HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

2. $\beta = \cos^{-1}\left(\cos\left(\frac{\pi}{2} - \cos^{-1} x\right)\right)$; $\beta = \cos^{-1}\{\cos(\sin^{-1}x)\}$

also $\alpha = \sin^{-1}[\cos(\sin^{-1}x)]$

$\alpha + \beta = \pi/2 \Rightarrow \tan\alpha = \cot\beta$

3. $(\sin^{-1} x + \sin^{-1} y)^2 = \pi^2$

$\Rightarrow \sin^{-1} x + \sin^{-1} y = \pm\pi$

$\Rightarrow \sin^{-1} x = \sin^{-1} y = \frac{\pi}{2}$

or $\sin^{-1} x = \sin^{-1} y = -\frac{\pi}{2}$

$\Rightarrow x^2 + y^2 = 2.$

4. $x = \sin 2\theta = 2\sin\theta\cos\theta = \frac{4}{5}$; $y = \sin\frac{\varphi}{2}$;

$y > 0, \tan\phi = \frac{4}{3}$

$y^2 = \sin^2\frac{\varphi}{2} = \frac{1-\cos\varphi}{2} = \frac{1}{5} = 1 - \frac{4}{5} = 1 - x$

$\Rightarrow y^2 = 1 - x$

7. $\tan\left[\sin^{-1}\frac{3}{5} + \tan^{-1}\frac{2}{3}\right]$

$\Rightarrow \tan\left[\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{6}{12}}\right] = \tan\tan^{-1}\frac{17}{6} = \frac{17}{6}$

8. $\sin^{-1} x - \cos^{-1} x = \cos^{-1}\frac{\sqrt{3}}{2}$

$\frac{\pi}{2} - 2\cos^{-1} x = \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$

$\Rightarrow \cos^{-1} x = \frac{\pi}{6}$

$x = \frac{\sqrt{3}}{2}$

9. $\frac{1}{\sqrt{2}}\left(\cos\frac{7\pi}{5} - \sin\frac{2\pi}{5}\right)$

$= \cos\frac{\pi}{4}\cos\frac{7\pi}{5} - \sin\frac{\pi}{4}\sin\frac{2\pi}{5}$

$= \cos\frac{\pi}{4}\cos\frac{7\pi}{5} + \sin\frac{\pi}{4}\sin\frac{7\pi}{5}$

11. $\sin^{-1}\left(-\sin\frac{50\pi}{9}\right) = -\sin^{-1}\sin\left(\frac{50\pi}{9}\right)$

$= -\sin^{-1}\sin\left(\frac{14\pi}{9}\right) = -\sin^{-1}\sin\left(2\pi - \frac{4\pi}{9}\right)$

$= -\sin^{-1}\sin\left(-\frac{4\pi}{9}\right) = \frac{4\pi}{9}$

$\cos^{-1}\cos\left(-\frac{31\pi}{9}\right) = \cos^{-1}\cos\left(\frac{31\pi}{9}\right)$

$= \cos^{-1}\cos\left(4\pi - \frac{5\pi}{9}\right) = \cos^{-1}\cos\frac{5\pi}{9} = \frac{5\pi}{9}$

Hence $\sec\left(\frac{4\pi}{9} + \frac{5\pi}{9}\right) = \sec\pi = -1$

12. $x = \frac{2\pi}{3} + \frac{\pi}{4} + \frac{\pi}{3} = \frac{5\pi}{4}$

$y = \cos\left(\frac{1}{2}\sin^{-1}\left(\sin\frac{5\pi}{8}\right)\right)$

$= \cos\left(\frac{1}{2}\left(\pi - \frac{5\pi}{8}\right)\right) = \cos\frac{3\pi}{16}.$

14. $\tan^{-1} 2 + \tan^{-1} 3 = \mathrm{cosec}^{-1} x$

$\Rightarrow \pi + \tan^{-1}(-1) = \mathrm{cosec}^{-1} x$

$\Rightarrow \pi - \frac{\pi}{4} = \mathrm{cosec}^{-1} x \Rightarrow \frac{3\pi}{4} = \mathrm{cosec}^{-1} x$

$\Rightarrow$ no solution. $\left\{-\frac{\pi}{2} \le \mathrm{cosec}^{-1} x \le \frac{\pi}{2}\right\}$

15. $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ and

$\sin^{-1}x - \cos^{-1}x = \sin^{-1}(3x-2)$

$-\quad +\quad -$

$2\cos^{-1}x = \cos^{-1}(3x-2)$

Also $\quad x \in [-1, 1]$

$\cos^{-1}(2x^2-1) = \cos^{-1}(3x-2)$

and $\quad (3x-2) \in [-1, 1]$ i.e. $-1 \le 3x-2 \le 1$

$2x^2 - 1 = 3x - 2$

hence $\quad x \in \left[\frac{1}{3}, 1\right]$

$2x^2 - 3x + 1 = 0 \quad \Rightarrow \quad x = 1$ or $1/2 \quad \Rightarrow$ A

$$= \cos\left(\frac{7\pi}{5} - \frac{\pi}{4}\right) = \cos\left(\frac{23\pi}{20}\right) = \cos\left(\pi + \frac{3\pi}{20}\right)$$

$$= \cos\left(\pi - \frac{3\pi}{20}\right) = \cos\frac{17\pi}{20}$$

$$\therefore \cos^{-1}\left\{\frac{1}{\sqrt{2}}\left(\cos\frac{7\pi}{5} - \sin\frac{2\pi}{5}\right)\right\} = \cos^{-1}\left(\cos\left(\frac{17\pi}{20}\right)\right)$$

$= \frac{17\pi}{20}$ (since $\frac{17\pi}{20}$ lies between 0 and π)

16. $\sin^{-1}\sqrt{1-x^2} + \cos^{-1}x = \cot^{-1}\frac{\sqrt{1-x^2}}{x} - \sin^{-1}x$

or $\quad \frac{\pi}{2} + \sin^{-1}\sqrt{1-x^2} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$

$\tan^{-1}\frac{\sqrt{1-x^2}}{x} + \sin^{-1}\sqrt{1-x^2} = 0$

$\Rightarrow \quad -1 \le x < 0 \cup \{1\} \quad \Rightarrow \quad$ C

18. $(\tan^{-1}x)^2 - 3\tan^{-1}x + 2 \ge 0$

$(\tan^{-1}x - 1)(\tan^{-1}x - 2) \ge 0$

we know that $\tan^{-1}x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

so $\quad \tan^{-1}x \ge 2$ (not possible) or $\tan^{-1}x \le 1$

$\Rightarrow \quad x \in (-\infty, \tan 1]$

19. $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$

$\Rightarrow \quad x = y = z = 1$

$\Rightarrow \quad x^{100} + y^{100} + z^{100} - \frac{9}{x^{101} + y^{101} + z^{101}} = 0$

20. $\cot^{-1}\left\{\frac{\sqrt{1-\sin x} + \sqrt{1+\sin x}}{\sqrt{1-\sin x} - \sqrt{1+\sin x}}\right\} \qquad \frac{\pi}{2} < x < \pi$

Rationalize the term in the bracket

$$= \cot^{-1}\left(\frac{2 + 2\sqrt{1-\sin^2 x}}{-2\sin x}\right) = \cot^{-1}\left(\frac{1-\cos x}{-\sin x}\right)$$

$$= \cot^{-1}\left(-\tan\frac{x}{2}\right) = \frac{\pi}{2} - \tan^{-1}\left(-\tan\frac{x}{2}\right)$$

$= \frac{\pi}{2} + \tan^{-1}\tan\frac{x}{2} \qquad$ since $\quad \frac{x}{2} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$$= \frac{\pi}{2} + \frac{x}{2}$$

21. $\sin^{-1}\left(\frac{3\sin 2\theta}{5 + 4\cos 2\theta}\right) = \frac{\pi}{2}$

Taking sin on both side $\frac{3\sin 2\theta}{5 + 4\cos 2\theta} = 1$

$3\sin 2\theta = 5 + 4\cos 2\theta$

$$\frac{6\tan\theta}{1 + \tan^2\theta} = 5 + 4\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right)$$

$\tan^2\theta - 6\tan\theta + 9 = 0$

$\tan\theta = 3$

22. Consider $\tan 65° - 2\tan 40°$

$\tan(45° + 20°) - 2\tan 40°$

$$\frac{1 + \tan 20°}{1 - \tan 20°} - \frac{4\tan 20°}{1 - \tan^2 20°}$$

$$\frac{(1 + \tan 20°)^2 - 4\tan 20°}{(1 - \tan 20°)(1 + \tan 20°)}$$

$$= \frac{(1-\tan 20°)(1-\tan 20°)}{(1-\tan 20°)(1+\tan 20°)} = \frac{(1-\tan 20°)}{(1+\tan 20°)}$$

$= \tan(45° - 20°) = \tan 25°$

$\therefore \quad \tan^{-1}(\tan 25°) = 25°$

23. $\cos^{-1}(2x) + \cos^{-1}(3x) = \pi - \cos^{-1}(x) = \cos^{-1}(-x)$

$\cos^{-1}[(2x)(3x) - \sqrt{1-4x^2}\sqrt{1-9x^2}\,] = \cos^{-1}(-x)$

$6x^2 - \sqrt{1-4x^2}\cdot\sqrt{1-9x^2} = -x$

$(6x^2+x)^2 = (1-4x^2)(1-9x^2)$

$\Rightarrow \quad x^2 + 12x^3 = 1 - 13x^2$

$\Rightarrow \quad 12x^3 + 14x^2 - 1 = 0$

$\therefore \quad a = 12; \quad b = 14; c = 0$

$\Rightarrow \quad a + b + c = 26$

24. $x = \frac{1}{2}; \qquad y = \frac{1}{3}$ (from 2nd relation)

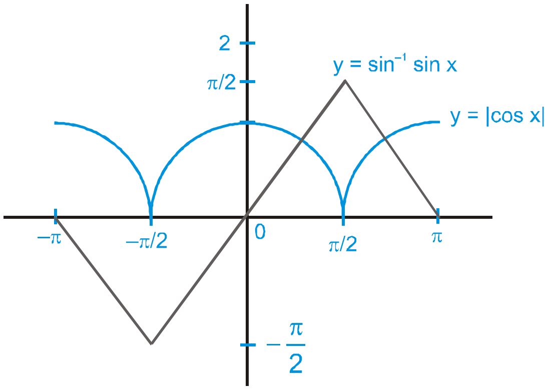
$\cot^{-1}\frac{1}{2} + \cot^{-1}\frac{1}{3} = \tan^{-1}2 + \tan^{-1}3 = \frac{3\pi}{4}$ Ans.]

25. $\tan\left(\frac{\pi}{4}+\alpha\right)$ when $\alpha = \tan^{-1}\left(\frac{\frac{1}{4}+\frac{1}{5}}{1-\frac{1}{20}}\right)$;

$$\alpha = \tan^{-1}\left(\frac{9}{19}\right) = \frac{1+\frac{9}{19}}{1-\frac{9}{19}} = \frac{28}{10} = \frac{14}{5} = \frac{a}{b}$$

$\Rightarrow \quad 14 + 5 = 19$

26. Given equation is $|\cos x| = \sin^{-1}(\sin x)$ $-\pi \le x \le \pi$



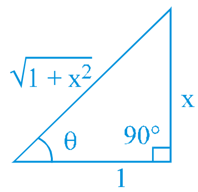
Number of solution = 2

27. Let $\tan^{-1}(x) = \theta$

$\Rightarrow \quad x = \tan\theta$

$\cos\theta = x$ (given)

$\frac{1}{\sqrt{1+x^2}} = x$



$x^2(1+x^2) = 1$

$\Rightarrow \quad x^2 = \frac{-1\pm\sqrt{5}}{2}$ (x^2 can not be – ve)

$\Rightarrow \quad x^2 = \frac{\sqrt{5}-1}{2} \quad \Rightarrow \quad \frac{x^2}{2} = \frac{\sqrt{5}-1}{4}$

$$\cos^{-1}\left(\frac{\sqrt{5}-1}{4}\right) = \cos^{-1}\left(\sin\frac{\pi}{10}\right) = \cos^{-1}\left(\cos\frac{2\pi}{5}\right) = \frac{2\pi}{5}$$

28. Let $x = \cos\theta, x \ge 0$

LHS $= 2\theta$

RHS $= \sin^{-1}(\,|2\cos\theta|\sin\theta)$

$\Rightarrow \quad \sin^2(\sin 2\theta)$

$\sin^{-1}(\sin 2\theta) \quad 0 \le 2\theta \le \pi/2$

$0 \le \theta \le \frac{\pi}{4} \quad \Rightarrow \quad \frac{1}{\sqrt{2}} \le x \le 1$

29. $x(x+1) \ge 0$ and $0 \le x^2 + x + 1 \le 1$

$\Rightarrow \quad x \ge 0$ or $x \le -1$ and $x(x+1) \le 0$

$x \le 0$ or $x \ge -1$

Hence $x = 0$ or $x = -1$

EXERCISE - 2

Part # I : Multiple Choice

1. Let $\tan^{-1}x = \alpha$ and $\tan^{-1}x^3 = \beta$

$\tan\alpha = x$ and $\tan\beta = x^3$

$$\therefore \quad 2\tan(\alpha+\beta) = \frac{2(\tan\alpha + \tan\beta)}{1-\tan\alpha\tan\beta}$$

$$= 2\left[\frac{x+x^3}{1-x^4}\right] = \frac{2x}{1-x^2} \qquad \Rightarrow \qquad \textbf{(A)}$$

Also $\dfrac{2x}{1-x^2} = \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \tan 2\alpha = \tan(2\tan^{-1}x)$

$\Rightarrow$ **(B)**

$$= \tan\left(2\left(\frac{\pi}{2} - \cot^{-1}x\right)\right) = \tan(\pi - \cot^{-1}x - \cot^{-1}x)$$

$$= \tan\left(\cot^{-1}(-x) - \cot^{-1}(x)\right)$$

$\Rightarrow$ **(C)**

2. Let $\tan^{-1}\dfrac{a}{x} = \alpha \qquad \Rightarrow \qquad \tan\alpha = \dfrac{a}{x}$ etc.

$$\alpha + \beta + \gamma + \delta = \frac{\pi}{2}$$

$$\tan(\alpha+\beta+\gamma+\delta) = \tan\frac{\pi}{2}$$

$$\frac{S_2 - S_3}{1 - S_2 + S_4} = \infty$$

$\Rightarrow \quad 1 - S_2 + S_4 = 0$

$\Rightarrow \quad S_4 - S_2 + 1 = 0$

How, $S_4 = \tan\alpha \cdot \tan\beta \cdot \tan\gamma \cdot \tan\delta = \dfrac{abcd}{x^4}$

$$S_2 = \sum \tan\alpha \cdot \tan\beta = \frac{\sum ab}{x^2}$$

$$\therefore \quad \frac{abcd}{x^4} - \frac{\sum ab}{x^2} + 1 = 0$$

$$x^4 - \sum ab\, x^2 + abcd = 0 \quad \to \quad x_1,\ x_2,\ x_3,\ x_4$$

$\therefore \quad x_1 + x_2 + x_3 + x_4 = 0$ ...... **(i)**

$\sum x_1x_2x_3 = 0$ ...... **(ii)**

$$\sum x_1x_2x_3 = \underbrace{x_1x_2x_3x_4}_{\text{non zero}}\left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right] = 0$$

$\Rightarrow$ **(B)**

$x_1x_2x_3x_4 = abcd$

$\Rightarrow$ **(C)**

3. $\cos\left(-\dfrac{14\pi}{5}\right) = \cos\dfrac{14\pi}{5} = \cos\dfrac{4\pi}{5}$

Hence $\cos\left(\dfrac{1}{2}\cos^{-1}\left(\cos\dfrac{4\pi}{5}\right)\right)$

$$= \cos\frac{4\pi}{10} = \cos\frac{2\pi}{5}$$

$\Rightarrow$ BCD

6. **for (A) and (B)**

$\cos(\cos^{-1}1) = 1$

$\Rightarrow \quad \cos\left(\cos(\cos^{-1}1)\right) = \cos 1$

$\sin^{-1}\left(\sin(\pi - 1)\right) = \pi - (\pi - 1) = 1$

$\Rightarrow \quad \sin\left(\sin^{-1}\left(\sin(\pi-1)\right)\right) = \sin 1$

$\cos^{-1}\left(\cos(2\pi - 2)\right) = \cos^{-1}(\cos 2) = 2$

$\Rightarrow \quad \sin\left(\cos^{-1}\left(\cos(2\pi-2)\right)\right) = \sin 2$

$(\tan\left(\cot^{-1}(\cot 1)\right) = \tan 1$

It is easy to compare

$\cos 1,\ \sin 1,\ \sin 2,\ \tan 1 \quad \cos 1 < \sin 1 < \sin 2 < \tan 1$

$\Rightarrow$ **(A)** is correct

for (C)

$\cos^{-1}\cos x$ is periodic and even

$\cos^{-1}\cos(2t\pi - 1) = \cos^{-1}(\cos 1) = 1\ (t \in I)$

$$\sum_{t=1}^{5000} \cos^{-1}\cos(2t\pi - 1) = 5000$$

now $\cot^{-1}\cot(t\pi + 2) = 2$

[$\cot^{-1}\cot x$ is periodic with period π]

7. $2x = \tan(2\tan^{-1}a) + 2\tan(\tan^{-1}a + \tan^{-1}a^3)$

$$2x = \frac{2a}{1-a^2} + \frac{2\left(a+a^3\right)}{1-a^4}$$

(Using $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$)

$\therefore\ a \neq \pm 1$

$\Rightarrow$ **(D)**

$$x = \frac{a}{1-a^2} + \frac{a}{1-a^2} = \frac{2a}{1-a^2}$$

$\Rightarrow\ x(1-a^2) = 2a$

$\Rightarrow\ a^2x + 2a = x$

$\Rightarrow$ **(A)**

Hence B & C are invalid

10. $\sum_{n=1}^{\infty} \tan^{-1}\frac{4n}{n^4 - 2n^2 + 2}$

$$= \lim_{k\to\infty}\sum_{n=1}^{k}\left\{\tan^{-1}(n+1)^2 - \tan^{-1}(n-1)^2\right\}$$

$$= \lim_{k\to\infty}\left\{\tan^{-1}(k+1)^2 + \tan^{-1}k^2 - \tan^{-1}1 - \tan^{-1}0\right\}$$

$$= \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{4} - 0 = \frac{3\pi}{4}$$

Also $\tan^{-1}2 + \tan^{-1}3 = \pi + \tan^{-1}\left(\frac{3+2}{1-3.2}\right)$

Since $xy = 6 > 1$

$$= \frac{3\pi}{4} \text{ and } \sec^{-1}\left(-\sqrt{2}\right) = \frac{3\pi}{4}$$

12. $\sum_{r=1}^{\infty} T_r = \cot^{-1}\left(r^2 + \frac{3}{4}\right) = \sum_{r=1}^{\infty}\tan^{-1}\left(\frac{1}{1 + r^2 - \frac{1}{4}}\right)$

$$= \sum_{r=1}^{\infty}\tan^{-1}\left(\frac{\left(r+\frac{1}{2}\right)-\left(r-\frac{1}{2}\right)}{1+\left(r-\frac{1}{2}\right)\left(r+\frac{1}{2}\right)}\right)$$

$$= \sum_{r=1}^{\infty}\tan^{-1}\left(r+\frac{1}{2}\right) - \sum_{r=1}^{\infty}\tan^{-1}\left(r-\frac{1}{2}\right)$$

Now it can be solved.

13. Let $y = \frac{2(x^2+1)+1}{(x^2+1)} = 2 + \frac{1}{(x^2+1)}$

$\Rightarrow\ 2 < y \leq 3$

Now $\sin^{-1}\sin y \leq \pi - \frac{5}{2}$

$\Rightarrow\ \pi - y \leq \pi - \frac{5}{2}$ $\qquad \Rightarrow\ y \geq \frac{5}{2}$

$\Rightarrow\ \frac{2x^2+3}{x^2+1} \geq \frac{5}{2}$

Now it can be solved

Part # II : Assertion & Reason

2. $x(x-2)(3x-7) = 2$

$\Rightarrow\ 3x^3 - 13x^2 + 14x - 2 = 0$

$s_1 = r + s + t = \frac{13}{3}$;

$s_2 = \frac{14}{3}$, $s_3 = \frac{2}{3}$

$$\tan^{-1}r + \tan^{-1}s + \tan^{-1}t = \pi + \tan^{-1}\left[\frac{s_1 - s_3}{1 - s_2}\right]$$

$$= \pi + \tan^{-1}[-1] = \frac{3\pi}{4}$$

Hence statement-I and statement-II both are true.

8. Using properties

$\therefore\ \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

$\Rightarrow\ \frac{a}{x} = \frac{x}{b}$ $\qquad \Rightarrow\ x = \sqrt{ab}$

Statement-1 is true

$$\tan^{-1}\left(\frac{m}{n}\right) + \tan^{-1}\left(\frac{1-\frac{m}{n}}{1+\frac{m}{n}}\right)$$

$$= \tan^{-1}\frac{m}{n} + \tan^{-1}1 - \tan^{-1}\frac{m}{n} = \frac{\pi}{4}$$

EXERCISE - 3

Part # I : Matrix Match Type

4. (A)

Let $x = \sqrt{\frac{a\,(a+b+c)}{bc}}$, $y = \sqrt{\frac{b\,(a+b+c)}{ac}}$,

$z =$, x, y, z > 0

$\Rightarrow$ $\theta = \tan^{-1}x + \tan^{-1}y + \tan^{-1}z$

Now $x + y + z =$

$$\sqrt{\frac{a\,(a+b+c)}{bc}} + \sqrt{\frac{b\,(a+b+c)}{ac}} + \sqrt{\frac{c\,(a+b+c)}{ab}}$$

$$= \frac{(a+b+c)^{3/2}}{\sqrt{abc}}$$

and $xyz = \frac{(a+b+c)^{3/2}}{\sqrt{abc}}$

$\Rightarrow$ $x + y + z = xyz$ $\Rightarrow$ $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$

Hence $\theta = \pi$

(B) Let $\alpha = \tan^{-1}(\cot A)$

$\beta = \tan^{-1}(\cot^3 A)$

$$\tan(\alpha+\beta) = \frac{\cot A + \cot^3 A}{1-\cot^4 A}$$

R.H.S. is negative $\Rightarrow$ $\pi < \alpha + \beta < \frac{\pi}{2}$

$$\tan(\alpha+\beta-\pi) = \frac{\cot A}{1-\cot^2 A} = -\frac{\tan 2A}{2}$$

$$\Rightarrow \alpha + \beta = \pi - \tan^{-1}\left(\frac{\tan 2A}{2}\right)$$

G.E. $= \pi$ independent of A.

(C) $x = \tan\theta$ $\theta < -\frac{\pi}{4}$ or $\theta > \frac{\pi}{4}$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}(\sin 2\theta) - \pi < 2\theta < -\frac{\pi}{2}$$

or $\frac{\pi}{2} < 2\theta$

$$= \begin{cases} -\pi - 2\theta \;;\; \theta < -\frac{\pi}{4} \\ \pi - 2\theta \;;\; \theta > \frac{\pi}{4} \end{cases}$$

$$= \begin{cases} -\pi - 2\tan^{-1}x \;;\; x < -1 \\ \pi - 2\tan^{-1}x \;;\; x > 1 \end{cases}$$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) + 2\tan^{-1}x = -\pi$$

(D) $\sin^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{12}{13}\right) + \cos^{-1}\left(\frac{16}{65}\right)$

$$= \sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left\{\frac{3}{5}\sqrt{1-\left(\frac{5}{13}\right)^2} - \frac{5}{13}\sqrt{1-\left(\frac{3}{5}\right)^2}\right\} + \cos^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left\{\frac{3}{5}.\frac{12}{13} - \frac{5}{13}.\frac{4}{5}\right\} + \cos^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left(\frac{16}{65}\right) + \cos^{-1}\left(\frac{16}{65}\right) = \frac{\pi}{2}$$

Part # II : Comprehension

Comprehension–1

1. (i) $\sin\left(\frac{\cos^{-1}x}{y}\right) = 1$

$\Rightarrow$ $\frac{\cos^{-1}x}{y} = 2n\pi + \frac{\pi}{2}$ & $y \neq 0$

$$\Rightarrow \cos^{-1}x = (4n+1)\frac{\pi}{2}y$$

when $n = 0 \quad \Rightarrow \quad \cos^{-1} x = \frac{\pi}{2} y$

when $y = 1, x = 0 \quad \{0 < \frac{\pi}{2} y \le \pi \; y = 2, x = -1 \Rightarrow 0 < y \le 2\}$

when

$n = 1$ or $> 1 \quad \cos^{-1} x = \frac{5\pi}{2}$ y or more(reject)

$n = -1$ or $< -1 \quad \cos^{-1}x = \frac{-3\pi}{2}$ y or more(reject)

(ii) $\cos\left(\frac{\sin^{-1} x}{y}\right) = 0$

$\Rightarrow \frac{\sin^{-1} x}{y} = (2n+1)\frac{\pi}{2} \quad \& \quad y \neq 0$

$n = 0 \qquad \sin^{-1} x = \frac{\pi}{2} y$

$\{\frac{-\pi}{2} \le \frac{\pi}{2} y \le \frac{\pi}{2} \Rightarrow -1 \le y \le 1\}$

When $\quad y = 1, x = 1 \quad \Rightarrow \quad y = -1, x = -1$

$n = -1 \quad \sin^{-1} x = -\frac{\pi}{2} y$

When $\quad y = 1, x = -1 \quad \Rightarrow \quad y = -1, x = 1$

Other values of n & y are out of range.

1. (0, 1) & (– 1, 2)
2. (1, 1), (1, –1), (–1, 1), (–1, –1)
3. one one onto

EXERCISE - 4

Subjective Type

1. (i) Let $\tan^{-1} x = \theta \quad \Rightarrow \quad \tan\theta = x \cot\theta = \frac{1}{x} \quad \forall \; x > 0$

$\theta = -\pi + \cot^{-1}\frac{1}{x} \quad \forall \; x < 0$

$\sin\theta = \frac{x}{\sqrt{1+x^2}} \quad \Rightarrow \quad \theta = \sin^{-1}\frac{x}{\sqrt{1+x^2}}$

$\cos\theta = \frac{1}{\sqrt{1+x^2}} \qquad x > 0$

$\theta = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \tan^{-1} x x > 0$

$\Rightarrow \tan^{-1} x = -\pi + \cot^{-1}\frac{1}{x}$

$= \sin^{-1}\frac{x}{\sqrt{1+x^2}} = -\cos^{-1}\frac{x}{\sqrt{1+x^2}}$

where $\quad x < 0$

(ii) Let

$\theta = \cos^{-1} x \quad$ given $\quad -1 < x < 0$

$\Rightarrow \cos\theta = x \theta \in (\frac{\pi}{2}, \pi)$

$\sec\theta = \frac{1}{x} \qquad \theta = \sec^{-1}\frac{1}{x}$

$\sin\theta = \sqrt{1-x^2} \qquad \Rightarrow \quad \theta = \pi - \sin^{-1}\sqrt{1-x^2}$

$\tan\theta = \frac{\sqrt{1-x^2}}{x} \qquad \Rightarrow \quad \theta = \pi + \tan^{-1}\frac{\sqrt{1-x^2}}{x}$

$\cot\theta = \frac{x}{\sqrt{1-x^2}} \qquad \Rightarrow \quad \theta = \cot^{-1}\frac{x}{\sqrt{1-x^2}}$

5. **(B)** $\sin(\sin^{-1}(\log_{1/2}x))+2|\cos(\sin^{-1}(x/2-1))|=0$

$$-1 \le \log_{1/2} x \le 1 \Rightarrow \frac{1}{2} \le x \le 2 \qquad \text{...... (i)}$$

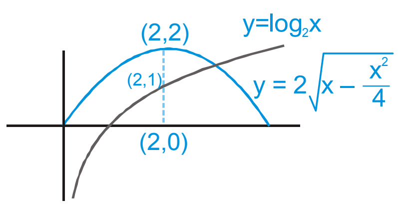
$$\text{and } -1 \le \frac{x}{2} - 1 \le 1$$

$$\Rightarrow \quad 0 \le \frac{x}{2} \le 2 \Rightarrow 0 \le x \le 4 \qquad \text{...... (ii)}$$

From (i) & (ii), $\frac{1}{2} \le x \le 2$

Also $\quad \log_{1/2} x + 2\left|\sqrt{x-\frac{x^2}{4}}\right| = 0$

$$2\sqrt{x-\frac{x^2}{4}} = \log_2 x \qquad \text{... (1)}$$



From graph it is clear that equation (1) does not have any solution in $\left[\frac{1}{2}, 2\right]$

8. $\left.\begin{array}{l} 0 \le (\tan^{-1} x)^2 \le \frac{\pi^2}{4} \\ 0 \le (\cos^{-1} y)^2 \le \pi^2 \end{array}\right] \Rightarrow (\tan^{-1} x)^2 + (\cos^{-1} x)^2 \le \frac{5\pi^2}{4}$

But $\quad (\tan^{-1} x)^2 + (\cos^{-1} x)^2 = \pi^2 k$

Hence $k\pi^2 \le \frac{5\pi^2}{4}, \quad k \le \frac{5}{4} \qquad$ ...... (i)

Now put $\quad \tan^{-1} x = \frac{\pi}{2} - \cos^{-1} y$

$$\left(\frac{\pi}{2} - \cos^{-1} y\right)^2 + (\cos^{-1} y)^2 = \pi^2 k$$

(where $\cos^{-1} y = t$)

$$2t^2 - \pi t + \left(\frac{\pi^2}{4} - k\pi^2\right) = 0$$

For real roots, $D \ge 0$

$$\pi^2 - 8\left(\frac{\pi^2}{4} - k\pi^2\right) \ge 0$$

$$\Rightarrow \quad 1 - 2 + 8k \ge 0, \; k \ge \frac{1}{8} \qquad \text{.......(ii)}$$

From (i) and (ii), $k = 1$

With

$$k = 1, \; t = \frac{\pi \pm \sqrt{8\pi^2 - \pi^2}}{4} = \frac{\pi + \sqrt{7}\pi}{4} = (1 \pm \sqrt{7})\,\frac{\pi}{4}.$$

or $\quad \cos^{-1} y = (\sqrt{7} + 1)\,\frac{\pi}{4}$ (as $0 \le \cos^{-1} y \le \pi$)

$$\therefore \quad y = \cos\,(\sqrt{7} + 1)\,\frac{\pi}{4}$$

$$\therefore \quad \tan^{-1} x = \frac{\pi}{2} - (\sqrt{7} + 1)\,\frac{\pi}{4} = \frac{\pi}{4}\,[(1 - \sqrt{7})]$$

$$\Rightarrow \quad x = \tan\,(1 - \sqrt{7})\,\frac{\pi}{4}.$$

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

1. Now, $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right)$

$= \tan^{-1}\left(\frac{17}{34}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

2. Given that $\cot^{-1}(\sqrt{\cos\alpha}) - \tan^{-1}(\sqrt{\cos\alpha}) = x$ ...... (i)

We know that, $\cot^{-1}(\sqrt{\cos\alpha}) + \tan^{-1}(\sqrt{\cos\alpha}) = \frac{\pi}{2}$ ...... (ii)

On adding equations (i) and (ii),

We get $2\cot^{-1}(\sqrt{\cos\alpha}) = \frac{\pi}{2} + x$

$\Rightarrow \sqrt{\cos\alpha} = \cot\left(\frac{\pi}{4}+\frac{x}{2}\right) \Rightarrow \sqrt{\cos\alpha}$

$= \frac{\cot\frac{x}{2}-1}{1+\cot\frac{x}{2}} \Rightarrow \sqrt{\cos\alpha} = \frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}$

$\Rightarrow \cos\alpha = \frac{1-\sin x}{1+\sin x}$

$\Rightarrow \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{1-\sin x}{1+\sin x}$

Applying componendo and dividendo rule,

We get $\sin x = \tan^2\left(\frac{\alpha}{2}\right)$

3. Given that, $\sin^{-1}x = 2\sin^{-1}\alpha$

Since, $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2} \le 2\sin^{-1}\alpha \le \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{4} \le \sin^{-1}\alpha \le \frac{\pi}{4}$

$\Rightarrow \sin\left(-\frac{\pi}{4}\right) \le \alpha \le \sin\left(\frac{\pi}{4}\right)$

$\Rightarrow -\frac{1}{\sqrt{2}} \le \alpha \le \frac{1}{\sqrt{2}} \Rightarrow |\alpha| \le \frac{1}{\sqrt{2}}$

4. Given that, $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$

$\Rightarrow \cos^{-1}\left(\frac{xy}{2}+\sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}\right) = \alpha$

$\Rightarrow \frac{xy}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}} = \cos\alpha$

$\Rightarrow 2\sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}} = 2\cos\alpha - xy$

On squaring both sides, we get

$\frac{4(1-x^2)(4-y^2)}{4} = 4\cos^2\alpha + x^2y^2 - 4xy\cos\alpha$

$\Rightarrow 4 - 4x^2 - y^2 + x^2y^2 = 4\cos^2\alpha + x^2y^2 - 4xy\cos\alpha$

$\Rightarrow 4x^2 - 4xy\cos\alpha + y^2 = 4\sin^2\alpha$

5. Since, $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{4}{5}\right)$

$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right) = \cos^{-1}\left(\frac{4}{5}\right) \Rightarrow \sin^{-1}\left(\frac{x}{5}\right) = \sin^{-1}\left(\frac{3}{5}\right)$

$\Rightarrow x = 3$

6. Since, $\operatorname{cosec}^{-1}\left(\frac{5}{3}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

$\cos\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right) = \cot\tan^{-1}\left[\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{1}{2}}\right]$

$= \cot\tan^{-1}\left[\frac{\left(\frac{17}{12}\right)}{\left(\frac{1}{2}\right)}\right] = \cos\left[\tan^{-1}\left(\frac{17}{6}\right)\right] = \frac{6}{17}$

7. $2y = x + z$

$2\tan^{-1} y = \tan^{-1} x + \tan^{-1}(z)$

$\Rightarrow \tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$

$\Rightarrow \frac{x+z}{1-y^2} = \frac{x+z}{1-xz}$

$\Rightarrow y^2 = xz$ or $x + z = 0$

$\Rightarrow x = y = z$

Part # II : IIT-JEE ADVANCED

1. $\tan^{-1}\sqrt{x(x+1)}$ is defined when $x^2 + x \geq 0$

$\sin^{-1}\sqrt{x^2 + x + 1}$ is defined when $0 \leq x^2 + x + 1 \leq 1$

Hence both will be defined when $x^2 + x = 0$

$\Rightarrow x = 0, -1$

2. $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} ....\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} + ...\right) = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}\left(\frac{x}{1+(x/2)}\right) + \cos^{-1}\left(\frac{x^2}{1+(x^2/2)}\right) = \frac{\pi}{2}$

$\frac{2x}{2+x} = \frac{2x^2}{2+x^2}$

$2x + x^3 = 2x^2 + x^3$

$x = 0, 1$ But $\therefore$ $|x| > 0$

so $x = 1$ is the only answer.

3. **Case-I :** $x \geq 0$

Let $\cot^{-1} x = \theta$ $\therefore$ $\theta \in \left(0, \frac{\pi}{2}\right]$

$\Rightarrow x = \cot\theta$

$\therefore \sin\theta = \frac{1}{\sqrt{1+x^2}}$

$\Rightarrow \sin^{-1}\sin\theta = \sin^{-1}\frac{1}{\sqrt{1+x^2}}$

$\Rightarrow \theta = \sin^{-1}\frac{1}{\sqrt{1+x^2}}$

Case-II : $x < 0$

Let $\cot^{-1} x = \theta$ $\therefore$ $\theta \in \left(\frac{\pi}{2}, \pi\right)$

$\Rightarrow \cot\theta = x$

$\therefore \sin\theta = \frac{1}{\sqrt{1+x^2}}$

$\Rightarrow \sin^{-1}\sin\theta = \sin^{-1}\frac{1}{\sqrt{1+x^2}}$

$\Rightarrow \pi - \theta = \sin^{-1}\frac{1}{\sqrt{1+x^2}} \Rightarrow \theta = \pi - \sin^{-1}\frac{1}{\sqrt{1+x^2}}$

Therefore,

$$\text{LHS} = \begin{cases} \cos\tan^{-1}\sin\sin^{-1}\frac{1}{\sqrt{1+x^2}} & \text{, if } x \geq 0 \\ \cos\tan^{-1}\sin\left(\pi - \sin^{-1}\frac{1}{\sqrt{1+x^2}}\right) & \text{, if } x < 0 \end{cases}$$

$= \cos\tan^{-1}\sin\sin^{-1}\frac{1}{\sqrt{1+x^2}}$; $x \in R = \cos\tan^{-1}\frac{1}{\sqrt{1+x^2}}$

Let $\phi = \tan^{-1}\frac{1}{\sqrt{1+x^2}}$

As $\frac{1}{\sqrt{1+x^2}} \in (0, 1]$ $\therefore$ $\phi \in \left(0, \frac{\pi}{4}\right]$

$\therefore \tan\phi = \frac{1}{\sqrt{1+x^2}}$ $\therefore \cos\phi = \sqrt{\frac{1+x^2}{2+x^2}}$

$\therefore$ LHS $= \cos\cos^{-1}\sqrt{\frac{1+x^2}{2+x^2}} = \sqrt{\frac{1+x^2}{2+x^2}} =$ RHS.

4. $\sin\cot^{-1}(1+x) = \cos(\tan^{-1} x)$

If $\alpha = \cot^{-1}(1+x)$ and $\beta = \tan^{-1} x$

Then $\frac{1}{\sqrt{x^2 + 2x + 2}} = \frac{1}{\sqrt{1+x^2}}$

$\Rightarrow x = -1/2$

5. $\sin^{-1}(ax) + \cos^{-1}y + \cos^{-1}(bxy) = \frac{\pi}{2}$

(A) $a = 1$, $b = 0$

$\Rightarrow \sin^{-1}(x) + \cos^{-1}(y) + \cos^{-1}(0) = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}x + \cos^{-1}y = 0$

$\Rightarrow \cos^{-1}y = -\sin^{-1}x$

$\Rightarrow \cos^{-1} y = \cos^{-1} \sqrt{1-x^2}$

$\Rightarrow x^2 + y^2 = 1$

(B) $\sin^{-1}(x) + \cos^{-1}y + \cos^{-1}(xy) = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}(y) + \cos^{-1}(xy) = \cos^{-1}x.$

$\Rightarrow \cos^{-1}\left(xy^2 - \sqrt{(1-y^2)(1-x^2y^2)}\right) = \cos^{-1}x.$

$\Rightarrow xy^2 - \sqrt{(1-y^2)(1-x^2y^2)} = x$

$\Rightarrow 1 - x^2 - y^2 + x^2y^2 = 0$

$\Rightarrow (1-x^2)(1-y^2) = 0$

(C) $\sin^{-1}(x) + \cos^{-1} y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}\left(2xy^2 - \sqrt{(1-y^2)(1-4x^2y^2)}\right) = \cos^{-1}x.$

$\Rightarrow 2xy^2 - \sqrt{(1-y^2)(1-4x^2y^2)} = x$

$\Rightarrow 2xy^2 - x = \sqrt{(1-y^2)(1-4x^2y^2)}$

$\Rightarrow 4x^2y^4 + x^2 - 4x^2y^2 = 1 - y^2 - 4x^2y^2 + 4x^2y^4$

$\Rightarrow x^2 + y^2 = 1.$

(D) $\sin^{-1}(2x) + \cos^{-1}y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}\left(2y^2x - \sqrt{(1-y^2)(1-4x^2y^2)}\right) = \cos^{-1}(2x)$

$\Rightarrow 2y^2x - \sqrt{1-y^2-4x^2y^2+4x^2y^4} = 2x.$

$\Rightarrow 1 - 4x^2 - y^2 + 4x^2y^2 = 0$

$\Rightarrow (1-4x^2)(1-y^2) = 0.$

6. $\sqrt{1+x^2}$

$$\left[\left\{x \cos\cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) + \sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right\}^2 - 1\right]^{1/2}$$

$$= \sqrt{1+x^2}\left[\left(\frac{x^2}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}}\right)^2 - 1\right]^{\frac{1}{2}}$$

$= \sqrt{1+x^2}\,.x$ Hence (C) is correct.

7. $\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right) = \sin^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$

$\therefore f(\theta) = \tan\theta$

$\therefore \frac{df}{d\tan\theta} = 1$

MOCK TEST

1. **(D)** Since $\cos^{-1}\left(\frac{4}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

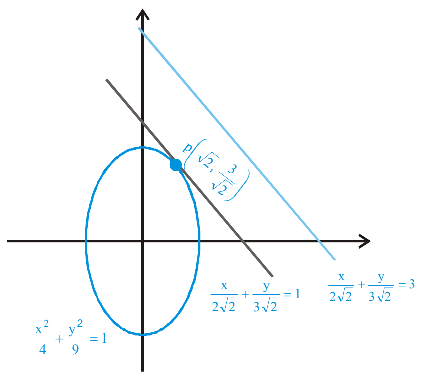
$$\therefore \quad \tan\left[\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right]$$

$$= \tan\left[\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right] = \frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}} = \frac{17}{6}$$

2. $-1 \le \frac{x^2}{4} + \frac{y^2}{9} \le 1$ represents interior and the boundary

of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ ....**(i)**

Also $\quad -1 \le \frac{x}{2\sqrt{2}} + \frac{y}{3\sqrt{2}} - 2 \le 1$



i.e. $\frac{x}{2\sqrt{2}} + \frac{y}{3\sqrt{2}} \ge 1$ and $\frac{x}{2\sqrt{2}} + \frac{y}{3\sqrt{2}} \le 3$

$\frac{x}{2\sqrt{2}} + \frac{y}{3\sqrt{2}} \ge 1$ represents the portion of xy plane which contains only one point viz :

$\left(\sqrt{2}, \frac{3}{\sqrt{2}}\right)$ of $\frac{x^2}{4} + \frac{y^2}{9} \le 1$

$$\therefore \quad \sin^{-1}\left(\frac{x^2}{4} + \frac{y^2}{9}\right) + \cos^{-1}\left(\frac{x}{2\sqrt{2}} + \frac{y}{3\sqrt{2}} - 2\right)$$

$$= \sin^{-1}\left(\frac{1}{2} + \frac{1}{2}\right) + \cos^{-1}\left(\frac{1}{2} + \frac{1}{2} - 2\right)$$

3. **(B)**

$\cos^{-1}(2x^2 - 1) = 2\pi - 2\cos^{-1}x \qquad$ (as $x < 0$)

$\cos^{-1}(2x^2 - 1) - 2\sin^{-1}x = 2\pi - 2\cos^{-1}x - 2\sin^{-1}x$

$= 2\pi - 2(\cos^{-1}x + \sin^{-1}x)$

$= 2\pi - 2\,\frac{\pi}{2} = \pi$

4. **(C)**

$\sin^{-1}(x - 1) \Rightarrow -1 \le x - 1 \le 1 \quad \Rightarrow 0 \le x \le 2$

$\cos^{-1}(x - 3) \Rightarrow -1 \le x - 3 \le 1 \quad \Rightarrow 2 \le x \le 4$

$\tan^{-1}\left(\frac{x}{2-x^2}\right) \Rightarrow x \in R,\ x \ne \sqrt{2}, -\sqrt{2}$

$\therefore \quad x = 2$

$\sin^{-1}(2 - 1) + \cos^{-1}(2 - 3) + \tan^{-1}\frac{2}{2-4} = \cos^{-1}k + \pi$

$\Rightarrow \sin^{-1}1 + \cos^{-1}(-1) + \tan^{-1}(-1) = \cos^{-1}k + \pi$

$\frac{\pi}{2} + \pi - \frac{\pi}{4} = \cos^{-1}k + \pi$

$\Rightarrow \cos^{-1}k = \frac{\pi}{4} \qquad \Rightarrow k = \frac{1}{\sqrt{2}}$

5. **(A)**

Let $\sin^{-1}a = A$,

$\sin^{-1}b = B$

$\sin^{-1}c = C$

$\therefore$ $\sin A = a$, $\sin B = b$, $\sin C = c$

and $A + B + C = \pi$, then

$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

Now $\quad a\sqrt{(1-a^2)} + b\sqrt{(1-c^2)} + c\sqrt{(1-c^2)}$

$= \sin A \cos A + \sin B \cos B + \sin C \cos C$

$= \frac{1}{2}[\sin 2A + \sin 2B + \sin 2C] = 2 \sin A \sin B \sin C = 2abc$

6. **(C)**

$0 \le \{x\} < 1 \qquad$ i.e. $-1 < -\{x\} \le 0$

$\therefore \quad \frac{\pi}{2} \le \cos^{-1}(-\{x\}) < \pi$

$\therefore$ the range is $\left[\frac{\pi}{2}, \pi\right)$

7. **(B)**

We have $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

$\Rightarrow \tan(2\tan^{-1}\cos x) = 2\operatorname{cosec} x$

$\Rightarrow \frac{2\cos x}{1-\cos^2 x} = 2\operatorname{cosec} x \qquad \Rightarrow \frac{2\cos x}{\sin^2 x} = 2\operatorname{cosec} x$

$\Rightarrow \sin x = \cos x \qquad \Rightarrow x = \frac{\pi}{4}$.

8. (C)

$\log_{1/2} \sin^{-1} x > \log_{1/2} \cos^{-1} x$

$\Leftrightarrow \cos^{-1} x > \sin^{-1} x, \quad 0 < x < 1$

$\Leftrightarrow \cos^{-1} x > \frac{\pi}{2} - \cos^{-1} x, \quad 0 < x < 1$

$\Leftrightarrow \cos^{-1} x > \frac{\pi}{4}, 0 < x < 1$

$\Leftrightarrow 0 < x < \frac{1}{\sqrt{2}}$

9. (A)

$S_1 : \sin^{-1}x - \frac{\pi}{2} + \sin^{-1}(-x) = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}(x) + \sin^{-1}(-x) = \pi$

$0 = \pi$ which is not possible

$\therefore$ no solution

$S_2 : \sin^{-1}(x^2 + 4x + 3) + \cos^{-1}(x^2 + 6x + 8) = \frac{\pi}{2}$

$= \sin^{-1}(x^2 + 4x + 3) + \cos^{-1}(x^2 + 4x + 3)$

$\Rightarrow x^2 + 6x + 8 = x^2 + 4x + 3$

$\Rightarrow 2x = -5 \quad \Rightarrow \quad x = -\frac{5}{2}$

$\because \; x^2 + 4x + 3 = (x+2)^2 - 1 \in [-1, 1]$ at $x = -\frac{5}{2}$

& $x^2 + 6x + 8 = (x+3)^2 - 1 \in [-1, 1]$ at $x = -\frac{5}{2}$

$\therefore \; x = -\frac{5}{2}$

$S_3 : \sin^{-1}\{\cos(\sin^{-1}x)\} + \cos^{-1}\{\sin(\cos^{-1}x)\} = \frac{\pi}{2}$

{As $\cos(\sin^{-1}x) = \sin(\cos^{-1}x) = \sqrt{1-x^2}$ }

$S_4 : 2\left[\tan^{-1}\frac{1+2}{1-2} + \pi + \tan^{-1}3\right]$

$= 2[\pi - \tan^{-1}3 + \tan^{-1}3] = 2\pi$

10. (C)

$\sin^{-1}\sin 5 = \sin^{-1}\sin(5 - 2\pi) = 5 - 2\pi$

$\left(\text{As } -\frac{\pi}{2} \le 5 - 2\pi \le \frac{\pi}{2}\right)$

$\therefore \; \sin^{-1}\sin 5 > x^2 - 4x$

$\Rightarrow 5 - 2\pi > x^2 - 4x \quad \Rightarrow \quad x^2 - 4x + 2\pi - 5 < 0$

sign sum of $(x^2 - 4x + 2\pi - 5)$

$2 - \sqrt{9 - 2\pi} < x < 2 + \sqrt{9 - 2\pi}$

Integral values of x are 1, 2, 3

Number of integral value of x = 3

11. Let $\tan^{-1}x = \theta$.

Then $-\frac{\pi}{2} < \theta < \frac{\pi}{2}, \; \theta \ne \pm\frac{\pi}{4}$ and $x = \tan\theta$

$\tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2} = \theta + \tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta}$

$= \theta + \tan^{-1}(\tan 2\theta)$, where $-\pi < 2\theta < \pi, \; 2\theta \ne \pm\frac{\pi}{2}$

$$= \begin{cases} \theta + \pi + 2\theta & \text{when } -\pi < 2\theta < -\frac{\pi}{2} \\ \theta + 2\theta & \text{when } -\frac{\pi}{2} < 2\theta < \frac{\pi}{2} \\ \theta - \pi + 2\theta & \text{when } \frac{\pi}{2} < 2\theta < \pi \end{cases}$$

$$= \begin{cases} \pi + 3\theta & \text{when } -\frac{\pi}{2} < \theta < -\frac{\pi}{4} \\ 3\theta & \text{when } -\frac{\pi}{4} < \theta < \frac{\pi}{4} \\ -\pi + 3\theta & \text{when } \frac{\pi}{4} < \theta < \frac{\pi}{2} \end{cases}$$

$$= \begin{cases} \pi + 3\tan^{-1}x & \text{when } x < -1 \\ 3\tan^{-1}x & \text{when } -1 < x < 1 \\ -\pi + 3\tan^{-1}x & \text{when } 1 < x \end{cases}$$

12. (A, C)

Domain of $f(x) = \ell n \cos^{-1} x$

is $x \in [-1, 1)$

$\therefore \; [\alpha] = -1$ or 0

13. If $-1 \le x < 0$, then $-\frac{\pi}{2} \le \sin^{-1}x < 0$

Also $0 < 2\cot^{-1}(y^2 - 2y) < 2\pi$

$\therefore\quad -\frac{\pi}{2} < \sin^{-1}x + 2\cot^{-1}(y^2-2y) < 2\pi$

$\therefore$ there is no solution in this case.

thus x can not be negative ....(i)

Now if $x \geq 0$, then $0 \leq \sin^{-1}x \leq \frac{\pi}{2}$

$\Rightarrow \frac{3\pi}{4} \leq \cot^{-1}(y^2-2y) < \pi$

$\Rightarrow y^2 - 2y \leq -1 \quad \Rightarrow y = 1$

since for $y = 1$, we have $2\cot^{-1}(y^2-2y) = 2\cot^{-1}(-1) = \frac{3\pi}{2}$

$\therefore \sin^{-1}x = \frac{\pi}{2}$ i.e. $x = 1$

$\therefore$ the solution is $x = 1, y = 1$

14. (A, B, C)

(A) $\sin\left(\tan^{-1}3 + \tan^{-1}\frac{1}{3}\right) = \sin\frac{\pi}{2} = 1$

(B) $\cos\left(\frac{\pi}{2} - \sin^{-1}\frac{3}{4}\right) = \cos\left(\cos^{-1}\frac{3}{4}\right) = \frac{3}{4}$

(C) $\sin\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)$

Let $\sin^{-1}\frac{\sqrt{63}}{8} = \theta$

so $\sin\theta = \frac{\sqrt{63}}{8}$ if $\cos\theta = \frac{1}{8}$

we have $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \frac{3}{4}$

$\sin\frac{\theta}{4} = \sqrt{\frac{1-\cos\frac{\theta}{2}}{2}} = \frac{1}{2\sqrt{2}}$

Now $\log_2 \sin\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right) = \log_2\frac{1}{2\sqrt{2}} = -\frac{3}{2}$

(D) $\cos^{-1}\frac{\sqrt{5}}{3} = \theta$

$\cos\theta = \frac{\sqrt{5}}{3}$

$\therefore \tan\frac{\theta}{2} = \frac{3-\sqrt{5}}{2}$ which is irrational

15. $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$

$\Rightarrow \frac{\pi}{2} - \cos^{-1}x + \frac{\pi}{2} - \cos^{-1}(1-x)$

$= \cos^{-1}x$

$\Rightarrow 2\cos^{-1}x = \pi - \cos^{-1}(1-x)$

$\Rightarrow \cos^{-1}(2x^2-1) = \cos^{-1}(x-1) \quad \Rightarrow 2x^2 - 1 = x - 1$

$\Rightarrow x(2x-1) = 0 \quad \Rightarrow x = 0, \frac{1}{2}$

17. Range of f is $\left\{\frac{\pi}{2}\right\}$ and domain of f is $\{0\}$.

Hence if domain of f is singleton then range has to be a singleton.

If S–2 and S–1 are reverse then the answer will be B.

18. (A)

(Moderate)

$\sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}} > \tan^{-1}x > \tan^{-1}y$

$\left\{\because x > y, \frac{x}{\sqrt{1-x^2}} > x\right\}$

Statement-II is true

$e < \pi$

$\frac{1}{\sqrt{e}} > \frac{1}{\sqrt{\pi}}$

by **Statement-II**

$\sin^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{\pi}}\right)$

Statement-I is true

19. (A)

$$f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\pi-2\tan^{-1}x,\ x\ge 1$$

$$f'(x)=-\frac{2}{1+x^2}\quad\Rightarrow\quad f'(2)=-\frac{2}{5}$$

Statement-I is True, Statement-II is True ; Statement-II is a correct explanation for statement-I.

20. (A)

$\operatorname{cosec}^{-1}x>\sec^{-1}x$

$\operatorname{cosec}^{-1}x>\frac{\pi}{2}-\operatorname{cosec}^{-1}x$

$\operatorname{cosec}^{-1}x>\frac{\pi}{4}$

$1\le x<\sqrt{2}$ and $\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right)\in[1,\sqrt{2})$

Statement 2 is true and explains statement 1

21. (A)→(r), (B)→(s), (C)→(p), (D)→(q)

(A) The difference $=2-(-2)=4$

(B) Let $f(x)=x^2-4x+3$

$f'(x)=2x-4=0\ \Rightarrow\ x=2$

$f(1)=0,\ f(2)=-1,\ f(3)=0$

$\therefore$ |greatest value – least value| = 1

(C) $\tan^{-1}\frac{1-x}{1+x}=\tan^{-1}1-\tan^{-1}x$

$\therefore$ greatest value $=\frac{\pi}{4}$

(D) $\therefore$ greatest value $=\frac{\pi}{2}$, least value $=\frac{\pi}{3}$

$\therefore$ difference $=\frac{\pi}{6}$

22. (A)→(q, s), (B)→(r, s, t), (C)→(r, s), (D)→(p, q)

(A) Given $\sin^{-1}x-\cos^{-1}x=\frac{\pi}{6}$

Also, $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$

Solving $x=\frac{\sqrt{3}}{2}$

(B) $\sec^2(\tan^{-1}2)+\operatorname{cosec}^2(\cot^{-1}3)$

$=1+(\tan(\tan^{-1}2))^2+1+(\cot(\cot^{-1}3))^2$

$=15$

(C) Given eqn. is $\frac{\pi}{2}-2\cos^{-1}x=\sin^{-1}(3x-2)$

or $3x-2=\cos(2\cos^{-1}x)=2\cos^2(\cos^{-1}x)-1$

$=2x^2-1$

$\Rightarrow\ 2x^2-3x+1=0\ \Rightarrow\ x=1$ or $\frac{1}{2}$

(D) $\sin 5=\sin(5-2\pi)$

$\Rightarrow\ \sin^{-1}(\sin 5)=\sin^{-1}(\sin(5-2\pi))$

$=5-2\pi$

24.

1. (B)

$$\sin^{-1}\left(\frac{4x}{x^2+4}\right)+2\tan^{-1}\left(-\frac{x}{2}\right)$$

$$=\sin^{-1}\left(\frac{2.\frac{x}{2}}{\left(\frac{x}{2}\right)^2+1}\right)-2\tan^{-1}\frac{x}{2}$$

$$=2\tan^{-1}\frac{x}{2}-2\tan^{-1}\frac{x}{2}=0$$

Here $\left|\frac{x}{2}\right|\le 1$

$|x|\le 2\ \Rightarrow\ -2\le x\le 2$

2. (A)

$$\cos^{-1}\frac{6x}{1+9x^2}=-\frac{\pi}{2}+2\tan^{-1}3x$$

$$\Rightarrow\ \frac{\pi}{2}-\sin^{-1}\frac{6x}{1+9x^2}=-\frac{\pi}{2}+2\tan^{-1}3x$$

$$\Rightarrow\ \sin^{-1}\frac{6x}{1+9x^2}=\pi-2\tan^{-1}3x$$

$$\Rightarrow\ \sin^{-1}\frac{2.\,3x}{1+(3x)^2}=\pi-2\tan^{-1}3x$$

Above is true when $3x>1$

$\Rightarrow \quad x > \frac{1}{3}$

$x \in \left(\frac{1}{3}, \infty\right)$

3. (C)

$(x-1)(x^2+1) > 0$

$\Rightarrow \quad x > 1$

$\therefore \quad \sin\left[\frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2}\right) - \tan^{-1}x\right]$

$= \sin\left[\frac{1}{2}(-\pi + 2\tan^{-1}x) - \tan^{-1}x\right] = \sin\left(-\frac{\pi}{2}\right) = -1$

25.

1. (B)

$A = (\tan^{-1}x)^3 + (\cot^{-1}x)^3$

$A = (\tan^{-1}x + \cot^{-1}x)^3 - 3\tan^{-1}x\cot^{-1}x(\tan^{-1}x + \cot^{-1}x)$

$\Rightarrow \quad A = \left(\frac{\pi}{2}\right)^3 - 3\tan^{-1}x\cot^{-1}x \cdot \frac{\pi}{2}$

$\Rightarrow \quad A = \frac{\pi^3}{8} - \frac{3\pi}{2}\tan^{-1}x\left(\frac{\pi}{2} - \tan^{-1}x\right)$

$\Rightarrow \quad A = \frac{\pi^3}{32} + \frac{3\pi}{2}\left(\tan^{-1}x - \frac{\pi}{4}\right)^2$

as $\quad x > 0$

$\frac{\pi^3}{32} \le A < \frac{\pi^3}{8}$

2. (C)

$B = (\sin^{-1}t)^2 + (\cos^{-1}t)^2$

$B = (\text{sizn}^{-1}t + \cos^{-1}t)^2 - 2\sin^{-1}t\cos^{-1}t$

$B = \frac{\pi^2}{4} - 2\sin^{-1}t\left(\frac{\pi}{2} - \sin^{-1}t\right)$

$B = \frac{\pi^2}{8} + 2\left(\sin^{-1}t - \frac{\pi}{4}\right)^2$

$B_{max} = \frac{\pi^2}{8} + 2.\frac{\pi^2}{16} = \frac{\pi^2}{4}$

3. (A)

$\lambda = \frac{\pi^3}{32} \qquad \mu = \frac{\pi^2}{4}$

$\frac{\lambda}{\mu} = \frac{\pi}{8}$

$\frac{\lambda - \mu\pi}{\mu} = \frac{\pi}{8} - \pi = \frac{-7\pi}{8}$

$\cot^{-1}\cot\left(\frac{\lambda - \mu\pi}{\mu}\right) = \cot^{-1}\cot\left(-\frac{7\pi}{8}\right) = \frac{\pi}{8}$

26. (1)

$(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = (\tan^{-1}x + \cot^{-1}x)^2 - 2\tan^{-1}x$

$\left(\frac{\pi}{2} - \tan^{-1}x\right) = \frac{\pi^2}{4} - \pi\tan^{-1}x + 2(\tan^{-1}x)^2 = \frac{5\pi^2}{8}$

$\therefore \quad \tan^{-1}x = \frac{2\pi}{3}, -\frac{\pi}{4}$

$\tan^{-1}x = -\frac{\pi}{4} \qquad \{\tan^{-1}x \ne \frac{2\pi}{3}\}$

$\therefore \quad x = -1$ is the solution

27. (1)

$\tan^{-1}\left[\frac{3\sin 2\alpha}{5 + 3\cos 2\alpha}\right] + \tan^{-1}\left[\frac{\tan\alpha}{4}\right]$

$= \tan^{-1}\left(\frac{6\tan\alpha}{8 + 2\tan^2\alpha}\right) + \tan^{-1}\left(\frac{\tan\alpha}{4}\right)$

$= \tan^{-1}\left(\frac{\frac{3\tan\alpha}{4+\tan^2\alpha} + \frac{\tan\alpha}{4}}{1 - \frac{3\tan^2\alpha}{16 + 4\tan^2\alpha}}\right)$

$\left\{\because \frac{3\tan^2\alpha}{16 + 4\tan^2\alpha} < 1\right\}$

$= \tan^{-1}(\tan\alpha) = \alpha$

28. $\tan\left(\frac{\pi}{4} + \alpha\right)$ when $\alpha = \tan^{-1}\left(\frac{\frac{1}{4} + \frac{1}{5}}{1 - \frac{1}{20}}\right)$;

$\alpha = \tan^{-1}\left(\frac{9}{19}\right) \qquad = \frac{1 + \frac{9}{19}}{1 - \frac{9}{19}} = \frac{28}{10} = \frac{14}{5} = \frac{a}{b}$

$\Rightarrow \quad 14 - 5 = 9$

29. $\cos^{-1}(2x) + \cos^{-1}(3x) = \pi - \cos^{-1}(x) = \cos^{-1}(-x)$

$\cos^{-1}[(2x)(3x) - \sqrt{1-4x^2}\sqrt{1-9x^2}\,] = \cos^{-1}(-x)$

$6x^2 - \sqrt{1-4x^2}\cdot\sqrt{1-9x^2} = -x$

$(6x^2 + x)^2 = (1 - 4x^2)(1 - 9x^2)$

$\Rightarrow$ $x^2 + 12x^3 = 1 - 13x^2$

$\Rightarrow$ $12x^3 + 14x^2 - 1 = 0$

$\therefore$ $a = 12;\ \ b = 14;\ c = 0$

$\Rightarrow$ $-a + b + c = 2$