 2(y')y'' + y'y'' + yy''' + fy''' = 0$

$\Rightarrow 3y'y'' + y'''\left[y - \frac{1}{y''} - \frac{(y')^2}{y''} - y\right] = 0$ (from (i))

$\Rightarrow 3y'(y'')^2 - y'''\left[1 + (y')^2\right] = 0$

(III) $(y' - 1)(x^2 + y^2 - 1) + 2(x + yy')(x - y) = 0$

(IV) $y = c_1 e^{3x} + c_2 e^{2x} + c_3 e^x$ .....(1)

$\frac{dy}{dx} = 3c_1 e^{3x} + 2c_2 e^{2x} + c_3 e^x$ .....(2)

$\frac{d^2y}{dx^2} = 9c_1 e^{3x} + 4c_2 e^{2x} + c_3 e^x$ .....(3)

$\frac{d^3y}{dx^3} = 27c_1 e^{3x} + 8c_2 e^{2x} + c_3 e^x$ .....(4)

Apply (4) – 6 × (3) + 11 × (2) – 6 × (1)

$\Rightarrow \frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

3. **(i)** $\frac{dT}{dt} = -k(T - S)$

$\Rightarrow \ell n(T - S) = -kt + C$

$\ell n(S_0 - S) = C$

$\Rightarrow \ell n\left(\frac{T - S}{S_0 - S}\right) = -kt$

(ii) $\frac{d}{dt}(4\pi r^2) = kt$

$\int d(4\pi r^2) = \int kt\,dt \Rightarrow 4\pi r^2 = \frac{kt^2}{2} + c$

$t = 0; r = 3 \Rightarrow c = 36\pi$

$t = 2; r = 5 \Rightarrow k = 32\pi$

$\therefore r^2 = 4t^2 + 9$

$\Rightarrow r = \sqrt{4t^2 + 9}$

(iii) $\frac{dy}{dx} = \frac{\lambda y}{x}$

$\ell ny = \lambda \ell nx + C$

$\therefore y = kx^{\lambda}$

4. **(A)** $\frac{du}{dx} + Pu = Q; \ \frac{dv}{dx} + Pv = Q$

$\Rightarrow \frac{d}{dx}(u - v) = -P(u - v)$

$\frac{d(u - v)}{u - v} = -P\,dx$

$\Rightarrow \ell n(u - v) = -\int P dx$

$\frac{dy}{dx} + Py = Qx$

I.F. $= \frac{1}{u - v}$

$y.\frac{1}{u - v} = \int \frac{Q}{u - v} + c$

$\Rightarrow \frac{u}{u - v} = \int \frac{Q}{u - v} + c'$ [u satisfies it]

$\therefore \frac{y}{u - v} = \frac{u}{u - v} + k$

$\Rightarrow y = u + k(u - v)$ ... (1)

(B) If $y = \alpha u + \beta v$ is a particular solution then compare with (1)

$\alpha = k + 1, \beta = -k$

$\Rightarrow \alpha + \beta = 1$

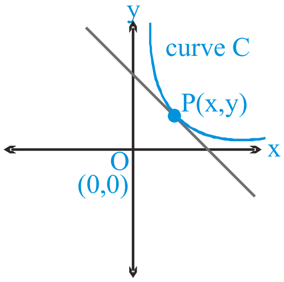
(C) If ω is a particular solution then it satisfies (1)

$\Rightarrow \omega = u + k(u - v)$

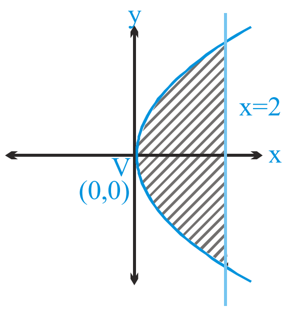
$\frac{u - v}{\omega - u}$ = constant

5. Let P (x, y) be any point on the curve C.

Now, $\frac{dy}{dx}=\frac{1}{y}$



$\Rightarrow$ $ydy=dx$



$\Rightarrow$ $\frac{y^2}{2}=x+k$

Since the curve passes through M (2,2), so k = 0

$\Rightarrow$ $y^2=2x$

Hence required area

$$=2\int_0^2\sqrt{2x}\,dx = 2\sqrt{2}\times\frac{2}{3}\left(x^{3/2}\right)_0^2=\frac{4}{3}\sqrt{2}\times2\sqrt{2} = \frac{16}{3}$$

(square unit)

$\Rightarrow$ $p+q=19$

6. $y-x=m(X-x)$

$X_{int}=x-\frac{y}{m}$

$$\frac{x-\frac{y}{m}+x}{2}=0$$

$2x=\frac{y}{m}$

$\frac{dy}{dx}=\frac{y}{2x}$

$\ell n\,y=\frac{1}{2}\ell nx+c \quad\Rightarrow\quad \frac{y^2}{x}=k$

$\Rightarrow$ $y^2=kx$

7.

(V) $\left[1+y(1+x^2)\right]dx=-x(1+x^2)dy$

$\frac{dy}{dx}+\frac{y}{x}=-\frac{1}{x(1+x^2)}$

I.F. $e^{\int 1/x}=x$

$\therefore$ $yx=-\int\frac{1}{1+x^2}dx$

$xy=-\tan^{-1}x+c$

(VI) $y-xy'=b+bx^2y'$

$\Rightarrow$ $y-b=(x+bx^2)\frac{dy}{dx}$

$\Rightarrow$ $\frac{dx}{x(1+bx)}=\frac{dy}{y-b}$

$\Rightarrow$ $\int\left(\frac{1}{x}-\frac{b}{1+bx}\right)dx=\log(y-b)+\log c$

$\Rightarrow$ $\log x-\log(1+bx)=\log(y-b)+\log c$

$\Rightarrow$ $\frac{x}{c}=(1+bx)(y-b)$

$\Rightarrow$ $b+(\frac{1}{c}+b^2)x=y(1+bx)$

$\Rightarrow$ $b+kx=y(1+bx)$

(X) $\frac{dy}{dx}-\frac{y}{x}=-\frac{y^4\cos x}{x^3}$

$\frac{1}{y^4}\frac{dy}{dx}-\frac{1}{y^3x}=-\frac{\cos x}{x^3}$

Put $-\frac{1}{y^3}=t \quad\Rightarrow\quad \frac{3}{y^4}\frac{dy}{dx}=\frac{dt}{dx}$

$\Rightarrow$ $\frac{dt}{dx}+\frac{3t}{x}=\frac{-3\cos x}{x^3}$

I.F. $e^{\int 3/x}=x^3$

$tx^3=-\int\frac{3\cos x}{x^3}x^3\,dx$

$\Rightarrow$ $-\frac{1}{y^3}x^3=-3\sin x+c$

$\Rightarrow$ $x^3y^{-3}=3\sin x-c$

(XI) $\frac{dy}{dx} - y = \frac{2xy^2}{e^x}$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = \frac{2x}{e^x}$

Put $-\frac{1}{y} = t$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \frac{dt}{dx} + t = \frac{2x}{e^x}$

I.F. is $e^{\int dx} = e^x$

$te^x = \int \frac{2x}{e^x} e^x \, dx$

$\Rightarrow -\frac{1}{y}e^x = x^2 + c$

$\Rightarrow y^{-1}e^x = -x^2 - c$

(XIII) $x(x^2+1)\frac{dy}{dx} + (x^2-1)y = x^3 \ell nx$

$$\frac{dy}{dx} + \frac{(x^2-1)y}{x(x^2+1)} = \frac{x^2 \ell n x}{x^2+1}$$

I.F. $= e^{\int \frac{x^2-1}{x(x^2+1)}dx} = \frac{x^2+1}{x}$

$$\left(\frac{x^2+1}{x}\right)y = \int \frac{x^2+1}{x} \times \frac{x^2}{x^2+1} \ell nx \, dx$$

$\Rightarrow \left(\frac{x^2+1}{x}\right)y = \int x \, \ell nx \, dx$

$\Rightarrow \left(\frac{x^2+1}{x}\right)y = \ell nx. \frac{x^2}{2} - \frac{x^2}{4} + c$

$\Rightarrow 4(x^2+1)y + x^3(1 - 2\ell nx) = cx$

8. $\frac{y^2 + y\sqrt{y^2 - x^2}}{x^2} = \ell n\left|y + \sqrt{y^2 - x^2}\,\frac{c^2}{x^3}\right|$, where same sign has to be taken.

9. $e^{-2\tan^{-1}\frac{y+2}{x-3}} = c.(y+2)$

10. $\frac{|y - mx|}{\sqrt{1+m^2}} = \frac{|my + x|}{\sqrt{1+m^2}}$

$|y - mx| = |my + x|$

$\Rightarrow y - mx = \pm(my + x)$ ...**(i)**

by taking positive sign

$y - mx = my + x$

$y - x = m(x + y)$

$\Rightarrow m = \frac{dy}{dx} = \frac{y - x}{y + x}$ let $y = tx$ $y = tx$

$t + x\frac{dt}{dx} = \frac{t-1}{t+1}$

$-\int \frac{t\,dt}{1+t^2} - \int \frac{dt}{1+t^2} = \ell n\, x + c$

$-\frac{1}{2}\ell n(1+t^2) - \tan^{-1} t = \ell n\, x + c$

$\ell n\sqrt{x^2+y^2} = -\tan^{-1}\frac{y}{x} + c$

$\Rightarrow \sqrt{x^2+y^2} = c_1 e^{-\tan^{-1} y/x}$

from equ. **(i)** by taking negative sign

$\frac{dy}{dx} = \frac{\frac{y}{x}+1}{\frac{-y}{x}+1}$ put $y = tx$

$\Rightarrow t + x\frac{dt}{dx} = \frac{t+1}{-t+1}$

$\Rightarrow \frac{1-t}{1+t^2}dt = \frac{dx}{x}$

$\Rightarrow \tan^{-1} t - \frac{1}{2}\ell n(1+t^2) = \ell nx + c$

$\ell n\sqrt{x^2+y^2} = +\tan^{-1}\frac{y}{x} + c$

$\Rightarrow \sqrt{x^2+y^2} = c_2 e^{\tan^{-1} y/x}$

then final solution $\sqrt{x^2+y^2} = c\, e^{\pm\tan^{-1} y/x}$

11. **(A)** $c(x-y)^{2/3}(x^2+xy+y^2)^{1/6} = \exp\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{x+2y}{x\sqrt{3}}\right]$

where $\exp x \equiv e^x$

(B) $y^2 - x^2 = c\,(y^2+x^2)^2$

12. $x^2+y^2-2x=0,\ x=1$

13. **(i)** $y' + P(x).y = Q(x)$

y_1 & y_2 are two solution of above equation so

$y_1' + P(x)\,y_1 = Q(x)$ ...(1)

$y_2' + P(x)\,y_2 = Q(x)$ ...(2)

Now if $y = y_1 + C(y_2 - y_1)$ is a solution of diff. equation then

$$\frac{d}{dx}(y_1 + c(y_2-y_1)) + P(x)\,(y_1 + c(y_2-y_1)) = Q(x)$$

$\Rightarrow\ y_1' + c(y_2' - y_1') + P(x)\,(y_1 + c(y_2-y_1)) = Q(x)$ ...(3)

from equation (1) & (2)

$(y_2' - y_1') = P(x)\,(y_1 - y_2)$

Now from equation (3)

$\Rightarrow\ y_1' + P(x)\,c(y_1-y_2) + P(x)y_1 + P(x)\,C(y_2-y_1) = Q(x)$

$\Rightarrow\ y_1' + P(x)y_1 = Q(x)$

which is true by equation (1)

so

$y = y_1 + c(y_2 - y_1)$ is a general solution of given diff. equation

(ii) multiply equation (1) by α and equation (2) by β then add

$(\alpha y_1' + \beta y_2') + P(x)\,(\alpha y_1 + \beta y_2) = Q(x)\,(\alpha+\beta)$

let $y = \alpha y_1 + \beta y_2$

$y' + P(x)y = Q(x)\,(\alpha+\beta)$

for y to be solution of diff. equation

$\alpha + \beta = 1$

14. $f(x) > 0\ \forall\, x \ge 2$

$$\frac{d}{dx}(x\,f(x)) \le -k\,f(x)$$

$$x\frac{dy}{dx} + y \le -ky \quad [f(x)=y]$$

$$x\frac{dy}{dx} \le -y\,(k+1)$$

$$\frac{dy}{dx} + \frac{(k+1)}{x}\,y \le 0$$

I.F. $= e^{\int\frac{k+1}{x}dx} = e^{\ell n\, x^{k+1}} = x^{k+1}$

$$x^{k+1}\cdot\frac{dy}{dx} + (k+1)\,.\,x^k\,y \le 0$$

$$\frac{d}{dx}(y.x^{k+1}) \le 0$$

$\Rightarrow\ g(x) = y\,.\,x^{k+1}$ decreases $\forall\ x \ge 2$

$\therefore\ g(x) \le f(2)\,.\,2^{k+1}$

$f(x)\,.\,x^{k+1} \le f(2)\,.\,2^{k+1}$

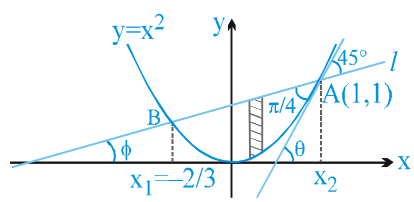
$f(x) \le A\,.\,x^{-k-1}$

15. $y = x^2$

$\left.\frac{dy}{dx}\right]_{A(1,1)} = 2x = 2;$ hence, $\tan\theta = 2$ ....(1)

Let the line l makes an angle ϕ with the x-axis

then, $\phi = \theta - \frac{\pi}{4}$; $\quad m_l = \tan\phi = \frac{\tan\theta - 1}{1+\tan\theta} = \frac{1}{3}$



$\therefore$ equation of the line l is,

$$y - 1 = \frac{1}{3}(x-1)$$

$\Rightarrow\ 3y - 3 = x - 1 \quad \Rightarrow \quad 3y = x + 2$

solving line and parabola $y = x^2$,

$3x^2 = x + 2 \quad \Rightarrow \quad 3x^2 - x - 2 = 0$

$x_1x_2 = -\frac{2}{3}$; but $x_2 = 1$;

$\therefore\ x_1 = -\frac{2}{3}$

$$\therefore\ A = \int_{-2/3}^{1}\left(\frac{x+2}{3} - x^2\right)dx$$

$$= \frac{1}{3}\int_{-2/3}^{1}\left(x + 2 - 3x^2\right)dx = \frac{1}{3}\left[\frac{x^2}{2} + 2x - x^3\right]_{-2/3}^{1}$$

$$= \frac{1}{3}\left[\left(\frac{1}{2} + 2 - 1\right) - \left(\frac{2}{9} - \frac{4}{3} + \frac{8}{27}\right)\right]$$

$$= \frac{1}{3}\left[\frac{3}{2} - \left(\frac{6-36+8}{27}\right)\right] = \frac{1}{3}\left[\frac{3}{2} + \frac{22}{27}\right]$$

$$= \frac{1}{3}\left[\frac{81+44}{54}\right] = \frac{125}{162} = \frac{a_1}{a_2}$$

$\Rightarrow \ a_1 + a_2 = 287$

16. **(A)** $x^2 + 2y^2 = c$, **(B)** $\sin y = ce^{-x}$

17. $\frac{x^2 - y^2}{x^2 + y^2}$ let the line from origin be $y = mx$

$\frac{dy}{dx} = \frac{1-m^2}{1+m^2}$ which is constant and independent of x, y

Hence $f'(x_1) = g'(x_1)$

18. $27\frac{7}{9}$ minutes

19.

(I). $y^2 + x\,\ell nax = 0$ **(II).** $y^2 = 3x^2 - 6x - x^3 + ce^{-x} + 4$

(III). $x\ell ny = e^x(x-1) + c$ **(IV).** $\sin y = (e^x + c)(1+x)$

(V). $cx^2 + 2xe^{-y} = 1$

(VI) $\left(\frac{dy}{dx} - y\right)\left(\frac{dy}{dx} - x\right) = 0$

$\frac{dy}{dx} = y$ or $\frac{dy}{dx} = x$

$\Rightarrow \ \ell ny = x + c$ or $y = \frac{x^2}{2} + c$

$\Rightarrow \ y = ke^x$

(VII). $y^2 = -1 + (x+1)\,\ell n \frac{c}{x+1}$ or $x + (x+1)\,\ell n \frac{c}{x+1}$

(VIII) $xy = t \ \Rightarrow \ x\frac{dy}{dx} + y = \frac{dt}{dx}$

$\Rightarrow \ x\frac{dy}{dx} = \frac{dt}{dx} - \frac{t}{x}$

$\Rightarrow \ (1 - t + t^2) = x\left(\frac{dt}{dx} - \frac{t}{x}\right)$

$\Rightarrow \ \frac{(1-t+t^2)}{x} + \frac{t}{x} = \frac{dt}{dx}$

$\Rightarrow \ \frac{1+t^2}{x} = \frac{dt}{dx}$

$\Rightarrow \ \frac{dt}{1+t^2} = \frac{dx}{x}$

$\Rightarrow \ \tan^{-1}t = \ell n|x| + \ell nc$

$\Rightarrow \ \tan^{-1}(xy) = \ell n|cx|$

$xy = \tan \ell n|xc|$

(IX) $e^y = c.\exp(-e^x) + e^x - 1$

(X) $y^2 = \frac{2}{3}\sin x + \frac{c}{\sin^2 x}$

22. $y^2 = \frac{x^4 + c^4}{2x^2}$ or $y^2 + 2x^2\ell nx = cx^2$

23. $\frac{y^2 \text{ int}}{\text{LSN}} = \frac{xy}{m^2 \text{ LST}}$

$$\frac{(y-mx)^2}{|my|} = \frac{xy}{m^2\left(\frac{y}{m}\right)}$$

$m \neq 0$ $y \neq 0$

[as $y = 0$ or $y = c$ is not a solution]

$(y - mx)^2 = xy$

$y - mx = \pm\sqrt{xy}$

$y \pm xy = mx \ \Rightarrow \ \frac{dy}{dx} = \frac{y \pm \sqrt{xy}}{x}$

$\Rightarrow \ \frac{dy}{dx} = \frac{y}{x} \pm \sqrt{\frac{y}{x}}$

put $\frac{y}{x} = t^2$

$\Rightarrow \ \frac{dy}{dx} = t^2 + 2xt\frac{dt}{dx}$

now diff. eq.

$t^2 + 2xt\frac{dt}{dx} = t^2 \pm \sqrt{t}$

after solving $x = e^{2\sqrt{y/x}}$; $x = e^{-2\sqrt{y/x}}$

24. $f(x) = e^{2x}$

25. We have $f'(x) = \ln x + 2 - \dfrac{f(x)}{x \ln x}$

$\Rightarrow \dfrac{dy}{dx} + \dfrac{y}{x \ln x} = \ln x + 2$ ......**(1)**

Integrating factor $= e^{\int \frac{1}{x \ln x} dx} = \ln x$

$\therefore$ Solution of the differential equation (1) is

$y \ln x = \int \ln x (\ln x + 2)\, dx + C$

$\Rightarrow y \ln x = \int \left(\ln^2 x + 2 \ln x\right) dx + C$

$\left(\int \left(f(x) + x f'(x)\right) dx = x f(x) + C\right)$

$\Rightarrow y \ln x = x \ln^2 x + C$

As, $x = 1$

$\Rightarrow C = 0$

Hence, $y = x \ln x$

$f(x) = x \ln x \Rightarrow f'(x) = x\left(\dfrac{1}{x}\right) + \ln x$

$f(e) = e,\ f'(e) = 1 + 1 = 2$

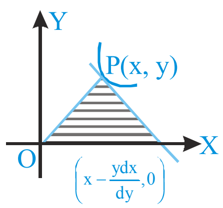
$f(e) - f'(e) = e - 2$

$\Rightarrow \left[f(e) - f'(e)\right] = 0$

26. Equation of tangent

$Y - y = \dfrac{dy}{dx}(X - x)$

when $Y = 0,\ X = x - y\dfrac{dx}{dy}$



$\left|\dfrac{1}{2}\left(x - y\dfrac{dx}{dy}\right) y\right| = a^2$

$xy - y^2 \dfrac{dx}{dy} = \pm 2a^2$

$\Rightarrow \dfrac{dx}{dy} - \dfrac{x}{y} = \mp \dfrac{2a^2}{y^2}$

$\Rightarrow \dfrac{x}{y} = \mp \int \dfrac{2a^2}{y^2} \dfrac{1}{y}\, dy$

$\Rightarrow \dfrac{x}{y} = \pm \dfrac{a^2}{y^2} + c \Rightarrow x = cy \pm \dfrac{a^2}{y}$

27. $x^2 + y^2 - 2x = 0$

29. Equation of tangent $Y - y = \dfrac{dy}{dx}(X - x)$

distance from origin $= \left|\dfrac{-x\dfrac{dy}{dx} + y}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}}\right|$

Equation of normal $Y - y = \dfrac{-1}{\dfrac{dy}{dx}}(X - x)$

$\Rightarrow Y\dfrac{dy}{dx} - y\dfrac{dy}{dx} = -X + x$

distance from origin $= \left|\dfrac{x + y\dfrac{dy}{dx}}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}}\right|$

Now, $\left|-x\dfrac{dy}{dx} + y\right| = \left|x + y\dfrac{dy}{dx}\right|$

either $-x\dfrac{dy}{dx} + y = x + y\dfrac{dy}{dx}$

or $x\dfrac{dy}{dx} - y = x + y\dfrac{dy}{dx}$

$\Rightarrow (x + y)\dfrac{dy}{dx} = y - x$

30. $\dfrac{dy_1}{dx} + Py_1 = Q,\ \dfrac{dy_2}{dx} + Py_2 = Q$

Put $y_2 = y_1 z$

$\Rightarrow \dfrac{dy_2}{dx} = y_1\dfrac{dz}{dx} + z\dfrac{dy_1}{dx}$

$\Rightarrow y_1\dfrac{dz}{dx} + z\dfrac{dy_1}{dx} = Q - Py_2$

$\Rightarrow y_1\dfrac{dz}{dx} + z\dfrac{dy_1}{dx} + Py_1 z = Q$

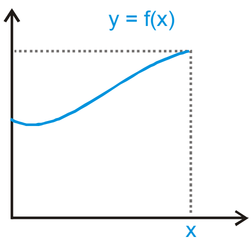
$\Rightarrow y_1\dfrac{dz}{dx} + zQ = Q \Rightarrow y_1\dfrac{dz}{dx} = Q(1 - z)$

$\Rightarrow \int \frac{dz}{1-z} = \int \frac{Q}{y_1} dx$

$\Rightarrow \ell n\,|z-1| = -\int \frac{Q}{y_1} dx + \lambda$

$\Rightarrow z = 1 + a\,e^{-\int \frac{Q}{y_1} dx}$

31. $y = f(x)\ f(x) \ge 0\ f(0) = 0$



$\int_0^x f(x)\,dx = K\,f(x)^{n+1}$

$f(x) = (n+1)\,K\,f(x)^n\ f'(x)$

$y^{1-n} = (n+1)\,K \frac{dy}{dx}$

$\int dx = (n+1)\,K \int y^{n-1}\,dy$

$\Rightarrow x + c = (n+1)k \frac{y^n}{n}$

$x = 0\,;\, y = 0 \quad \Rightarrow \quad c = 0$

$x = 1, y = 1 \quad \Rightarrow \quad k = \frac{n}{n+1}$

$x = y^n \quad \Rightarrow \quad y = (x)^{1/n}$

32. $\frac{dm}{dt} = -\lambda m$

$\frac{1}{m} dm = -\lambda dt$

$\ell n\ m = -\lambda t + c$

$m = ke^{-\lambda t}$ (at $t = 0,\ m = m_0$)

$\Rightarrow k = m_0$

$m = m_0 e^{-\lambda t}$ (at $t = t_0,\ m = m_0 - \frac{\alpha m_0}{100}$)

$\Rightarrow \lambda = \frac{-1}{t_0} \ell n\left(1 - \frac{\alpha}{100}\right)$

or $(x-y)\frac{dy}{dx} = x + y$

$\Rightarrow \frac{dy}{dx} = \frac{y-x}{y+x}$ or $\frac{dy}{dx} = \frac{x+y}{x-y}$

Put $y = vx$

$\Rightarrow v + x\frac{dv}{dx} = \frac{v-1}{v+1}$ or $v + x\frac{dv}{dx} = \frac{1+v}{1-v}$

$\Rightarrow \int \frac{v+1}{1+v^2} dv = \int \frac{-dx}{x}$ or $\int \frac{v-1}{1+v^2} dv = \int \frac{-dx}{x}$

$\Rightarrow \frac{1}{2} \ell n\,|1+v^2| + \tan^{-1} v = -\ell nx + \ell nc$

or $\frac{1}{2} \ell n\,|1+v^2| - \tan^{-1} v = \ell nc - \ell nx$

Hence solution will be

$\frac{1}{2} \ell n\,|1+v^2| + \ell nx = \pm \tan^{-1} v + \ell nc$

$x\sqrt{1+v^2} = ke^{\pm \tan^{-1} v}$

$\Rightarrow \sqrt{x^2+y^2} = ce^{\pm \tan^{-1} y/x}$

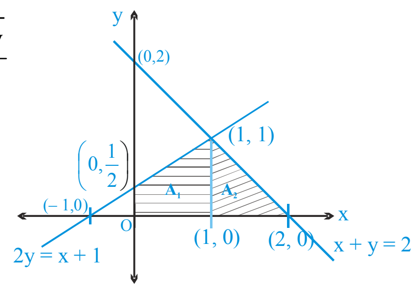
33. We have, $\left|x - y\frac{dx}{dy}\right| = 2\left|\frac{dy}{dx}\right|$

$\Rightarrow x - y\frac{dx}{dy} = \pm 2\frac{dy}{dx}$

Taking positive sign,

$\Rightarrow 2\left(\frac{dy}{dx}\right)^2 - x\left(\frac{dy}{dx}\right) + y = 0 \Rightarrow \frac{dy}{dx}$

$= \frac{x \pm \sqrt{x^2 - 8y}}{4}$



But no solution exist, as it does not pass through (1, 1).

Now, taking negative sign,

$\Rightarrow 2\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) - y = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-x \pm \sqrt{x^2 + 8y}}{4}$

Let $x^2 + 8y = v^2 \quad \Rightarrow \quad 2x + 8\frac{dy}{dx} = 2v\frac{dv}{dx}$

So, $v\frac{dv}{dx} - x = -x \pm v \Rightarrow \frac{dv}{dx} = \pm 1$

$\Rightarrow v = \pm x + c \quad \Rightarrow \quad \sqrt{x^2 + 8y} = \pm x + C$

So, curves are

$\sqrt{x^2 + 8y} = x + 2$ and $\sqrt{x^2 + 8y} = -x + 4$

$\therefore$ On squaring, we get $2y = x + 1$ and $x + y = 2$.

$\therefore$ Clearly, required area $= A_1 + A_2 = \frac{3}{4} + \frac{1}{2} = \frac{5}{4} = \frac{m}{n}$ (given)

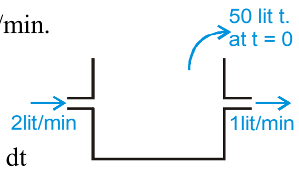
So, $(m + n)_{Least} = 5 + 4 = 9$

34. Input rate = 10 gm/min

After time t volume of tank = 50 + (2 – 1)t

Concentration of salt in time t = $\frac{m}{50+t}$ gm/lit

Output rate = $\frac{m}{50+t}$ .1 gm/min.



$$\frac{dm}{dt} = \frac{-m}{50+t} + 10$$

$m(50 + t) = \int 10(50 + t)\, dt$

$\Rightarrow$ $m(50 + t) = 500\, t + 5t^2 + c$

$\Rightarrow$ ($\because$ at t = 0, m = 0 $\Rightarrow$ c = 0)

$\Rightarrow$ $m(50 + t) = 5\,(100\, t + t^2)$

$\Rightarrow$ $m = 5t\left(\frac{100+t}{50+t}\right)$ gm

35. $\int f(x)dx = F(x) + c$

$f(x) = F'(x)$

let $F(x) = y$

$$\frac{dy}{dx} + \cos x \,.\, y = \frac{\sin 2x}{(1+\sin x)^2}$$

I.F. = $e^{\sin x}$

$$y.e^{\sin x} = 2\int \frac{e^{\sin x}.\sin x \cos x}{(1+\sin x)^2}dx + c$$

$$y\, e^{\sin x} = 2\int \frac{e^t(t+1-1)}{(1+t)^2}dt \qquad [\text{Put } \sin x = t]$$

$$= 2\int e^t\left\{\frac{1}{t+1} + \frac{-1}{(1+t)^2}\right\} + c$$

$$y\, e^{\sin x} = \frac{2e^t}{t+1} + c \Rightarrow y\, e^{\sin x} = \frac{2e^{\sin x}}{\sin x + 1} + c$$

$$\Rightarrow y = \frac{2}{\sin x + 1} + c.\, e^{-\sin x}$$

$$f(x) = \frac{dy}{dx} = \frac{-2\cos x}{(\sin x + 1)^2} - c.e^{-\sin x}\cos x$$

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

1. $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$ Put $y = vx$

$\Rightarrow$ $\frac{dy}{dx} = V + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x.vx} = \frac{v^2-1}{2v}$$

or $\frac{xdv}{dx} = \frac{v^2-1}{2v} - v$

$x\frac{dv}{dx} = \frac{-v^2-1}{2v}$ or $-\int\frac{2v\,dv}{1+v^2} = \int\frac{dx}{x}$

$-\log(1 + v^2) = \log x + c$

$\log x + \log(1 + v^2) = \log c$

$\log x.\left(1+\frac{y^2}{x^2}\right) = \log c$ or $x\left(\frac{x^2+y^2}{x^2}\right) = c$

$\frac{x^2+y^2}{x} = c$ or $x^2 + y^2 = cx$

2. $y = e^{cx}$

$\log y = cx$ ..... (i)

$\frac{1}{y}y' = c$ $\Rightarrow$ $y' = cy$

$c = \frac{y'}{y}$ put in equation (i) $\log y = \frac{y'}{y}.x$

or $y \log y = xy'$

3. Given $\frac{dy}{dx} = \frac{y-1}{x(x-1)}$ or $\int\frac{dy}{y-1} = \int\frac{dx}{x(x+1)}$

$\log(y - 1) = \log\left(\frac{x}{x+1}\right) + \log C$

or $y - 1 = \frac{cx}{x+1}$ ..... (i)

Equation (i) passes through (1, 0)

$-1 = \frac{C}{2}$ $\Rightarrow$ $C = -2$ Put in (i)

$(y - 1) = \frac{-2x}{x+1}$ $\quad (y - 1)(x + 1) + 2x = 0$

4. Equation of given parabola is $y^2 = Ax + B$ where A and B are parameters

$$2y\frac{dy}{dx} = A \qquad y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

This is the equation of given parabola order = 2, degree 1

5. $(1 + y^2) = (e^{\tan^{-1}y} - x)\frac{dy}{dx}$ or $(1+y^2)\frac{dx}{dy} + x + e^{\tan^{-1}y}$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{(1+y^2)}$$

Now I.F. = $e^{\int\frac{dy}{1+y^2}} = e^{\tan^{-1}y}$

$\therefore$ solution $x(e^{\tan^{-1}y}) = \int\frac{e^{\tan^{-1}y}}{1+y^2}e^{\tan^{-1}y}dy + C$

$$xe^{\tan^{-1}y} = \frac{e^{2\tan^{-1}y}}{2} + C$$

or $2xe^{\tan^{-1}y} = e^{2\tan^{-1}y} + K$

6. Given family of curves is

$x^2 + y^2 - 2ay = 0$ ..... **(1)**

$2x + 2yy' - 2ay' = 0$ ..... **(2)**

Now put the value of 2a from (1) to in (2)

$$2x + 2yy' - \frac{x^2+y^2}{y}.y' = 0$$

$2xy + (y^2 - x^2)y' = 0$ or $(x^2 - y^2)y' = 2xy$

7. $ydx + (x + x^2y)dy = 0 \quad ydx + xdy = -x^2y\,dy$

$$\int\frac{d(xy)}{(xy)^2} = -\int\frac{dy}{y} \Rightarrow -\frac{1}{xy} = -\log y + c$$

$$\Rightarrow \frac{-1}{xy} + \log y = c$$

8. $y^2 = 2c(x + \sqrt{c})$ ..... **(1)**

$y^2 = 2cx + 2c\sqrt{c}$

$$2y\frac{dy}{dx} = 2c$$

$\Rightarrow yy_1 = C$ Put in equation(1)

$\Rightarrow y^2 = 2yy_1(x + \sqrt{yy_1})$

$y^2 = -2yy_1x = 2yy_1\sqrt{yy_1}$

or $(y^2 - 2yy_1x)^2 = 4y^3y_1^3$

Degree = 3 order = 1

9. $\frac{dy}{dx} = \frac{y}{x}\left(\log\frac{y}{x} + 1\right)$ which is homogeneous equ.

Put $y = vx$, $\frac{dy}{dx} = v + \frac{xdv}{dx}$

$$v + x\frac{dv}{dx} = \frac{vx}{x}\left(\log\frac{vx}{x} + 1\right)$$

$$\frac{xdv}{dx} = v(\log v + 1) - v = v\log v + v - v$$

$$\int\frac{dv}{v\log v} = \int\frac{dx}{x} \Rightarrow \log(\log v) = \log x + \log c$$

$$\Rightarrow \log\frac{y}{x} = cx$$

10. Given $Ax^2 + By^2 = 1$ Divide by B

$\frac{A}{B}x^2 + y^2 = \frac{1}{B}$ Differentiate w.r.t x

$$2x\frac{A}{B} + 2y\frac{dy}{dx} = 0 \qquad \text{..... (i)}$$

Again Differentiate w.r.t. x

$$2\frac{A}{B} + 2\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 0 \qquad \text{..... (ii)}$$

Put $\frac{A}{B} = -\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]$ in equation **(i)**

$$-2x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] + 2y\frac{dy}{dx} = 0$$

or $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

It have second order and first degree.

11. Let the centre of circle is (h, 0) and radius will be also h

$\therefore$ equation of circle $(x - h)^2 + (y - 0)^2 = h^2$

$\Rightarrow x^2 - 2hx + h^2 + y^2 = h^2$

$\Rightarrow x^2 - 2hx + y^2 = 0$ ..... **(i)**

Equation **(i)** passes through origin differentiating it w.r.t. x

$$2x - 2h + 2y\frac{dy}{dx} = 0$$

$\Rightarrow h = x + y\frac{dy}{dx}$ put in equation **(i)**

$$x^2 - 2x\left(x + y\frac{dy}{dx}\right) + y^2 = 0 \Rightarrow y^2 = x^2 + 2xy\frac{dy}{dx}$$

12. $\frac{dy}{dx} = 1 + \frac{y}{x}$ put $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = 1 + \frac{vx}{x} \Rightarrow x\frac{dv}{dx} = 1$

$\int dv = \int \frac{dx}{x}$

$\Rightarrow v = \log x + c$ or $\frac{y}{x} = \log x + c$ ... **(i)**

Given $y(1) = 1 \Rightarrow 1 = \log 1 + c \Rightarrow c = 1$ put **(i)**

$y = x\ell nx + x$

13. Equation of circle $(x-h)^2 + (y-2)^2 = 25$ .....**(i)**

Differentiate w.r.t. x

$2(x-h) + 2(y-2)\frac{dy}{dx} = 0$

$(x-h) = -(y-2)\frac{dy}{dx}$ put in **(i)**

$(y-2)^2\left(\frac{dy}{dx}\right)^2 + (y-2)^2 = 25$

or $(y-2)^2(y')^2 + (y-2)^2 = 25$

14. $y = c_1e^{c_2x}$ ....**(1)**

$y' = c_1c_2e^{c_2x}$ ....**(2)**

$y'' = c_1c_2^2e^{c_2x}$

$y'' = c_2y'$ ....**(3)**

Now $\frac{(2)}{(1)}$

$\frac{y'}{y} = c_2$

$\Rightarrow$ Put in (3)

$y'' = \frac{y'}{y}.y' \Rightarrow y''\, y = (y')^2$

15. $\cos x\, dy = y(\sin x - y)dx$

$\Rightarrow \frac{dy}{dx} = \frac{y\sin x - y^2}{\cos x} \Rightarrow \frac{dy}{dx} = y\tan x - y^2\sec x$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{1}{y}\tan x - \sec x$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$

$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\tan x = \sec x$ .....**(1)**

Put $\frac{1}{y} = t$ in equation (1)

$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$ .....**(2)**

From equation (1) & (2), we get,

$\Rightarrow \frac{dt}{dx} + t.\tan x = \sec x$

$\therefore$ I.F. $= e^{\int \tan x\, dx}$

$= e^{\log|\sec x|} = \sec x$

$\therefore$ Solution of differential equation is :

$t.\sec x = \int \sec x.\sec x.dx + c$

$\frac{1}{y}\sec x = \tan x + c$

$\sec x = y(\tan x + c)$

16. $\frac{dy}{dx} = y + 3 > 0$ $y(0) = 2$, $y(\log 2) = ?$

$\int \frac{dy}{y+3} = \int dx$

$\log|y+3| = x + c$

$y(0) = 2$

$\log|2+3| = 0 + c \Rightarrow c = \log 5.$

$y.(\log 2) = ?$

$\log|y+3| = \log 2 + \log 5$

$\log|y+3| = \log 10$

$y + 3 = 10$

$y = 7$

17. $\frac{dV}{dt} = -k(T-t)$

$\int dV = \int -K(T-t)dt$

$V = -K\left[Tt - \frac{t^2}{2}\right] + C$

At $t = 0$ $V = I \Rightarrow C = I$

$V = -Kt\left(T - \frac{t}{2}\right) + I$

$V(T) = -KT\left(T - \frac{T}{2}\right) + I = \frac{-KT^2}{2} + I$

18. Equation of tangent at (x_1, y_1) is

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1)$$

$$x\text{– intercept} = x_1 - y_1\frac{dx_1}{dy_1}$$

According to question

$$x_1 = \frac{x_1 - y_1\frac{dx_1}{dy_1}}{2}$$

$$\Rightarrow \quad x_1 = -y_1\frac{dx_1}{dy_1}$$

$$\int\frac{dy}{y} = \int -\frac{dx}{x}$$

$$\Rightarrow \quad \ell ny = -\ell nx + \ell nc$$

$$\Rightarrow \quad y = \frac{c}{x} \quad \Rightarrow \quad xy = c$$

Now at $x = 2, y = 3$

$\Rightarrow \quad c = 6$

$$\therefore \quad xy = 6 \quad \Rightarrow \quad y = \frac{6}{x}$$

19. $y^2dx + \left(x - \frac{1}{y}\right)dy = 0$

$$\Rightarrow \quad y^2\frac{dx}{dy} + x = \frac{1}{y} \quad \Rightarrow \quad \frac{dx}{dy} + \frac{x}{y^2} + \frac{1}{y^3}$$

$\therefore$ Integrating factor (I.F.) = $e^{\int\frac{1}{y^2}dy} = e^{-1/y}$

$\therefore$ General solution is –

$$x\,.\,e^{-1/y} = \int\frac{1}{y^3}e^{-1/y}dy + c$$

Let $I_1 = \int\frac{1}{y^3}e^{-1/y}dy$

put $\frac{-1}{y} = t$

$y^{-2}\,dy = dt$

$\therefore \quad I_1 = -\int te^t dt$

$= -e^t(t - 1)$

$= e^t(1 - t)$

$\therefore$ General solution is

$$xe^{-1/y} = e^{-1/y}\left(1 + \frac{1}{y}\right) + C$$

$$\Rightarrow \quad x = 1 + \frac{1}{y} + Ce^{1/y}$$

Put $x = 1, y = 1$

$$\therefore \quad 1 = 1 + \frac{1}{1} + Ce^{1/1}$$

$\Rightarrow \quad C = -1/e$

$$\therefore \quad x = 1 + \frac{1}{y} - \frac{e^{1/y}}{e}$$

20. $\frac{dP(t)}{dt} = \frac{1}{2}P(t) - 450$

integrate

$$\int\frac{dP}{P - 900} = \int\frac{1}{2}dt$$

$$\ell n|(P - 900)| = \frac{1}{2}t + C \quad ....(1)$$

given $t = 0 \to P = 850$

$\therefore \quad C = \ell n\, 50$

from (1)

$$\ln|(P - 900)| = \frac{1}{2}t + \ell n50$$

$$\frac{1}{2}t = \ell n\left|\left(\frac{P - 900}{50}\right)\right|$$

$$t = 2\,\ell n\left|\left(\frac{P - 900}{50}\right)\right|$$

at $\quad P = 0$

$$t = 2\ell n\frac{900}{50}$$

$$t = 2\ell n18$$

21. $P = 100x - 12\,x^{3/2}.\frac{2}{3} + C$

$x = 0, \quad P = 2000$

$\qquad C = 2000$

$P_{(x = 25)} = 2500 - 1000 + 2000 = 3500$

24. $y(1+xy)\,dx = xdy$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + y^2 \Rightarrow \frac{dy}{dx} - \frac{y}{x} + y^2$

Bernaulli's DE

$n = 2$

$$LF = \underset{e}{\int}(1-2)\left(-\frac{1}{x}\right)dx = \underset{e}{\int}\frac{1}{x}dx = x\ ,$$

$$\text{solution } y^{1-2}\,x = \int(1-2).x.1.dx$$

$$\Rightarrow \frac{x}{y} = -\frac{x^2}{2} + C$$

Given $f(1) = -1$

$$\Rightarrow \frac{1}{-1} = -\frac{1}{2} + C \quad \Rightarrow \quad C = -\frac{1}{2}$$

$$\therefore \text{ equation } \frac{x}{y} = -\frac{x^2}{2} - \frac{1}{2}$$

when $x = -\frac{1}{2}$, we have $-\frac{1}{2y} = -\frac{1}{4\times 2} - \frac{1}{2}$

$$\Rightarrow -\frac{1}{y} = -\frac{5}{4} \quad \Rightarrow \quad y = \frac{4}{5}$$

Part # II : IIT-JEE ADVANCED

1. Let X_0 be initial population of the country and Y_0 be its initial food production.

Let the average consumption be a units. Therefore, food required initially aX_0. It is given

$$Y_0 = aX_0\left(\frac{90}{100}\right) = 0.9\ aX_0 \qquad .....(1)$$

Let X be the population of the country in year t.

Then $\frac{dX}{dt}$ = rate of change of population

$$= \frac{3}{100}X = 0.03\ X$$

$$\therefore \frac{dX}{X} = 0.03\ dt$$

Integrating $\int\frac{dX}{X} = \int 0.03dt$

$\Rightarrow \log X = 0.03t + c$

$\Rightarrow X = A\,.\,e^{0.03t}$ where $A = e^c$

At $t = 0$, $X = X_0$, thus $X_0 = A$

$\therefore X = X_0\,e^{0.03t}$

Let Y be the food production in year t.

Then $Y = Y_0\left(1+\frac{4}{100}\right)^t = 0.9aX_0\,(1.04)^t$

($\because Y_0 = 0.9aX_0$ from (1))

Food consumption in the year t is $aX_0\,e^{0.03\,t}$.

Again for no food deficit, $Y - X \geq 0$

$\Rightarrow 0.9\,X_0\,a\,(1.04)^t > a\,X_0\,e^{0.03\,t}$

$$\Rightarrow \frac{(1.04)^t}{e^{0.03t}} > \frac{1}{0.9} = \frac{10}{9}$$

Taking log on both sides,

$t[\ell n\,(1.04) - 0.03] \geq \ell n\,10 - \ell n\,9$

$$\Rightarrow t \geq \frac{\ell n10 - \ell n9}{\ell n(1.04) - 0.03}$$

Thus the least integral values of the year n, when the country becomes self sufficient, is the smallest integer greater than or equal to $\frac{\ell n10 - \ell n9}{\ell n(1.04) - 0.03}$

3. $\frac{dy}{dt} - \frac{t}{1+t}y = \frac{1}{1+t}$ I.F. $= e^{\int\frac{t}{1+t}dt} = e^{-t}(1+t)$

$$\therefore \text{ solution is } ye^{-t}(1+t) = \int e^{-t}\,(1+t)\frac{1}{1+t} + C$$

$ye^{-t}(1+t) = -e^{-t} + c$ given $y(0) = -1 \quad \Rightarrow \quad c = 0$

$y(1+t) = -1$ or $y = -\frac{1}{1+t}$

and $y(1) = -\frac{1}{1+1} = -\frac{1}{2}$

5. Given : liquid evaporates at a rate proportional to its surface area.

$\Rightarrow \dfrac{dv}{dt} \propto -S$ .....**(1)**

We know, volume of liquid = $\dfrac{1}{3}\pi r^2 h$

and surface area = πr^2 (of liquid in contact with air)

or $V = \dfrac{1}{3}\pi r^2 h$ and $S = \pi r^2$ .....**(2)**

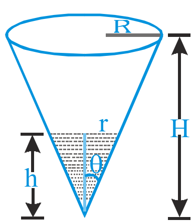
Also, $\tan\theta = \dfrac{R}{H} = \dfrac{r}{h}$ .....**(3)**

From (2) and (3),

$V = \dfrac{1}{3}\pi r^3 \cot\theta$ and $S = \pi r^2$ .....**(4)**

Substituting (4) in (1), we get

$\dfrac{1}{3}\pi\cot\theta.\ 3r^2 . \dfrac{dr}{dt} = -K\pi r^2$



$\Rightarrow \cot\theta \int_R^0 dr = -K\int_0^T dt$

where T is required time after which the cone is empty.

$\Rightarrow \cot\theta\,(0 - R) = -K(T - 0)$

$\Rightarrow R\cot\theta = KT$

$\Rightarrow H = KT$ (using **(3)**)

$\Rightarrow T = \dfrac{H}{K}$

6. $\dfrac{dy}{dx} = \dfrac{-\cos x(1+y)}{2+\sin x}$ or $\int\dfrac{dy}{1+y} = \int\dfrac{-\cos x dx}{2+\sin x}$

$\log(1+y) = -\log(2+\sin x) + c$

or $\log(1+y) + \log(2+\sin x) = c$

Given $y(0) = 1$ means when $x = 0, y = 1$

$\Rightarrow \log 2 + \log 2 = c \log c \Rightarrow c = 4$

$\Rightarrow 1 + y = \dfrac{4}{2+\sin x}$ or $y = \dfrac{4}{2+\sin x} - 1$

or $y\left(\dfrac{\pi}{2}\right) = \dfrac{4}{3} - 1 = \dfrac{1}{3}$

8. (A) $\dfrac{dy}{dx} = \dfrac{xy}{x^2+y^2}$ This is homogeneous so put

$y = vx,\ \dfrac{dy}{dx} = v + \dfrac{dv}{dx}.x$

$v + \dfrac{dv}{dx}x = \dfrac{xvx}{x^2+(vx)^2} = \dfrac{v}{1+v^2}$

or $x\dfrac{dv}{dx} = \dfrac{v}{1+v^2} - \dfrac{v}{1} = \dfrac{-v^3}{1+v^2}$

$\int -\dfrac{(1+v^2)dv}{v^3} = \int\dfrac{dx}{x}$ or $-\int\dfrac{dv}{v^3} - \int\dfrac{dv}{v} = \int\dfrac{dx}{x}$

or $\dfrac{1}{2v^2} - \log v = \log x + c$

put $v = \dfrac{y}{x}$

$\Rightarrow \dfrac{x^2}{2y^2} = \log\dfrac{y}{x} + \log x + c = \log y + c$ given $y(1) = 1$

$\dfrac{1}{2} = c \Rightarrow c = 2 \qquad \Rightarrow \dfrac{x^2}{2y^2} = \log y + 2$

Now put $x = x_0,\ y = e$

$\dfrac{x_0^2}{2e^2} = 1 + \dfrac{1}{2} = \dfrac{3}{2}$ or $x_0^2 = \dfrac{6e^2}{2} = 3e^2$

$\Rightarrow x_0 = \sqrt{3}.e$

(B) $\dfrac{xdy - ydx}{y^2} = dy$ or $\dfrac{ydx - xdy}{y^2} = -dy$

or $\int d\left(\dfrac{x}{y}\right) = -\int dy$

$\dfrac{x}{y} = -y + c$ Given $y(1) = 1$

$\Rightarrow 1 = -1 + c \Rightarrow c = 2$

$\dfrac{x}{y} = -y + 2$ Now $y(-3),\ \dfrac{-3}{y} = -y + 2$

or $y^2 - 2y - 3 = 0$

$y^2 - 3y + y - 3 = 0 = (y-3)(y+1) = 0$

$y = 3$

or $y = -1$ But $y > 0$

$\therefore y = 3$

11. **(A)** $\lim_{t\to x}\frac{t^2 f(x)-x^2 f(t)}{t-x}=1$

Applying L-hospital, we get

$\Rightarrow$ $x^2 f'(x)-2x f(x)+1=0$

$$\Rightarrow \frac{x^2 f'(x)-2x f(x)}{(x^2)^2}+\frac{1}{x^4}=0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{f(x)}{x^2}\right)=-\frac{1}{x^4} \Rightarrow f(x)=cx^2+\frac{1}{3x}$$

Also $f(1)=1 \Rightarrow c=\frac{2}{3}$

$$\Rightarrow f(x)=\frac{2x^2}{3}+\frac{1}{3x}$$

(B) $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{y}$

$$\Rightarrow \int\frac{y}{\sqrt{1-y^2}}dy=\int dx$$

$$\Rightarrow -\sqrt{1-y^2}=x+c \quad \Rightarrow (x+c)^2+y^2=1$$

Centre (–c,0); radius = 1

12. $\int\frac{dx}{x\sqrt{x^2-1}}=\int\frac{dy}{y\sqrt{y^2-1}}$

$\sec^{-1}x=\sec^{-1}y+c \quad \because y(2)=\frac{2}{\sqrt{3}} \quad \therefore c=\frac{\pi}{6}$

$\sec^{-1}x=\sec^{-1}y+\frac{\pi}{6} \quad \Rightarrow y=\sec(\sec^{-1}x-\frac{\pi}{6})$

Now $\cos^{-1}\frac{1}{x}=\cos^{-1}\frac{1}{y}+\frac{\pi}{6}$

$$\cos^{-1}\frac{1}{y}=\cos^{-1}\frac{1}{x}-\cos^{-1}\frac{\sqrt{3}}{2}$$

$$\frac{1}{y}=\frac{\sqrt{3}}{2x}-\sqrt{1-\frac{1}{x^2}}\left(\frac{1}{2}\right)$$

$$\frac{2}{y}=\frac{\sqrt{3}}{x}-\sqrt{1-\frac{1}{x^2}}$$

Hence S(I) is true and S(II) is false.

13. **(A)** $\frac{dy}{dx}=-\frac{y}{(x-3)^2}$

$\Rightarrow \ln y=\frac{1}{x-3}+c \Rightarrow y=e^{\frac{1}{x-3}+c}, x\neq 3.$

(B) $I=\int_1^5(x-1)(x-2)(x-3)(x-4)(x-5)\,dx$

Applying $x\to 6-x$

$$I=\int_1^5(5-x)(4-x)(3-x)(2-x)(1-x)\,dx=-I$$

$\Rightarrow I=0.$

(C) $f(x)=\cos^2x+\sin x$

$f'(x)=-2\cos x\sin x+\cos x$

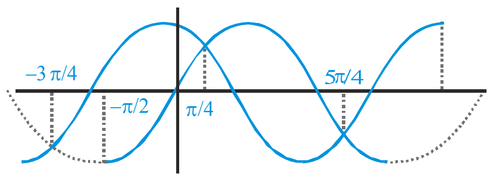
$\Rightarrow \cos x(-2\sin x+1)=0$

$\cos x=0$ or $\sin x=\frac{1}{2}$

sign of $f'(x)$ changes from –ve to +ve while $f(x)$ passes through $x=\frac{\pi}{6},\frac{5\pi}{6}$.

(D) $f(x)=\tan^{-1}(\sin x+\cos x)$

$$f(x)=\frac{\cos x-\sin x}{1+(\sin x+\cos)^2}>0$$



$x\in(-3\pi/4,\pi/4)$

14. Given $y=f(x)$

Tangent at point P(x, y)

$$Y-y=\left(\frac{dy}{dx}\right)_{(x,y)}(X-x)$$

Now y–intercept

$$\Rightarrow Y=y-x\frac{dy}{dx}$$

Given that, $y - x\frac{dy}{dx} = x^3$

$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = -x^2$ is a linear differential equation

with I.F. $= e^{\int -\frac{1}{x}dx} = e^{-\ell n x} = e^{\ell n\left(\frac{1}{x}\right)} = \frac{1}{x}$

Hence, solution is $\frac{y}{x} = \int -x^2 . \frac{1}{x} dx + C$

or $\frac{y}{x} = -\frac{x^2}{2} + C$

Given $f(1) = 1$

Substituting we get, $C = \frac{3}{2}$

so $y = -\frac{x^3}{2} + \frac{3}{2}x$

Now $f(-3) = \frac{27}{2} - \frac{9}{2} = 9$

15. (A) (Bonus)

(Comment : The given relation does not hold for x =1, therefore it is not an identity. Hence there is an error in given question. The correct identity must be-)

$6\int_1^x f(t)dt = 3x f(x) - x^3 - 5, \ \forall x \geq 1$

Now applying Newton Leibnitz theorem

$6f(x) = 3xf'(x) - 3x^2 + 3f(x)$

$\Rightarrow 3f(x) = 3xf'(x) - 3x^2$

Let $y = f(x)$

$\Rightarrow x\frac{dy}{dx} - y = x^2 \Rightarrow \frac{xdy - ydx}{x^2} = dx$

$\Rightarrow \int d\left(\frac{y}{x}\right) = \int dx$

$\Rightarrow \frac{y}{x} = x + C$ (where C is constant)

$\Rightarrow y = x^2 + Cx$

$\therefore f(x) = x^2 + Cx$

Given $f(1) = 2 \Rightarrow C = 1$

$\therefore f(2) = 2^2 + 2 = 6$

(B) Given y(0) = 0, g(0) = g(2) = 0

Let $y'(x) + y(x) . g'(x) = g(x) g'(x)$

$\Rightarrow y'(x) + (y(x) - g(x)) g'(x) = 0$

$\Rightarrow \frac{y'(x)}{g'(x)} + y(x) = g(x)$

$\Rightarrow \frac{dy(x)}{dg(x)} + y(x) = g(x)$

$\Rightarrow$ I.F. $= e^{\int d(g(x))} = e^{g(x)}$

$\Rightarrow y(x).e^{g(x)} = \int e^{g(x)} g(x).dg(x)$

$y(x).e^{g(x)} = g(x).e^{g(x)} - e^{g(x)} + c$

put x = 0

$\Rightarrow 0 = 0 - 1 + c \Rightarrow c = 1$

$\Rightarrow y(2) . e^{g(2)} = g(2)e^{g(2)} - e^{g(2)} + 1$

$\Rightarrow y(2) = 0 - e^0 + 1 \Rightarrow y(2) = 0$

16. $\frac{dy}{dx} - y\tan x = 2x\sec x$

I.F. $= e^{\int -\tan x\, dx} = \cos x$

$\therefore$ Equation reduces to

$y.\cos x = \int 2x.\sec x.\cos x dx$

$\Rightarrow y \cos x = x^2 + C$

$\because y(0) = 0 \Rightarrow 0 = 0 + C$

$\therefore y \cos x = x^2 \Rightarrow y(x) = x^2 \sec x$

$\therefore y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16}\sqrt{2} = \frac{\pi^2}{8\sqrt{2}}$ ($\therefore$ **(A)** is correct)

$y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}.2 = \frac{2\pi^2}{9}$ ($\therefore$ **(C)** is wrong)

Also $y'(x) = 2x \sec x + x^2 \sec x \tan x$

$\Rightarrow y'\left(\frac{\pi}{4}\right) = \frac{\pi}{2}.\sqrt{2} + \frac{\pi^2\sqrt{2}}{16}$ ($\therefore$ **(B)** is wrong)

and $y'\left(\frac{\pi}{3}\right) = 2.\frac{\pi}{3}.2 + \frac{\pi^2}{9}.2.\sqrt{3}$

$= \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$ ($\therefore$ **(D)** is correct)

17. $f'(x) - 2f(x) < 0$

Multiply both side by e^{-2x}

$e^{-2x} f'(x) - 2e^{-2x} f(x) < 0$

$\frac{d}{dx}\left(e^{-2x} f(x)\right) < 0$

Now, $g(x) = e^{-2x} f(x)$

$\therefore$ g(x) is a decreasing function.

$x > \frac{1}{2}$

$g(x) < g\left(\frac{1}{2}\right)$

$\Rightarrow e^{-2x} f(x) < \frac{1}{e} \quad \Rightarrow f(x) < e^{2x-1}$

$\Rightarrow \int_{1/2}^{1} f(x)dx < \frac{1}{e}\int_{1/2}^{1} e^{2x}dx$

$= \left[\frac{1}{2e}e^{2x}\right]_{1/2}^{1} = \frac{1}{2e}\left(e^2 - e\right) = \frac{1}{2}\left(e-1\right)$

$\Rightarrow \int_{1/2}^{1} f(x)dx < \frac{e-1}{2}$

obviously $f(x)$ is positive

$\therefore \int_{1/2}^{1} f(x)dx > 0$

18. $\frac{dy}{dx} = \frac{y}{x} + \sec\frac{y}{x}$

Let $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + \frac{xdv}{dx} = v + \sec v$

$\cos v\, dv = \frac{dx}{x}$

$\sin v = \ell nx + c$

$\sin\left(\frac{y}{x}\right) = \ell nx + c$

$\because$ passing through $\left(1, \frac{\pi}{6}\right)$

$\Rightarrow \sin\frac{\pi}{6} = c \Rightarrow c = \frac{1}{2}$

$\therefore \sin\frac{y}{x} = \ell nx + \frac{1}{2}$

Paragraph for Question 19 and 20

19. $e^{-x}(f''(x) - 2f'(x) + f(x)) \geq 1$

$D((f'(x) - f(x))e^{-x}) \geq 1$

$\Rightarrow D((f'(x) - f(x)e^{-x}) \geq 0$

$\Rightarrow (f'(x) - f(x))e^{-x}$ is an increasing function.

As we know that $e^{-x}f(x)$ has local minima at $x = \frac{1}{4}$

$e^{-x}(f'(x) - f(x)) = 0$ at $x = \frac{1}{4}$

Let $F(x) = e^{-x}(f'(x) - f(x))$

$F(x) < 0$ in $\left(0, \frac{1}{4}\right)$

$e^{-x}(f'(x) - f(x) < 0$ in $\left(0, \frac{1}{4}\right)$

$f'(x) < f(x)$ in $\left(0, \frac{1}{4}\right)$

option C

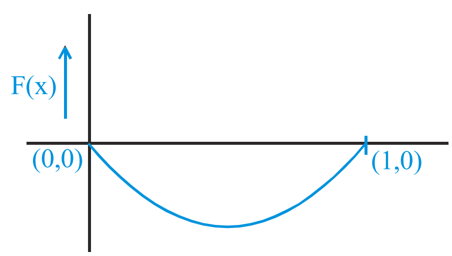
20. $D(e^{-x}(f'(x) - f(x)) \geq 0\ \forall\ x \in (0, 1)$

$D(D(e^{-x}f(x)) \geq 0\ \forall\ x \in (0, 1)$

$D^2(e^{-x}f(x)) \geq 0$

Let $F(x) = e^{-x}f(x)$

$F''(x) > 0$ means it is concave upward.



$F(0) = F(1) = 0$

$F(x) < 0\ \forall\ x \in (0, 1)$

$e^{-x}f(x) < 0\ \forall\ x \in (0, 1)$

$f(x) < 0$

Option D is possible

$(x+2)^2 + y(x+2) = y^2 . \frac{dx}{dy}$

$\frac{dx}{dy} = \frac{(x+2)^2}{y^2} + \frac{x+2}{y}$

$$\frac{1}{(x+2)^2}\frac{dx}{dy}=\frac{1}{y^2}+\frac{1}{y(x+2)}$$

$$\therefore \frac{1}{(x+2)^2}\frac{dx}{dy}-\frac{1}{(x+2)y}=\frac{1}{y^2}$$

$$-\frac{dt}{dy}-\frac{t}{y}=\frac{1}{y^2}$$

$\therefore$ Put $\frac{1}{x+2}=t, -\frac{1}{(x+2)^2}\frac{dx}{dy}=\frac{dt}{dy}$

$$\frac{dt}{dy}+\frac{t}{y}=-\frac{1}{y^2} \qquad \text{I.F} = e^{\int\frac{1}{y}dy}=y$$

$$t.y=C+\int y\left(-\frac{1}{y^2}\right)dy$$

$$t.y=C-\log y$$

$$\therefore \frac{1}{x+2}.y=C-\log y$$

It passes $(1, 3) \Rightarrow 1 = C + \log 3 \Rightarrow C = 1 + \log(3)$

$$\frac{y}{x+2}=1+\log 3-\log y$$

[A] option is correct.

For option **(C)**

$$\frac{(x+2)^2}{x+2}=1-\log\left(\frac{y}{3}\right)$$

$$x+1=\log\left(\frac{y}{3}\right)$$

$\therefore y = 3e^{-x-1}$

$\Rightarrow$ Intersect

For option **(D)**

$$\frac{(x+3)^2}{4+2}-1=-\log\left(\frac{(x+3)^2}{3}\right)$$

$$\therefore \frac{(x+3)^2-1}{x+2}=-\log\left\{\frac{(x+3)^2}{3}\right\}$$

$$3e\left(\frac{(x+3)^2-1}{-x-2}\right)=(x+3)^2$$

$\Rightarrow$ will intersect.

$\Rightarrow$ **(D)** is not correct.

MOCK TEST

1. (A)

$$x^2=e^{\left(\frac{x}{y}\right)^{-1}\left(\frac{dy}{dx}\right)} \Rightarrow x^2=e^{\left(\frac{y}{x}\right)\left(\frac{dy}{dx}\right)}$$

$$\Rightarrow \ln x^2=\frac{y}{x}\frac{dy}{dx}$$

$$\Rightarrow \int x\ell nx^2dx=\int y\,dy$$

Put $x^2 = t \Rightarrow 2xdx = dt$

$$\frac{1}{2}\int \ell nt\,dt=\frac{y^2}{2}$$

$\Rightarrow c + t\,\ell n\,t - t = y^2$

$\Rightarrow y^2 = x^2\,\ell nx^2 - x^2 + c$

2. (D)

$$\frac{dy}{dx}=\frac{(x+y)+1}{2(x+y)+1}$$

put $x+y=t,\ 1+\frac{dy}{dx}=\frac{dt}{dx}$

$$\Rightarrow \frac{dt}{dx}=\frac{t+1}{2t+1}+1=\frac{3t+2}{2t+1}$$

$$\Rightarrow \int\frac{2t+1}{3t+2}dt=\int dx$$

$$\Rightarrow \frac{2t}{3}-\frac{1}{9}\ell n(3t+2)=x+c$$

$\Rightarrow 6(x+y) - \ell n(3x+3y+2) = 9x + c$

$\Rightarrow \ell n(3x+3y+2) = 6y - 3x + c$

Since it passes through (0, 0) hence equation of curve is

$$6y-3x=\ell n\left|\frac{3x+3y+2}{2}\right|$$

3. (A)

$$y_1y_3=3y_2^2 \Rightarrow \int\frac{y_3}{y_2}=\int\frac{3y_2}{y_1}$$

$\Rightarrow \ell ny_2 = 3\,\ell ny_1 + \ell nc$

$$\Rightarrow y_2=cy_1^3 \Rightarrow \int\frac{y_2}{y_1^2}=\int cy_1$$

$$\Rightarrow -\frac{1}{y_1}=cy+d$$

$\Rightarrow -dx = (cy + d)\,dy$

$$\Rightarrow -x=\frac{cy^2}{2}+d\,y+e$$

4. $\dfrac{d^2y}{dx^2}(x^2+1) = 2x\dfrac{dy}{dx}$

$\Rightarrow \displaystyle\int \frac{\frac{d^2y}{dx^2}}{\frac{dy}{dx}}\,dx = \int \frac{2x}{x^2+1}\,dx$

$\Rightarrow \ell n\left(\dfrac{dy}{dx}\right) = \ell n\,(x^2+1) + \ell n\,c$

$\Rightarrow \dfrac{dy}{dx} = c(x^2+1) \quad \Rightarrow \quad c = 3$ as at $x = 0,\ \dfrac{dy}{dx} = 3$

$\Rightarrow \dfrac{dy}{dx} = 3(x^2+1)\,dx$

$\Rightarrow y = x^3 + 3x + 1$

5. **(B)**

$\dfrac{dy}{dx} = \dfrac{1}{x\cos y + 2\sin y\cos y}$

$\therefore \dfrac{dx}{dy} = x\cos y + 2\sin y\cos y$

$\Rightarrow \dfrac{dx}{dy} + (-\cos y)\,x = 2\sin y\cos y$

$\therefore$ I.F. $= e^{-\int \cos y\,dy} = e^{-\sin y}$

$\therefore$ The solution is

$x.\ e^{-\sin y} = 2\int e^{-\sin y}\ .\ \sin y\cos y\,dy$

$\Rightarrow x.\ e^{-\sin y} = -2\sin y\,e^{-\sin y} - 2\int(-e^{-\sin y})\cos y\,dy$

$\Rightarrow x.\ e^{-\sin y} = -2\sin y\,e^{-\sin y} + 2\int e^{-\sin y}\cos y\,dy$

$= -2\sin y\,e^{-\sin y} - 2e^{-\sin y} + c$

$\Rightarrow x = -2\sin y - 2 + c\,e^{\sin y} = ce^{\sin y} - 2(1 + \sin y)$

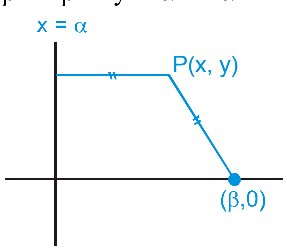
$\therefore k = 2$

6. **(A)**

Directrix $\perp$ to x axis, Let $x = \alpha$ and focus on x axis Let $(\beta, 0)$, Now

$(x-\beta)^2 + y^2 = (x-\alpha)^2$

$\beta^2 - 2\beta x + y^2 = \alpha^2 - 2\alpha x$



$y^2 = 2(\beta-\alpha)\,x + \alpha^2 - \beta^2$

In general $y^2 = mx + c$ (Two arbitrary constant m and c)

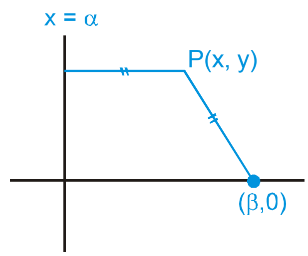
$2y\dfrac{dy}{dx} = m$

$2y\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 = 0$

7. Directrix $\perp$ to x axis, Let $x = \alpha$ and focus on x axis Let $(\beta,0)$, Now

$(x-\beta)^2 + y^2 = (x-\alpha)^2$

$\beta^2 - 2\beta x + y^2 = \alpha^2 - 2\alpha x$



$y^2 = 2(\beta-\alpha)\,x + \alpha^2 - \beta^2$

In general $y^2 = mx + c$ (Two arbitnary constant m and c)

$2y\dfrac{dy}{dx} = m$

$2y\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 = 0$

8. **(A)**

Put $x = \sin\theta$, $y = \sin\phi$, so that the given equation is reduced to

$\cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$

$\Rightarrow 2\cos\dfrac{\theta+\varphi}{2}\cos\dfrac{\theta-\varphi}{2} = 2a\cos\dfrac{\theta+\varphi}{2}\sin\dfrac{\theta-\varphi}{2}$

$\Rightarrow \cot\dfrac{\theta-\varphi}{2} = a$

$\Rightarrow \theta - \phi = 2\cot^{-1}a$

$\Rightarrow \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a$

Differentiating we get

$\dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{\sqrt{1-y^2}}\dfrac{dy}{dx} = 0$

So the degree is one.

9. (A)

$$y\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y\frac{dy}{dx} - x = 0$$

$$y\frac{dy}{dx}\left(\frac{dy}{dx}-1\right) + x\left(\frac{dy}{dx}-1\right) = 0$$

$$\left(y\frac{dy}{dx}+x\right)\left(\frac{dy}{dx}-1\right) = 0$$

$\therefore$ either $ydy + xdx = 0$ or $y - x = c$

since the curves pass through the point (3, 4)

$\therefore$ $x^2 + y^2 = 25$ or $x - y + 1 = 0$

10. (A)

The given differential equation is

$$(x^2-1)\frac{dy}{dx} + 2xy = \frac{1}{x^2-1}$$

$$\Rightarrow \frac{dy}{dx} + \frac{2x}{x^2-1}y = \frac{1}{(x^2-1)^2} \qquad \text{.......(i)}$$

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = \frac{2x}{x^2-1}$ and $Q = \frac{1}{(x^2-1)^2}$

$$\therefore \text{ I.F.} = e^{\int P\,dx} = e^{\int \frac{2x}{x^2-1}dx} = e^{\log(x^2-1)} = (x^2-1)$$

$\therefore$ Solution is $y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx$

$$\Rightarrow y(x^2-1) = \int \frac{1}{x^2-1}dx$$

$$\Rightarrow y(x^2-1) = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$$

This is the required solution.

11. (A, B)

$y = e^{-x}\cos x$

$$y_1 = -e^{-x}\cos x - e^{-x}\sin x = -\sqrt{2}\,e^{-x}\cos\left(x-\frac{\pi}{4}\right)$$

$$y_2 = +\left(\sqrt{2}\right)^2 e^{-x}\cos\left(x-\frac{\pi}{2}\right)$$

$$y_3 = \left(-\sqrt{2}\right)^3 e^{-x}\cos\left(x-\frac{3\pi}{4}\right)$$

$$y_4 = +\left(\sqrt{2}\right)^4 e^{-x}\cos(x-\pi) = -4e^{-x}\cos x$$

$\Rightarrow$ $y_4 + 4y = 0$ $\quad k_4 = 4$

Differentiating it again 4 times

$y_8 + 4y_4 = 0 \quad \Rightarrow \quad y_8 - 16y = 0 \quad \Rightarrow \quad k_8 = -16$

$y_{12} + 4y_8 = 0 \quad \Rightarrow \quad y_{12} + 64y = 0 \quad \Rightarrow \quad k_{12} = 64$

Similarly $k_{16} = -256$

12. (A) $\frac{dy}{dx} = -c_1\sin x + \sqrt{c_2}\cos x$

$$\frac{d^2y}{dx^2} = -c_1\cos x - \sqrt{c_2}\sin x = 2 - y$$

$$\frac{d^2y}{dx^2} + y + 2 = 0$$

(B) $\frac{dy}{dx} = \cos x\frac{\sec^2 x/2}{2\tan\frac{x}{2}} - \sin x\,\ell n\left(\tan\frac{x}{2}\right)$

$$\frac{dy}{dx} = \cot x - \sin x\,\ell n\left(\tan\frac{x}{2}\right)$$

$$\frac{d^2y}{dx^2} = -\text{cosec}^2 x - \left(\sin x.\frac{1}{\sin x} + \cos x\,\ell n\left(\tan\frac{x}{2}\right)\right)$$

$$\frac{d^2y}{dx^2} = -\cot^2 x - 2 - \cos x\,\ell n\left(\tan\frac{x}{2}\right)$$

$$\frac{d^2y}{dx^2} + y + \cot^2 x = 0$$

(C) $\frac{dy}{dx} = -c_1\sin x + c_2\cos x + \frac{d}{dx}\left(\cos x\,\ell n\left(\tan\frac{x}{2}\right)\right)$

$$\frac{d^2y}{dx^2} = -(c_1\cos x + c_2\sin x) + \frac{d^2}{dx^2}\left(\cos x\,\ell n\left(\tan\frac{x}{2}\right)\right)$$

$$\frac{d^2y}{dx^2} = -c_1\cos x - c_2\sin x - \cot^2 x - 2 - \cos x\,\ell n\left(\tan\frac{x}{2}\right)$$

$$\frac{d^2y}{dx^2} + y + \cot^2 x = 0$$

13. (B,C)

(A) $\frac{dy}{dx} = -c_1 \sin x + \sqrt{c_2}\cos x$

$\frac{d^2y}{dx^2} = -c_1 \cos x - \sqrt{c_2}\sin x = 2 - y$

$\frac{d^2y}{dx^2} + y - 2 = 0$

(B) $\frac{dy}{dx} = \cos x \frac{\sec^2 x/2}{2\tan\frac{x}{2}} - \sin x\, \ell n\left(\tan\frac{x}{2}\right)$

$\frac{dy}{dx} = \cot x - \sin x\, \ell n\left(\tan\frac{x}{2}\right)$

$\frac{d^2y}{dx^2} = -\text{cosec}^2 x - \left(\sin x.\frac{1}{\sin x} + \cos x\, \ell n\left(\tan\frac{x}{2}\right)\right)$

$\frac{d^2y}{dx^2} = -\cot^2 x - 2 - \cos x\, \ell n\left(\tan\frac{x}{2}\right)$

$\frac{d^2y}{dx^2} + y + \cot^2 x = 0$

(C) $\frac{dy}{dx} = -c_1 \sin x + c_2 \cos x + \frac{d}{dx}\left(\cos x\, \ell n\left(\tan\frac{x}{2}\right)\right)$

$\frac{d^2y}{dx^2} = -(c_1 \cos x + c_2 \sin x) + \frac{d^2}{dx^2}\left(\cos x\, \ell n\left(\tan\frac{x}{2}\right)\right)$

$\frac{d^2y}{dx^2} = -c_1 \cos x - c_2 \sin x - \cot^2 x - 2 - \cos x\, \ell n\left(\tan\frac{x}{2}\right)$

$\frac{d^2y}{dx^2} + y + \cot^2 x = 0$

14. $x^2 + y^2 + 2gx + 2fy + c = 0$

(There are three arbitrary constants g, f and c)

$2x + 2y\cdot\frac{dy}{dx} + 2g + 2f\cdot\frac{dy}{dx} = 0$ .... **(i)**

$2 + 2y_1^2 + 2yy_2 + 2fy_2 = 0$ .... **(ii)**

$4y_1y_2 + 2y_1y_2 + 2yy_3 + 2fy_3 = 0$ .... **(iii)**

from **(i)** , **(ii)** and **(iii)**

$y_3\left(1 + y_1^2\right) - 3y_1y_2^2 = 0$

15. (A, B, D)

$\frac{dy}{dx}(x^2y^3 + xy) = 1 \Rightarrow \frac{dx}{dy} = x^2y^3 + xy$

$\Rightarrow \frac{dx}{dy} - xy = x^2y^3 \Rightarrow -\frac{1}{x^2}\frac{dx}{dy} + \frac{y}{x} = -y^3$

put, $\frac{1}{x} = t$ and differentiating with respect to y $-\frac{1}{x^2}$

$\frac{dx}{dy} = \frac{dt}{dy}$

$\Rightarrow \frac{dt}{dy} + ty = -y^3 \Rightarrow e^{\frac{y^2}{2}} t = \int -y^3 e^{y^2/2}\, dy$

$\Rightarrow \frac{1}{x} = 2 - y^2 + c\, e^{-\frac{y^2}{2}}$

16. (B)

Statement-II $\ell n\frac{x}{y} - \frac{1}{xy} = c$

i.e $\ell nx - \ell ny - \frac{1}{xy} = c$

diff.

$\frac{1}{x} - \frac{1}{y}\frac{dy}{dx} + \frac{y + x\frac{dy}{dx}}{(xy)^2} = 0$

$xy^2\,dx - x^2y\,dy + ydx + xdy = 0$

$(1 + xy)\,ydx + x(1 - xy)\,dy = 0$

$\therefore$ statement is true.

Statement-I $x - \frac{y^2}{2} + \frac{1}{3}(x^2 + y^2)^{3/2} + c = 0$

$1 - y\frac{dy}{dx} + \frac{1}{2}\sqrt{x^2 + y^2}\left(2x + 2y\frac{dy}{dx}\right) = 0$

$dx - y\,dy + x\sqrt{x^2 + y^2}\,dx + y\sqrt{x^2 + y^2}\,dy = 0$

$(1 + x\sqrt{x^2 + y^2})\,dx + y(-1 + \sqrt{x^2 + y^2})\,dy = 0$

$\therefore$ Statement is true

17. (B)

$\frac{dy}{dx} = x + \frac{1}{x^2}$

$dy = xdx + \frac{1}{x^2}.dx$

$y = \frac{x^2}{2} - \frac{1}{x} + c$

$x = 3, y = 9$

$9 = \frac{9}{2} - \frac{1}{3} + c \Rightarrow \frac{9}{2} + \frac{1}{3} = c$

$c = \frac{27 + 2}{6} = \frac{29}{6}$

$y = \frac{x^2}{2} - \frac{1}{x} + \frac{29}{6}$

Statement-I is true

Statement-II

$\frac{dy}{dx} = t$

$t^2 - t(e^x + e^{-x}) + 1 = 0$

$e^x t^2 - t(e^{2x} + 1) + e^x = 0$

$e^x t(t - e^x) - (t - e^x) = 0$

$(t - e^x)(te^x - 1) = 0$

$dy = e^x dx$

$y = e^x$ ....**(i)**

$e^x \frac{dy}{dx} = 1$

$y = -e^{-x}$ ....**(ii)**

$y = c_1e^x + c_2e^{-x}$ can not be the solution of given diff. Equation.

18. (B)

Statement-I : Equation of all circles can be given by $x^2 + y^2 + 2gx + 2fy + c = 0$, will be of order 3

Statement-II : is obviously true but it does not explain statement-1

19. (C)

Statement-I $y^3 \frac{dy}{dx} + (x + y^2) = 0$

$y^2 = t$

$2y\frac{dy}{dx} = \frac{dt}{dx}$

$\frac{1}{2}\frac{dt}{dx} . t + x + t = 0$

is homogenous equation

Statement-II is obviously false

20. (B)

Statement-1 is obviously true

Statement-2 is also obviously true

but statement-2 does not explain statement-1

21. (A) Variable separable

(B) Variable separable

(C) Variable separable

(D) Put $\tan y = t \Rightarrow \sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$

$\tan x \frac{dt}{dx} + t\sec^2 x = 1$

(Linear Diff. equation)

22. (A)→(q), (B)→(r), (C)→(t), (D)→(p)

(A) $xdy = ydx + y^2 dy \Rightarrow \frac{x\,dy - y\,dx}{y^2} = dy$

$\Rightarrow -d\left(\frac{x}{y}\right) = dy$

$-\frac{x}{y} = y + c$ put $x = 1, y = 1 \Rightarrow c = -2$

$-\frac{x}{y} = y - 2$ Now $\frac{x_0}{3} = -5 \Rightarrow x_0 = -15$

(B) $\frac{dy}{dt} - \frac{t}{t+1}y = \frac{1}{t+1}$

$I.f = e^{-\int \frac{t+1-1}{t+1}dt} = e^{-t+\ell n(t+1)} = (t+1)e^{-t}$

solution is $(t+1)e^{-t}\ y = -e^{-t} + c$

put $t = 0$ and $y = -1 \Rightarrow c = 0$

Now at $t = 1$

$\therefore\ 2e^{-1} y = -e^{-1}$

$y = -\frac{1}{2}$

(C) $(x^2 + y^2)\,dy = xy\,dx$

$\frac{dy}{dx} = \frac{xy}{x^2+y^2}$ put $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore\ \frac{v}{1+v^2} = v + x\frac{dv}{dx} \Rightarrow \ell n\, v - \frac{1}{2v^2} = -\ell n\, x + c$

put $x = 1, y = 1, v = 1$, then $c = -\frac{1}{2}$

$\therefore\ \ell n\frac{y}{x} - \frac{1}{2}\frac{x^2}{y^2} = -\ell n\, x - \frac{1}{2}$ put $y = e$

$\therefore\ x_0 = \sqrt{3}\, e$

(D) $\frac{dy}{dx} + 2\frac{y}{x} = 0$

$x^2 y = C$ put $x = 1, y = 1$

and we get $C = 1$

put $x = 2 \Rightarrow y = \frac{1}{4}$

23.

1. (B)

$\frac{dx}{x} = \frac{y\,dy}{1+y^2} \Rightarrow \ell n\, x = \frac{1}{2} . \ell n(1 + y^2) + c$

from the given condition $c = 0$ $\therefore\ x^2 - y^2 = 1$

2. (A)

$$\frac{dy}{dx} + \frac{1+y^2}{\sqrt{1-x^2}} = 0 \quad \Rightarrow \quad \frac{dy}{1+y^2} + \frac{dx}{\sqrt{1-x^2}} = 0$$

$\Rightarrow \tan^{-1} y + \sin^{-1} x = c$

3. (B) $\frac{dy}{dx} = (1+x).(1+y)$ gives $y = e^{\frac{(1+x)^2}{2}} - 1$

24.

1. $\int_{200}^{f(t)} \frac{dy}{10-y} = \int_0^t k\, dt$

$[\ln|10-y|]_{f(t)}^{200} = kt$

$\ln(190) - \ln|10-f(t)| = kt$

$\ln|10-f(t)| = \ln 190 - kt$

$|10-f(t)| = e^{\ln 190}. e^{-kt}$

$f(t) - 10 = 190e^{-kt}$ $\quad (f(t) > 0$ as $M(t) = 10)$

$f(t) = 10 + 190\, e^{-kt}$.

2. $\frac{dy}{dt} + ky = kM(t)$

Let $M(t) = 10^o$ (constant)

$\int_{200}^{100} \frac{dy}{10-y} = \int_0^{40} k\, dt$

$\left[\ln|(10-y)|\right]_{100}^{200} = k[t]_0^{40}$

$\ln(190) - \ln(90) = 40\,k$

$k = \frac{\ln(19) - \ln(9)}{40}$

3. $\int_{400}^{200} \frac{dy}{10-y} = k\int_0^t dt$

$\left[\ln(y-10)\right]_{200}^{400} = kt$

$t = \frac{\ln 390 - \ln 190}{k}$

$t = 40\left(\frac{\ln 39 - \ln 19}{\ln 19 - \ln 9}\right)$

25.

1. (B)

Obviously option B is not a solution

2. (B)

Among the four option B is the solution of the differential equation

3. (D)

Among the four option D is satisfying

26. (3)

The given equation contains one constant.

Differentiating the equation once, we get

$2x - 2yy' = 2c\,(x^2 + y^2)\,(2x + 2yy')$

But $c = \frac{x^2 - y^2}{(x^2+y^2)^2}$

Substituting for c, we get

$$(x - yy') = \frac{(x^2+y^2)(x^2-y^2)}{(x^2+y^2)^2} . 2\,(x + yy')$$

$\Rightarrow (x^2+y^2)\,(x - yy') = 2\,(x^2 - y^2)\,(x + yy')$

$\Rightarrow yy'\,[(x^2+y^2) + 2\,(x^2-y^2)] = x\,(x^2+y^2) - 2x\,(x^2-y^2)$

$\Rightarrow yy'\,(3x^2 - y^2) = x\,(3y^2 - x^2)$

$\Rightarrow y' = \frac{x(3y^2 - x^2)}{y(3x^2 - y^2)}$

27. (2)

Suppose the point at which normal is drawn is $p \equiv (x, y)$

Equation of the normal is $Y - y = -\frac{dx}{dy}(X - x)$

at X axis $Y = 0$

$\therefore$ coordinate of $X = \frac{y\,dy}{dx} + x$

$\therefore$ Point where it cuts the X axis are $\left(y\frac{dy}{dx} + x, 0\right)$, say A

Mid point of PA are $\left(x + \frac{y\frac{dy}{dx}}{2}, \frac{y}{2}\right)$, which lies on the curve $2y^2 = x$

$\therefore\quad y^2 = 2x + y\dfrac{dy}{dx}$

$\Rightarrow\quad y\dfrac{dy}{dx} + 2x = y^2$ ......(i)

put $y^2 = t$

and differentiating with respect to x $2y\dfrac{dy}{dx} = \dfrac{dt}{dx}$

Now from equation (i) we get $\dfrac{1}{2}\dfrac{dt}{dx} + 2x = t$

$\Rightarrow\quad \dfrac{dt}{dx} - 2t = -4x$

I.F. $= e^{\int -2\,dx} = e^{-2x} \quad \Rightarrow \quad te^{-2x} = -\int e^{-2x}\,4x + c$

$\Rightarrow\quad y^2 e^{-2x} = -\left\{\dfrac{4x\,e^{-2x}}{-2} + \dfrac{1}{2}\int 4e^{-2x}\,dx\right\} + c$

$\Rightarrow\quad y^2 e^{-2x} = +\,2xe^{-2x} + e^{-2x} + c$

at $x = 0$ $y = 0$ $\Rightarrow c = -1$

$\therefore\quad y^2 = 2x - e^{2x} + 1$

28. (1)

For reservoir A,

$\dfrac{dV_A}{dt} = -k_1 V_A$

$\Rightarrow\quad \dfrac{dV_A}{V_A} = -k_1\,dt \qquad \Rightarrow\quad \text{In } V_A = -k_1 t + c$

initially $t = 0$, $V_A = V_{OA}$

$\therefore\quad \text{In } V_{OA} = c \qquad \Rightarrow\quad \text{In } V_A = -k_1 t + \text{In } V_{OA}$

$\Rightarrow\quad \text{In}\left(\dfrac{V_A}{V_{OA}}\right) = -k_1 t \qquad \Rightarrow\quad V_A = V_{OA}\,e^{-k_1 t}$ ......(i)

For reservoir B,

$\dfrac{dV_B}{dt} = -k_2 V_B$

Similarly $V_B = V_{OB}\,e^{-k_2 t}$ ......(ii)

Given that at $t = 0$, $V_{OA} = 2V_{OB}$

At $t = 1$ hr., $V_A = \dfrac{3}{2}V_B$

$\Rightarrow\quad V_{OA}\,e^{-k_1} = \dfrac{3}{2}V_{OB}\,e^{-k_2}$

$\Rightarrow\quad 2V_{OB}\,e^{-k_1} = \dfrac{3}{2}V_{OB}\,e^{-k_2} \quad (\because\ V_{OA} = 2V_{OB})$

$\Rightarrow\quad e^{(k_1 - k_2)} = \dfrac{4}{3}$

$\Rightarrow\quad (k_1 - k_2) = \text{In } 4/3$ ......(iii)

After time t : $V_A = V_B$

$\Rightarrow\quad V_{OA}\,e^{-k_1 t} = V_{OB}\,e^{-k_2 t}$

$\Rightarrow\quad 2V_{OB}\,e^{-k_1 t} = V_{OB}\,e^{-k_2 t} \quad (\because\ V_{OA} = 2V_{OB})$

$\Rightarrow\quad 2 = e^{(k_1 - k_2)t}$

$\Rightarrow\quad \text{In } 2 = (k_1 - k_2)t$ ......(iv)

from (iii) and (iv) we get $t = \dfrac{\text{In } 2}{\text{In } (4/3)}$

$\Rightarrow\quad t = \log_{4/3} 2$

29. (1)

Since given differential equation is

$y(x + y^3)\,dx = x(y^3 - x)\,dy$

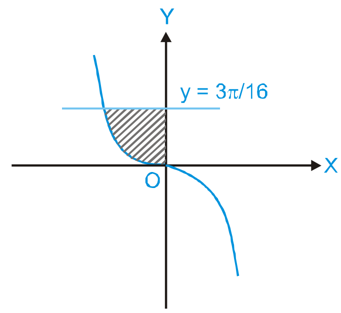
$\Rightarrow\quad (xy\,dx + x^2\,dy) + y^4\,dx - y^3 x\,dy = 0$

$\Rightarrow\quad x(y\,dx + x\,dy) + y^3(y\,dx - x\,dy) = 0$

$\Rightarrow\quad xd(xy) = y^3(x\,dy - y\,dx)$

$\Rightarrow\quad xd(xy) = x^2y^3\,d\left(\dfrac{y}{x}\right)$

$\Rightarrow\quad \dfrac{d(xy)}{(xy)^2} = \left(\dfrac{y}{x}\right)d\left(\dfrac{y}{x}\right)$



On integrating $-\dfrac{1}{xy} = \dfrac{1}{2}\left(\dfrac{y}{x}\right)^2 + c$

at $(4, -2)$

$\Rightarrow\quad \dfrac{1}{8} = \dfrac{1}{2}\left(-\dfrac{2}{4}\right)^2 + c$

$\Rightarrow\quad c = 0 \qquad \therefore\quad y^3 = -2x \qquad \Rightarrow\quad y = (-2x)^{1/3}$

Since $y = g(x) = \int_{1/8}^{\sin^2 x} \sin^{-1}\sqrt{t}\, dt + \int_{1/8}^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt$

$\therefore \quad \frac{dy}{dx} = g'(x) = x \,.\, \sin 2x + x \,.\, (-\sin 2x) = 0$

$\therefore \quad y = c_1$ (constant)

put $\sin x = \cos x = \frac{1}{\sqrt{2}}$

$\therefore \quad c_1 = \int_{1/8}^{1/2} (\sin^{-1}\sqrt{t} + \cos^{-1}\sqrt{t}\,)\, dt$

$= \frac{\pi}{2}\left(\frac{1}{2} - \frac{1}{8}\right) = \frac{3\pi}{16} \qquad \therefore \quad y = g(x) = \frac{3\pi}{16}$

Hence area between $y = \frac{3\pi}{16}$ and $y = (-2x)^{1/3}$

$= \int_0^{3\pi/16} x\, dy = \left|\int_0^{3\pi/16} \left(-\frac{y^3}{2}\right) dy\right| = \frac{1}{8} \cdot \left(\frac{3\pi}{16}\right)^4$ sq. units.

30. (4)

$(x^2 + 4y^2 + 4xy)\, dy = (2x + 4y + 1)\, dx.$

$$\frac{dy}{dx} = \frac{2(x+2y)+1}{(x+2y)^2} \qquad \text{........ (i)}$$

put, $x + 2y = t$ and differentiating with respect to x

$$1 + \frac{2dy}{dx} = \frac{dt}{dx}$$

From equation (i) we get $\left(\frac{dt}{dx} - 1\right)\frac{1}{2} = \frac{2t+1}{t^2}$

$\Rightarrow \quad \frac{dt}{dx} - 1 = \frac{4t+2}{t^2}$

$\Rightarrow \quad \int \frac{t^2}{t^2+4t+2}\, dt = \int dx$

$\Rightarrow \quad \int dt - \int \frac{4t+2}{t^2+4t+2}\, dt = x + c_1$

$\Rightarrow \quad t - \int \frac{2(2t+4)-6}{t^2+4t+2}\, dt = x + c_1$

$\Rightarrow \quad t - 2\,\ell n\,(t^2+4t+2) + 6\,\frac{1}{2\sqrt{2}}\,\ell n\left|\frac{t+2-\sqrt{2}}{t+2+\sqrt{2}}\right| = x + c_1$

$\Rightarrow \quad y = \ell n\,((x+2y)^2 + 4(x+2y) + 2) - \frac{3}{2\sqrt{2}}\,\ell n \left|\frac{x+2y+2-\sqrt{2}}{x+2y+2+\sqrt{2}}\right| + c$, where $c = c_1/2$he

dictionary definition of physics is "the study of matter, energy,