Continuity and Differentiability

[TOPIC 1] Continuity

1.1 Continuity at a Point

A function $f(x)$ is said to be continuous at a point $x = a$, if

$$(\text{LHL})_{x=a} = (\text{RHL})_{x=a} = f(a) \text{ or } \lim_{x \to a} f(x) = f(a)$$

where, $(\text{LHL})_{x=a} = \lim_{x \to a^-} f(x)$

and $(\text{RHL})_{x=a} = \lim_{x \to a^+} f(x)$.

NOTE To evaluate LHL and RHL of a function $f(x)$ at $x = a$, put $x = a - h$ and $x = a + h$ respectively, where $h \to 0$.

1.2 Discontinuity of a Function

A function $f(x)$ is said to be discontinuous at $x = a$, if it is not continuous at $x = a$, i.e. when any of the following cases arise:

(*i*) $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a} f(x)$ does not exist.

(*ii*) $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

(*iii*) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$ or $\lim_{x \to a} f(x) \neq f(a)$

1.3 Continuity in an Interval

A function $y = f(x)$ is said to be continuous in an interval (a, b) iff $f(x)$ is continuous at every point in that interval and f is said to be continuous in the interval $[a, b]$ iff f is continuous in the interval (a, b) and also at the point a from the right and at the point b from the left.

Note A function is said to be continuous, if it is continuous on the whole of its domain.

Useful Results for Continuity

(*i*) Every identity function is continuous.
(*ii*) Every constant function is continuous.
(*iii*) Every polynomial function is continuous.
(*iv*) Every rational function is continuous.
(*v*) All trigonometric functions are continuous in their domain.
(*vi*) Modulus function is continuous.

Standard Results of Limits

(*i*) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(*ii*) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(*iii*) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

(*iv*) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(*v*) $\lim_{x \to \infty} \frac{1}{x^p} = 0, p \in (0, \infty)$

(*vi*) $\lim_{x \to 0} \frac{\log (1 + x)}{x} = 1$

(*vii*) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

(*viii*) $\lim_{x \to 0} (1 + x)^{1/x} = e$

(*ix*) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

(*x*) $\lim_{x \to \infty} \sin x$ and $\lim_{x \to \infty} \cos x$ oscillate between -1 to 1.

1.4 Algebra of Continuous Functions

Suppose f and g are two real functions, continuous at real number c. Then,

(*i*) $f + g$ is continuous at $x = c$.

(*ii*) $f - g$ is continuous at $x = c$.

(*iii*) $f \cdot g$ is continuous at $x = c$.

(*iv*) kf is continuous at $x = c$, where k is any constant.

(*v*) $\left(\frac{f}{g}\right)$ is continuous at $x = c$ [provided $g(c) \neq 0$].

1.5 Composition of Two Continuous Functions

Suppose f and g are two real valued functions such that (fog) is defined at c. If g is continuous at c and f is continuous at $g(c)$, then (fog) is continuous at c.

[TOPIC 2] Differentiability

2.1 Differentiability

A function $f(x)$ is said to be differentiable at a point $x = a$, if Left hand derivative at $(x = a)$ equals to Right hand derivative at $(x = a)$ i.e. LHD at $(x = a)$ = RHD (at $x = a$), where Right hand derivative, $Rf'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ and

Left hand derivative, $Lf'(a) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$

NOTE (i) The common value of $Rf'(a)$ and $Lf'(a)$ is denoted by $f'(a)$ and it is known as the derivative of $f(x)$ at $x = a$.

(ii) Every differentiable function is continuous but every continuous function need not be differentiable.

Useful Results for Differentiability

(*i*) Every polynomial, exponential and constant functions are differentiable.

(*ii*) Logarithmic function is differentiable in their domain.

(*iii*) Trigonometric and inverse trigonometric functions are differentiable in their domain.

(*iv*) Modulus function is differentiable everywhere except at that point where it is zero.

2.2 Differentiation

The process of finding derivative of a function is called differentiation.

Rules of Derivative

(*i*) **Sum and Difference Rule** Let $y = f(x) \pm g(x)$. Then, by using sum and difference rule, its derivative is written as $\frac{dy}{dx} = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$.

(*ii*) **Product Rule** Let $y = f(x)\, g(x)$. Then, by using product rule, it's derivative is written as

$$\frac{dy}{dx} = \left[\frac{d}{dx} f(x)\right] g(x) + \left[\frac{d}{dx} g(x)\right] f(x).$$

(*iii*) **Quotient Rule** Let $y = \frac{f(x)}{g(x)}$; $g(x) \neq 0$, then by using quotient rule, its derivative is written as

$$\frac{dy}{dx} = \frac{g(x) \times \frac{d}{dx}[f(x)] - f(x) \times \frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

(*iv*) **Chain Rule** Let $y = f(u)$ and $u = f(x)$, then by using chain rule, we may write $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, when $\frac{dy}{du}$ and $\frac{du}{dx}$ both exist.

Some Standard Derivatives

(*i*) $\frac{d}{dx}(\text{constant}) = 0$

(*ii*) $\frac{d}{dx}(x^n) = nx^{n-1}$

(*iii*) $\frac{d}{dx}(e^x) = e^x$

(*iv*) $\frac{d}{dx}(\log_e x) = \frac{1}{x}, x > 0$

(*v*) $\frac{d}{dx}(a^x) = a^x \log_e a, a > 0$

(*vi*) $\frac{d}{dx}(\sin x) = \cos x$

(*vii*) $\frac{d}{dx}(\cos x) = -\sin x$

(*viii*) $\frac{d}{dx}(\tan x) = \sec^2 x$

(*ix*) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(*x*) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(*xi*) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(*xii*) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$

(*xiii*) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$

(*xiv*) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$

(xv) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

(xvi) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

(xvii) $\frac{d}{dx}(\text{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

Derivative of Implicit Function

Suppose $f(x, y) = 0$ is a function of x and y, which cannot express in the form of $y = \phi(x)$. Then, such function is called implicit function. For differentiation of this type of function, we differentiate it and simplifying such that $\frac{dy}{dx}$ or $\frac{dx}{dy}$ is on left hand side and the other variable is on right hand side.

Logarithmic Differentiation

Let $\quad y = [f(x)]^{g(x)} \quad \ldots$(i)

Then, by taking log (to base e), we can write Eq. (i) as $\log y = g(x)\log f(x)$.

Now, by using chain rule

$$\frac{dy}{dx} = [f(x)]^{g(x)}\left[\frac{g(x)}{f(x)} f'(x) + g'(x)\log f(x)\right].$$

NOTE The logarithmic function $\log_a x$ ($a > 0$ and $a \neq 1$) has the following properties:

(i) $\log_a(mn) = \log_a m + \log_a n$

(ii) $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

(iii) $\log_a m^n = n\log_a m$ (iv) $\log_a m = \frac{\log m}{\log a}$

(v) $\log_a a = 1$ (vi) $\log_a 1 = 0$

Differentiation of Parametric Function

If $x = f(t)$, $y = g(t)$, where t is a parameter, then

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}, \text{ whenever } \frac{dx}{dt} \neq 0$$

NOTE dy/dx is expressed in terms of parameter only without directly involving the main variables x and y.

Differentiation of a Function with Respect to Another Function

Suppose $y = f(x)$ and $z = g(x)$ are two functions. Then, differentiation of y with respect to z is

$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx}.$$

Second Order Derivative

It is the derivative of the first order derivative,

i.e. $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

2.3 Rolle's and Mean Value Theorem

Rolle's Theorem

Let $f:[a, b] \to R$ be continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, where a and b are some real numbers. Then, there exists atleast one number c in (a, b) such that $f'(c) = 0$.

Mean Value Theorem

Let $f:[a, b] \to R$ be continuous function on $[a, b]$ and differentiable on (a, b). Then, there exists atleast one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

NOTE Mean value theorem is an expansion of Rolle's theorem.

Some Useful Substitutions for Finding Derivatives

Expression	Substitution
(i) $a^2 + x^2$	$x = a\tan\theta$ or $x = a\cot\theta$
(ii) $a^2 - x^2$	$x = a\sin\theta$ or $x = a\cos\theta$
(iii) $x^2 - a^2$	$x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$
(iv) $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
(v) $\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$	$x^2 = a^2\cos 2\theta$