

OSCILLATIONS

- 14.1 Introduction
- 14.2 Periodic and oscillatory motions
- 14.3 Simple harmonic motion
- 14.4 Simple harmonic motion and uniform circular motion
- 14.5 Velocity and acceleration in simple harmonic motion
- 14.6 Force law for simple harmonic motion
- 14.7 Energy in simple harmonic motion
- 14.8 Some systems executing SHM
- 14.9 Damped simple harmonic motion
- 14.10 Forced oscillations and resonance

Summary
Points to ponder
Exercises
Additional Exercises
Appendix

14.1 INTRODUCTION

In our daily life we come across various kinds of motions. You have already learnt about some of them, e.g. rectilinear motion and motion of a projectile. Both these motions are non-repetitive. We have also learnt about uniform circular motion and orbital motion of planets in the solar system. In these cases, the motion is repeated after a certain interval of time, that is, it is periodic. In your childhood you must have enjoyed rocking in a cradle or swinging on a swing. Both these motions are repetitive in nature but different from the periodic motion of a planet. Here, the object moves to and fro about a mean position. The pendulum of a wall clock executes a similar motion. There are leaves and branches of a tree oscillating in breeze, boats bobbing at anchor and the surging pistons in the engines of cars. All these objects execute a periodic to and fro motion. Such a motion is termed as oscillatory motion. In this chapter we study this motion.

The study of oscillatory motion is basic to physics; its concepts are required for the understanding of many physical phenomena. In musical instruments like the sitar, the guitar or the violin, we come across vibrating strings that produce pleasing sounds. The membranes in drums and diaphragms in telephone and speaker systems vibrate to and fro about their mean positions. The vibrations of air molecules make the propagation of sound possible. Similarly, the atoms in a solid oscillate about their mean positions and convey the sensation of temperature. The oscillations of electrons in the antennas of radio, TV and satellite transmitters convey information.

The description of a periodic motion in general, and oscillatory motion in particular, requires some fundamental concepts like period, frequency, displacement, amplitude and phase. These concepts are developed in the next section.

14.2 PERIODIC AND OSCILLATORY MOTIONS

Fig 14.1 shows some periodic motions. Suppose an insect climbs up a ramp and falls down it comes back to the initial point and repeats the process identically. If you draw a graph of its height above the ground versus time, it would look something like Fig. 14.1 (a). If a child climbs up a step, comes down, and repeats the process, its height above the ground would look like that in Fig 14.1 (b). When you play the game of bouncing a ball off the ground, between your palm and the ground, its height versus time graph would look like the one in Fig 14.1 (c). Note that both the curved parts in Fig 14.1 (c) are sections of a parabola given by the Newton’s equation of motion (see section 3.6),

$$h = ut + \frac{1}{2}gt^2 \text{ for downward motion, and}$$

$$h = ut - \frac{1}{2}gt^2 \text{ for upward motion,}$$

with different values of u in each case. These are examples of periodic motion. Thus, a motion that repeats itself at regular intervals of time is called **periodic motion**.

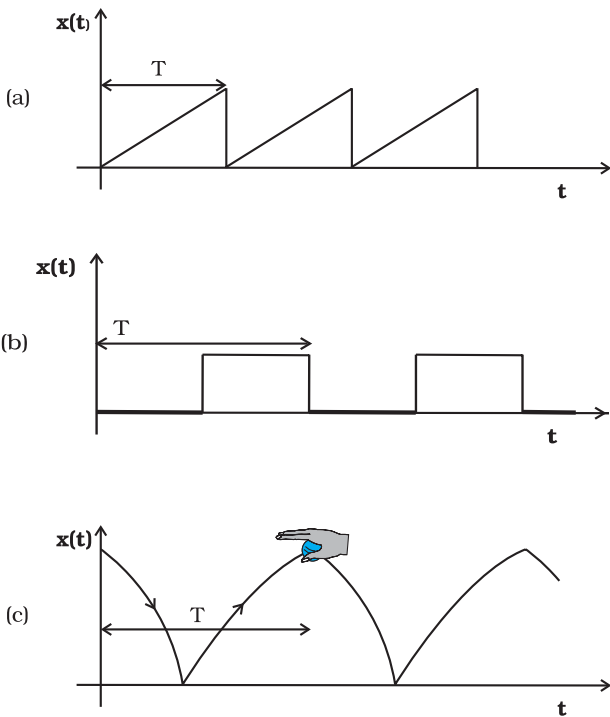


Fig 14.1 Examples of periodic motion. The period T is shown in each case.

Very often the body undergoing periodic motion has an equilibrium position somewhere inside its path. When the body is at this position no net external force acts on it. Therefore, if it is left there at rest, it remains there forever. If the body is given a small displacement from the position, a force comes into play which tries to bring the body back to the equilibrium point, giving rise to **oscillations** or **vibrations**. For example, a ball placed in a bowl will be in equilibrium at the bottom. If displaced a little from the point, it will perform oscillations in the bowl. Every oscillatory motion is periodic, but every periodic motion need not be oscillatory. Circular motion is a periodic motion, but it is not oscillatory.

There is no significant difference between oscillations and vibrations. It seems that when the frequency is small, we call it oscillation (like the oscillation of a branch of a tree), while when the frequency is high, we call it vibration (like the vibration of a string of a musical instrument).

Simple harmonic motion is the simplest form of oscillatory motion. This motion arises when the force on the oscillating body is directly proportional to its displacement from the mean position, which is also the equilibrium position. Further, at any point in its oscillation, this force is directed towards the mean position.

In practice, oscillating bodies eventually come to rest at their equilibrium positions, because of the damping due to friction and other dissipative causes. However, they can be forced to remain oscillating by means of some external periodic agency. We discuss the phenomena of damped and forced oscillations later in the chapter.

Any material medium can be pictured as a collection of a large number of coupled oscillators. The collective oscillations of the constituents of a medium manifest themselves as waves. Examples of waves include water waves, seismic waves, electromagnetic waves. We shall study the wave phenomenon in the next chapter.

14.2.1 Period and frequency

We have seen that any motion that repeats itself at regular intervals of time is called **periodic motion**. The **smallest interval of time after which the motion is repeated is called its period**. Let us denote the period by the symbol T . Its SI unit is second. For periodic motions,

which are either too fast or too slow on the scale of seconds, other convenient units of time are used. The period of vibrations of a quartz crystal is expressed in units of microseconds (10^{-6} s) abbreviated as μs . On the other hand, the orbital period of the planet Mercury is 88 earth days. The Halley's comet appears after every 76 years.

The reciprocal of T gives the number of repetitions that occur per unit time. This quantity is called the **frequency of the periodic motion**. It is represented by the symbol ν . The relation between ν and T is

$$\nu = 1/T \quad (14.1)$$

The unit of ν is thus s^{-1} . After the discoverer of radio waves, Heinrich Rudolph Hertz (1857-1894), a special name has been given to the unit of frequency. It is called hertz (abbreviated as Hz). Thus,

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ oscillation per second} = 1\text{s}^{-1} \quad (14.2)$$

Note, that the frequency, ν , is not necessarily an integer.

► **Example 14.1** On an average a human heart is found to beat 75 times in a minute. Calculate its frequency and period.

Answer The beat frequency of heart = $75/(1 \text{ min})$
 $= 75/(60 \text{ s})$
 $= 1.25 \text{ s}^{-1}$
 $= 1.25 \text{ Hz}$
 The time period T = $1/(1.25 \text{ s}^{-1})$
 $= 0.8 \text{ s}$ ◀

14.2.2 Displacement

In section 4.2, we defined displacement of a particle as the change in its position vector. In

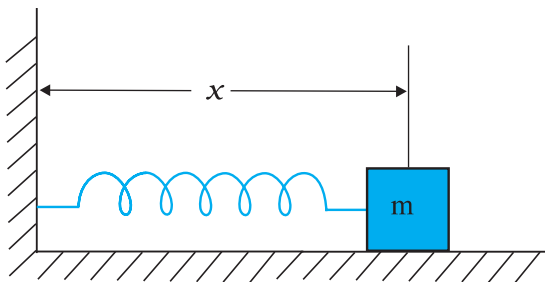


Fig. 14.2(a) A block attached to a spring, the other end of which is fixed to a rigid wall. The block moves on a frictionless surface. The motion of the block can be described in terms of its distance or displacement x from the wall.

this chapter, we use the term displacement in a more general sense. It refers to change with time of any physical property under consideration. For example, in case of rectilinear motion of a steel ball on a surface, the distance from the starting point as a function of time is its position displacement. The choice of origin is a matter of convenience. Consider a block attached to a spring, the other end of which is fixed to a rigid wall [see Fig. 14.2(a)]. Generally it is convenient to measure displacement of the body from its equilibrium position. For an oscillating simple pendulum, the angle from the vertical as a function of time may be regarded as a displacement variable [see Fig. 14.2(b)]. The term displacement is not always to be referred

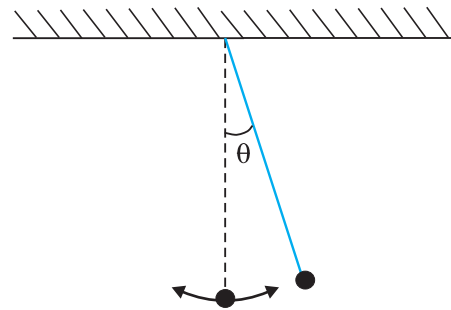


Fig. 14.2(b) An oscillating simple pendulum; its motion can be described in terms of angular displacement θ from the vertical.

in the context of position only. There can be many other kinds of displacement variables. The voltage across a capacitor, changing with time in an a.c. circuit, is also a displacement variable. In the same way, pressure variations in time in the propagation of sound wave, the changing electric and magnetic fields in a light wave are examples of displacement in different contexts. The displacement variable may take both positive and negative values. In experiments on oscillations, the displacement is measured for different times.

The displacement can be represented by a mathematical function of time. In case of periodic motion, this function is periodic in time. One of the simplest periodic functions is given by

$$f(t) = A \cos \omega t \quad (14.3a)$$

If the argument of this function, ωt , is increased by an integral multiple of 2π radians,

the value of the function remains the same. The function $f(t)$ is then periodic and its period, T , is given by

$$T = \frac{2\pi}{\omega} \tag{14.3b}$$

Thus, the function $f(t)$ is periodic with period T ,

$$f(t) = f(t+T)$$

The same result is obviously correct if we consider a sine function, $f(t) = A \sin \omega t$. Further, a linear combination of sine and cosine functions like,

$$f(t) = A \sin \omega t + B \cos \omega t \tag{14.3c}$$

is also a periodic function with the same period T . Taking,

$$A = D \cos \phi \text{ and } B = D \sin \phi$$

Eq. (14.3c) can be written as,

$$f(t) = D \sin (\omega t + \phi) , \tag{14.3d}$$

Here D and ϕ are constant given by

$$D = \sqrt{A^2 + B^2} \text{ and } \tan^{-1} \frac{B}{A}$$

The great importance of periodic sine and cosine functions is due to a remarkable result proved by the French mathematician, Jean Baptiste Joseph Fourier (1768-1830): **Any periodic function can be expressed as a superposition of sine and cosine functions of different time periods with suitable coefficients.**

► **Example 14.2** Which of the following functions of time represent (a) periodic and (b) non-periodic motion? Give the period for each case of periodic motion [ω is any positive constant].

- (i) $\sin \omega t + \cos \omega t$
- (ii) $\sin \omega t + \cos 2 \omega t + \sin 4 \omega t$
- (iii) $e^{-\omega t}$
- (iv) $\log (\omega t)$

Answer

- (i) $\sin \omega t + \cos \omega t$ is a periodic function, it can also be written as $\sqrt{2} \sin (\omega t + \pi/4)$.
Now $\sqrt{2} \sin (\omega t + \pi/4) = \sqrt{2} \sin (\omega t + \pi/4 + 2\pi)$

$$= \sqrt{2} \sin [\omega(t + 2\pi/\omega) + \pi/4]$$

The periodic time of the function is $2\pi/\omega$

- (ii) This is an example of a periodic motion. It can be noted that each term represents a periodic function with a different angular frequency. Since period is the least interval of time after which a function repeats its value, $\sin \omega t$ has a period $T_0 = 2\pi/\omega$; $\cos 2 \omega t$ has a period $\pi/\omega = T_0/2$; and $\sin 4 \omega t$ has a period $2\pi/4\omega = T_0/4$. The period of the first term is a multiple of the periods of the last two terms. Therefore, the smallest interval of time after which the sum of the three terms repeats is T_0 , and thus the sum is a periodic function with a period $2\pi/\omega$
- (iii) The function $e^{-\omega t}$ is not periodic, it decreases monotonically with increasing time and tends to zero as $t \rightarrow \infty$ and thus, never repeats its value.
- (iv) The function $\log(\omega t)$ increases monotonically with time t . It, therefore, never repeats its value and is a non-periodic function. It may be noted that as $t \rightarrow \infty$, $\log(\omega t)$ diverges to ∞ . It, therefore, cannot represent any kind of physical displacement. ◀

14.3 SIMPLE HARMONIC MOTION

Let us consider a particle vibrating back and forth about the origin of an x-axis between the limits $+A$ and $-A$ as shown in Fig. 14.3. In between these extreme positions the particle

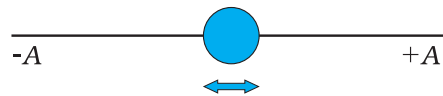


Fig. 14.3 A particle vibrating back and forth about the origin of x-axis, between the limits $+A$ and $-A$.

moves in such a manner that its speed is maximum when it is at the origin and zero when it is at $\pm A$. The time t is chosen to be zero when the particle is at $+A$ and it returns to $+A$ at $t = T$. In this section we will describe this motion. Later, we shall discuss how to achieve it. To study the motion of this particle, we record its positions as a function of time

by taking ‘snapshots’ at regular intervals of time. A set of such snapshots is shown in Fig. 14.4. The position of the particle with reference to the origin gives its displacement at any instant of time. For such a motion the displacement $x(t)$ of the particle from a certain chosen origin is found to vary with time as,

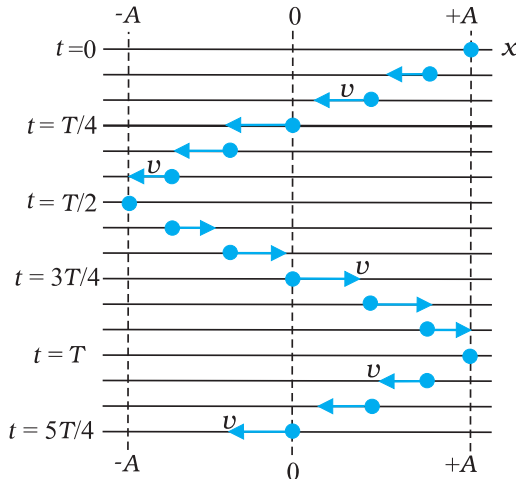


Fig. 14.4 A sequence of ‘snapshots’ (taken at equal intervals of time) showing the position of a particle as it oscillates back and forth about the origin along an x -axis, between the limits $+A$ and $-A$. The length of the vector arrows is scaled to indicate the speed of the particle. The speed is maximum when the particle is at the origin and zero when it is at $\pm A$. If the time t is chosen to be zero when the particle is at $+A$, then the particle returns to $+A$ at $t = T$, where T is the period of the motion. The motion is then repeated. It is represented by Eq. (14.4) for $\phi = 0$.

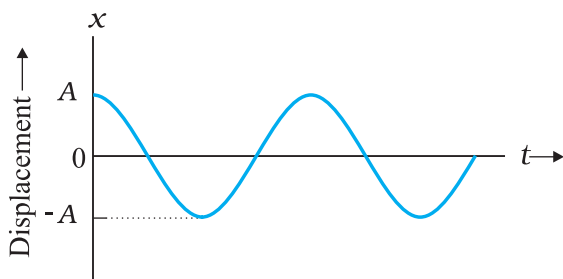


Fig. 14.5 A graph of x as a function of time for the motion represented by Eq. (14.4).

$$x(t) = A \cos(\omega t + \phi) \quad (14.4)$$

in which A , ω and ϕ are constants.

$$x(t) = A \cos(\omega t + \phi)$$

\uparrow Displacement \uparrow Amplitude \uparrow Angular frequency \uparrow Phase constant

Phase

Fig. 14.6 A reference of the quantities in Eq. (14.4).

The motion represented by Eq. (14.4) is called **simple harmonic motion** (SHM); a term that means the periodic motion is a sinusoidal function of time. Equation (14.4), in which the sinusoidal function is a cosine function, is plotted in Fig. 14.5. The quantities that determine

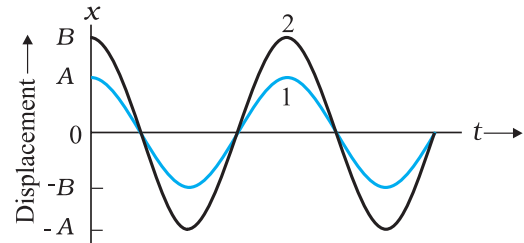


Fig. 14.7 (a) A plot of displacement as a function of time as obtained from Eq. (14.4) with $\phi = 0$. The curves 1 and 2 are for two different amplitudes A and B .

the shape of the graph are displayed in Fig. 14.6 along with their names.

We shall now define these quantities.

The quantity A is called the **amplitude** of the motion. It is a positive constant which represents the magnitude of the maximum displacement of the particle in either direction. The cosine function in Eq. (14.4) varies between the limits ± 1 , so the displacement $x(t)$ varies between the limits $\pm A$. In Fig. 14.7 (a), the curves 1 and 2 are plots of Eq. (14.4) for two different amplitudes A and B . The difference between these curves illustrates the significance of amplitude.

The time varying quantity, $(\omega t + \phi)$, in Eq. (14.4) is called the **phase** of the motion. It describes the state of motion at a given time. The constant ϕ is called the **phase constant** (or **phase angle**). The value of ϕ depends on the displacement and velocity of the particle at $t = 0$. This can be understood better by considering Fig. 14.7(b). In this figure, the curves 3 and 4 represent plots of Eq. (14.4) for two values of the phase constant ϕ .

It can be seen that the phase constant signifies the initial conditions.

The constant ω called the angular frequency of the motion, is related to the period T . To get

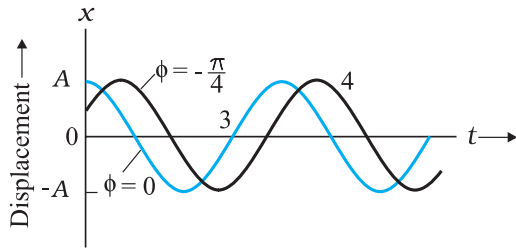


Fig. 14.7 (b) A plot obtained from Eq. 14.4. The curves 3 and 4 are for $\phi = 0$ and $-\pi/4$ respectively. The amplitude A is same for both the plots.

their relationship, let us consider Eq. (14.4) with $\phi = 0$; it then reduces to,

$$x(t) = A \cos \omega t \tag{14.5}$$

Now since the motion is periodic with a period T , the displacement $x(t)$ must return to its initial value after one period of the motion; that is, **$x(t)$ must be equal to $x(t + T)$** for all t . Applying this condition to Eq. (14.5) leads to,

$$A \cos \omega t = A \cos \omega(t + T) \tag{14.6}$$

As the cosine function first repeats itself when its argument (the phase) has increased by 2π , Eq. (14.6) gives,

$$\omega(t + T) = \omega t + 2\pi$$

or $\omega T = 2\pi$

Thus, the angular frequency is,

$$\omega = 2\pi / T \tag{14.7}$$

The SI unit of angular frequency is radians per second. To illustrate the significance of period T , sinusoidal functions with two different periods are plotted in Fig. 14.8.

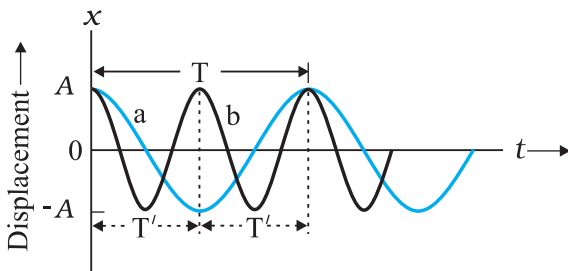


Fig. 14.8 Plots of Eq. (14.4) for $\phi = 0$ for two different periods.

In this plot the SHM represented by curve a , has a period T and that represented by curve b , has a period $T' = T/2$.

We have had an introduction to simple harmonic motion. In the next section we will discuss the simplest example of simple harmonic motion. It will be shown that the projection of uniform circular motion on a diameter of the circle executes simple harmonic motion.

▶ Example 14.3 Which of the following functions of time represent (a) simple harmonic motion and (b) periodic but not simple harmonic? Give the period for each case.

(1) $\sin \omega t - \cos \omega t$
 (2) $\sin^2 \omega t$

Answer

(a) $\sin \omega t - \cos \omega t$
 $= \sin \omega t - \sin (\pi/2 - \omega t)$
 $= 2 \cos (\pi/4) \sin (\omega t - \pi/4)$
 $= \sqrt{2} \sin (\omega t - \pi/4)$

This function represents a simple harmonic motion having a period $T = 2\pi/\omega$ and a phase angle $(-\pi/4)$ or $(7\pi/4)$

(b) $\sin^2 \omega t$
 $= \frac{1}{2} - \frac{1}{2} \cos 2\omega t$

The function is periodic having a period $T = \pi/\omega$ It also represents a harmonic motion with the point of equilibrium occurring at $\frac{1}{2}$ instead of zero. ◀

14.4 SIMPLE HARMONIC MOTION AND UNIFORM CIRCULAR MOTION

In 1610, Galileo discovered four principal moons of the planet Jupiter. To him, each moon seemed to move back and forth relative to the planet in a simple harmonic motion; the disc of the planet forming the mid point of the motion. The record of his observations, written in his own hand, is still available. Based on his data, the position of the moon Callisto relative to Jupiter is plotted in Fig. 14.9. In this figure, the circles represent Galileo’s data points and the curve drawn is a best fit to the data. The curve obeys Eq. (14.4), which is the displacement function for SHM. It gives a period of about 16.8 days.

It is now well known that Callisto moves with essentially a constant speed in an almost circular orbit around Jupiter. Its true motion is uniform circular motion. What Galileo saw and what we can also see, with a good pair of binoculars, is the projection of this uniform circular motion on a line in the plane of motion. This can easily be visualised by performing a

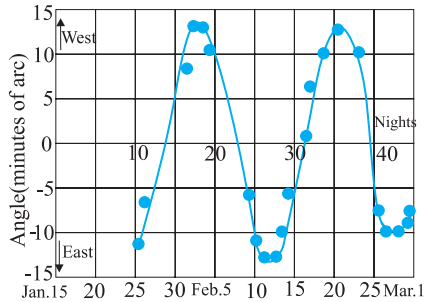


Fig. 14.9 The angle between Jupiter and its moon Callisto as seen from earth. The circles are based on Galileo's measurements of 1610. The curve is a best fit suggesting a simple harmonic motion. At Jupiter's mean distance, 10 minutes of arc corresponds to about 2×10^6 km.

simple experiment. Tie a ball to the end of a string and make it move in a horizontal plane about a fixed point with a constant angular speed. The ball would then perform a uniform circular motion in the horizontal plane. Observe the ball sideways or from the front, fixing your attention in the plane of motion. The ball will appear to execute to and fro motion along a horizontal line with the point of rotation as the midpoint. You could alternatively observe the shadow of the ball on a wall which is perpendicular to the plane of the circle. In this process what we are observing is the motion of the ball on a diameter of the circle normal to the direction of viewing. This experiment provides an analogy to Galileo's observation.

In Fig. 14.10, we show the motion of a **reference particle** P executing a uniform circular motion with (constant) angular speed ω in a **reference circle**. The radius A of the circle is the magnitude of the particle's position vector. At any time t , the angular position of the particle

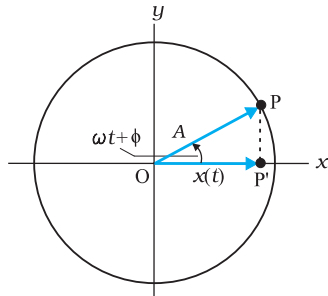


Fig. 14.10 The motion of a reference particle P executing a uniform circular motion with (constant) angular speed ω in a reference circle of radius A .

is $\omega t + \phi$, where ϕ is its angular position at $t = 0$. The projection of particle P on the x -axis is a point P' , which we can take as a second particle. The projection of the position vector of particle P on the x -axis gives the location $x(t)$ of P' . Thus we have,

$$x(t) = A \cos(\omega t + \phi)$$

which is the same as Eq. (14.4). This shows that if the reference particle P moves in a uniform circular motion, its projection particle P' executes a simple harmonic motion along a diameter of the circle.

From Galileo's observation and the above considerations, we are led to the conclusion that circular motion viewed edge-on is simple harmonic motion. In a more formal language we can say that : **Simple harmonic motion is the projection of uniform circular motion on a diameter of the circle in which the latter motion takes place.**

► **Example 14.4** Fig. 14.11 depicts two circular motions. The radius of the circle, the period of revolution, the initial position and the sense of revolution are indicated on the figures. Obtain the simple harmonic motions of the x -projection of the radius vector of the rotating particle P in each case.

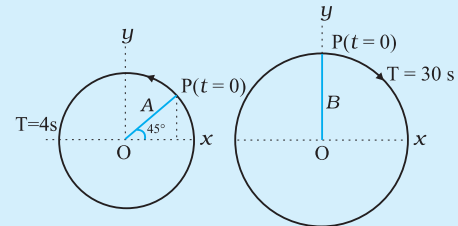


Fig. 14.11

Answer

- (a) At $t = 0$, OP makes an angle of $45^\circ = \pi/4$ rad with the (positive direction of) x -axis. After time t , it covers an angle $\frac{2\pi}{T}t$ in the anticlockwise sense, and makes an angle of $\frac{2\pi}{T}t + \frac{\pi}{4}$ with the x -axis. The projection of OP on the x -axis at time t is given by,

$$x(t) = A \cos \left(\frac{2}{T}t + \frac{\pi}{4} \right)$$

For $T = 4$ s,

$$x(t) = A \cos \left(\frac{2}{4}t + \frac{\pi}{4} \right)$$

which is a SHM of amplitude A , period 4 s,

and an initial phase* = $\frac{\pi}{4}$.

(b) In this case at $t = 0$, OP makes an angle of $90^\circ = \frac{\pi}{2}$ with the x -axis. After a time t , it

covers an angle of $\frac{2}{T}t$ in the clockwise

sense and makes an angle of $\frac{\pi}{2} - \frac{2}{T}t$ with the x -axis. The projection of OP on the x -axis at time t is given by

$$\begin{aligned} x(t) &= B \cos \left(\frac{\pi}{2} - \frac{2}{T}t \right) \\ &= B \sin \left(\frac{2}{T}t \right) \end{aligned}$$

For $T = 30$ s,

$$x(t) = B \sin \frac{2}{15}t$$

Writing this as $x(t) = B \cos \left(\frac{\pi}{2} - \frac{2}{15}t \right)$, and comparing with Eq. (14.4). We find that this represents a SHM of amplitude B , period 30 s,

and an initial phase of $\frac{\pi}{2}$.

14.5 VELOCITY AND ACCELERATION IN SIMPLE HARMONIC MOTION

It can be seen easily that the magnitude of velocity, \mathbf{v} , with which the reference particle P (Fig. 14.10) is moving in a circle is related to its angular speed, ω as

$$v = \omega A \tag{14.8}$$

where A is the radius of the circle described by the particle P . The magnitude of the velocity vector \mathbf{v} of the projection particle is ωA ; its projection on the x -axis at any time t , as shown in Fig. 14.12, is

$$v(t) = -\omega A \sin (\omega t + \phi) \tag{14.9}$$

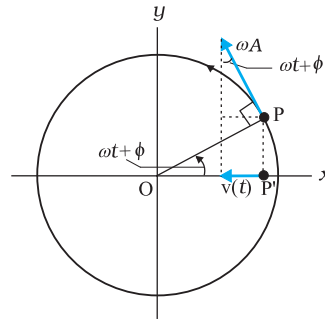


Fig. 14.12 The velocity, $v(t)$, of the particle P' is the projection of the velocity v of the reference particle, P .

The negative sign appears because the velocity component of P is directed towards the left, in the negative direction of x . Equation (14.9) expresses the instantaneous velocity of the particle P' (projection of P). Therefore, **it expresses the instantaneous velocity of a particle executing SHM**. Equation (14.9) can also be obtained by differentiating Eq. (14.4) with respect to time as,

$$v(t) = \frac{d}{dt} x(t) \tag{14.10}$$

* The natural unit of angle is radian, defined through the ratio of arc to radius. Angle is a dimensionless quantity. Therefore it is not always necessary to mention the unit 'radian' when we use π , its multiples or submultiples. The conversion between radian and degree is not similar to that between metre and centimetre or mile. If the argument of a trigonometric function is stated without units, it is understood that the unit is radian. On the other hand, if degree is to be used as the unit of angle, then it must be shown explicitly. For example, $\sin(15^\circ)$ means sine of 15 degree, but $\sin(15)$ means sine of 15 radians. Hereafter, we will often drop 'rad' as the unit, and it should be understood that whenever angle is mentioned as a numerical value, without units, it is to be taken as radians.

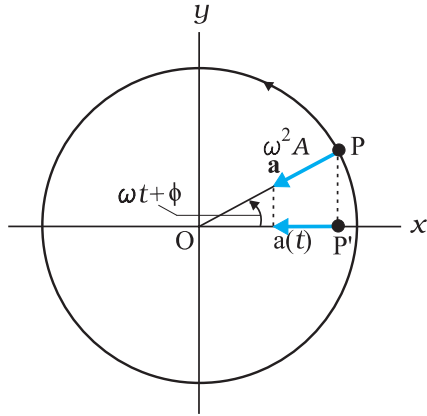


Fig. 14.13 The acceleration, $a(t)$, of the particle P' is the projection of the acceleration \mathbf{a} of the reference particle P .

We have seen that a particle executing a uniform circular motion is subjected to a radial acceleration \mathbf{a} directed towards the centre. Figure 14.13 shows such a radial acceleration, \mathbf{a} , of the reference particle P executing uniform circular motion. The magnitude of the radial acceleration of P is $\omega^2 A$. Its projection on the x -axis at any time t is,

$$\begin{aligned} a(t) &= -\omega^2 A \cos(\omega t + \phi) \\ &= -\omega^2 x(t) \end{aligned} \quad (14.11)$$

which is the acceleration of the particle P' (the projection of particle P). Equation (14.11), therefore, represents the instantaneous acceleration of the particle P' , which is executing **SHM**. Thus Eq. (14.11) **expresses the acceleration of a particle executing SHM**. It is an important result for **SHM**. It shows that in **SHM, the acceleration is proportional to the displacement and is always directed towards the mean position**. Eq. (14.11) can also be obtained by differentiating Eq. (14.9) with respect to time as,

$$a(t) = \frac{d}{dt} v(t) \quad (14.12)$$

The inter-relationship between the displacement of a particle executing simple harmonic motion, its velocity and acceleration can be seen in Fig. 14.14. In this figure (a) is a plot of Eq. (14.4) with $\phi = 0$ and (b) depicts Eq. (14.9) also with $\phi = 0$. Similar to the amplitude A in Eq. (14.4), the positive quantity ωA in Eq. (14.9) is called the **velocity amplitude** v_m . In Fig. 14.14(b), it can be seen that the velocity of the

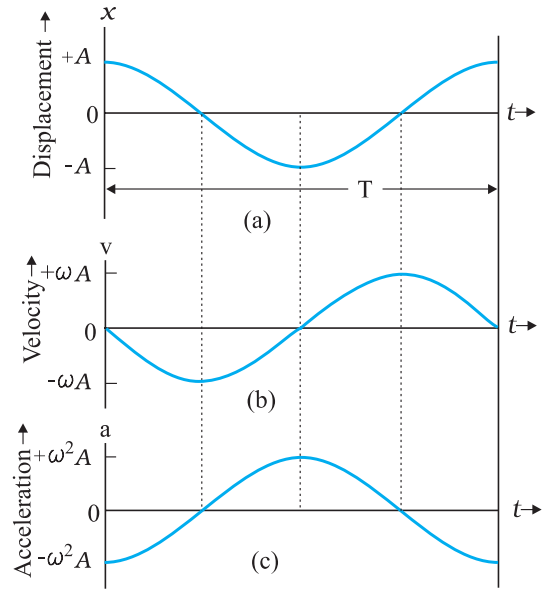


Fig. 14.14 The particle displacement, velocity and acceleration in a simple harmonic motion. (a) The displacement $x(t)$ of a particle executing SHM with phase angle ϕ equal to zero. (b) The velocity $v(t)$ of the particle. (c) The acceleration $a(t)$ of the particle.

oscillating particle varies between the limits $\pm v_m = \pm \omega A$. Note that the curve of $v(t)$ is shifted (to the left) from the curve of $x(t)$ by one quarter period and thus the particle velocity lags behind the displacement by a phase angle of $\pi/2$; when the magnitude of displacement is the greatest, the magnitude of the velocity is the least. When the magnitude of displacement is the least, the velocity is the greatest. Figure 14.14(c) depicts the variation of the particle acceleration $a(t)$. It is seen that when the displacement has its greatest positive value, the acceleration has its greatest negative value and vice versa. When the displacement is zero, the acceleration is also zero.

► **Example 14.5** A body oscillates with SHM according to the equation (in SI units),

$$x = 5 \cos [2\pi t + \pi/4].$$

At $t = 1.5$ s, calculate the (a) displacement, (b) speed and (c) acceleration of the body.

Answer The angular frequency ω of the body $= 2\pi \text{ s}^{-1}$ and its time period $T = 1$ s.

At $t = 1.5$ s

(a) displacement $= (5.0 \text{ m}) \cos [(2\pi \text{ s}^{-1}) \times 1.5 \text{ s} + \pi/4]$

$$\begin{aligned}
 &= (5.0 \text{ m}) \cos [(3\pi + \pi/4)] \\
 &= -5.0 \times 0.707 \text{ m} \\
 &= -3.535 \text{ m}
 \end{aligned}$$

(b) Using Eq. (14.9), the speed of the body

$$\begin{aligned}
 &= - (5.0 \text{ m})(2\pi \text{ s}^{-1}) \sin [(2\pi \text{ s}^{-1}) \times 1.5 \text{ s} \\
 &\quad + \pi/4] \\
 &= - (5.0 \text{ m})(2\pi \text{ s}^{-1}) \sin [(3\pi + \pi/4)] \\
 &= 10\pi \times 0.707 \text{ m s}^{-1} \\
 &= 22 \text{ m s}^{-1}
 \end{aligned}$$

(c) Using Eq.(14.10), the acceleration of the body

$$\begin{aligned}
 &= -(2\pi \text{ s}^{-1})^2 \times \text{displacement} \\
 &= - (2\pi \text{ s}^{-1})^2 \times (-3.535 \text{ m}) \\
 &= 140 \text{ m s}^{-2}
 \end{aligned}$$

14.6 FORCE LAW FOR SIMPLE HARMONIC MOTION

In Section 14.3, we described the simple harmonic motion. Now we discuss how it can be generated. Newton’s second law of motion relates the force acting on a system and the corresponding acceleration produced. Therefore, if we know how the acceleration of a particle varies with time, this law can be used to learn about the force, which must act on the particle to give it that acceleration. If we combine Newton’s second law and Eq. (14.11), we find that for simple harmonic motion,

$$\begin{aligned}
 F(t) &= ma \\
 &= -m\omega^2 x(t)
 \end{aligned}$$

or $F(t) = -kx(t)$ (14.13)

where $k = m\omega^2$ (14.14a)

or

$$\omega = \sqrt{\frac{k}{m}} \tag{14.14b}$$

Equation (14.13) gives the force acting on the particle. It is proportional to the displacement and directed in an opposite direction. Therefore, it is a restoring force. Note that unlike the centripetal force for uniform circular motion that is constant in magnitude, the restoring force for SHM is time dependent. The force law expressed by Eq. (14.13) can be taken as an alternative definition of simple harmonic motion. It states : **Simple harmonic motion is the motion executed by a particle subject to a force, which is proportional to the displacement of the particle and is directed towards the mean position.**

Since the force F is proportional to x rather than to some other power of x , such a system is also referred to as a **linear harmonic oscillator**. Systems in which the restoring force is a non-linear function of x are termed as non-linear harmonic or anharmonic oscillators.

► **Example 14.6** Two identical springs of spring constant k are attached to a block of mass m and to fixed supports as shown in Fig. 14.15. Show that when the mass is displaced from its equilibrium position on either side, it executes a simple harmonic motion. Find the period of oscillations.

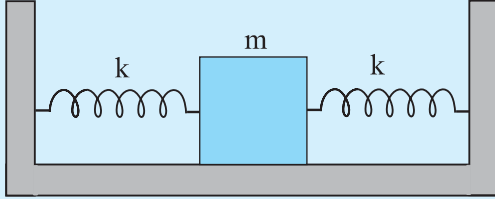


Fig. 14.15

Answer Let the mass be displaced by a small distance x to the right side of the equilibrium position, as shown in Fig. 14.16. Under this situation the spring on the left side gets

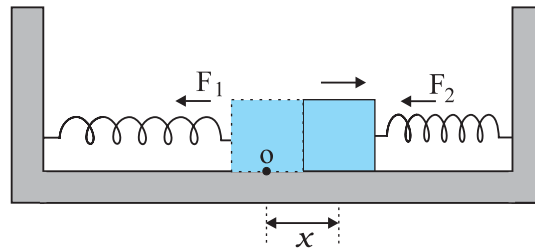


Fig. 14.16

elongated by a length equal to x and that on the right side gets compressed by the same length. The forces acting on the mass are then,

$$F_1 = -kx \text{ (force exerted by the spring on the left side, trying to pull the mass towards the mean position)}$$

$$F_2 = -kx \text{ (force exerted by the spring on the right side, trying to push the mass towards the mean position)}$$

The net force, F , acting on the mass is then given by,

$$F = -2kx$$

Hence the force acting on the mass is proportional to the displacement and is directed towards the mean position; therefore, the motion executed by the mass is simple harmonic. The time period of oscillations is,

$$T = 2\pi \sqrt{\frac{m}{2k}}$$

14.7 ENERGY IN SIMPLE HARMONIC MOTION

A particle executing simple harmonic motion has kinetic and potential energies, both varying between the limits, zero and maximum.

In section 14.5 we have seen that the velocity of a particle executing SHM, is a periodic function of time. It is zero at the extreme positions of displacement. Therefore, the kinetic energy (K) of such a particle, which is defined as

$$\begin{aligned} K &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) \\ &= \frac{1}{2}k A^2 \sin^2(\omega t + \phi) \end{aligned} \quad (14.15)$$

is also a periodic function of time, being zero when the displacement is maximum and maximum when the particle is at the mean position. Note, since the sign of v is immaterial in K , the period of K is $T/2$.

What is the potential energy (PE) of a particle executing simple harmonic motion? In Chapter 6, we have seen that the concept of potential energy is possible only for conservative forces. The spring force $F = -kx$ is a conservative force, with associated potential energy

$$U = \frac{1}{2}k x^2 \quad (14.16)$$

Hence the potential energy of a particle executing simple harmonic motion is,

$$\begin{aligned} U(x) &= \frac{1}{2}k x^2 \\ &= \frac{1}{2}k A^2 \cos^2(\omega t + \phi) \end{aligned} \quad (14.17)$$

Thus the potential energy of a particle executing simple harmonic motion is also periodic, with period $T/2$, being zero at the mean position and maximum at the extreme displacements.

It follows from Eqs. (14.15) and (14.17) that the total energy, E , of the system is,

$$E = U + K$$

$$\begin{aligned} &= \frac{1}{2}k A^2 \cos^2(\omega t + \phi) + \frac{1}{2}k A^2 \sin^2(\omega t + \phi) \\ &= \frac{1}{2}k A^2 [\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)] \end{aligned}$$

The quantity within the square brackets above is unity and we have,

$$E = \frac{1}{2}k A^2 \quad (14.18)$$

The total mechanical energy of a harmonic oscillator is thus independent of time as expected for motion under any conservative force. The time and displacement dependence of the potential and kinetic energies of a linear simple harmonic oscillator are shown in Fig. 14.17.

It is observed that in a linear harmonic oscillator, all energies are positive and peak twice during every period. For $x = 0$, the energy is all kinetic and for $x = \pm A$ it is all potential.

In between these extreme positions, the potential energy increases at the expense of kinetic energy. This behaviour of a linear harmonic oscillator suggests that it possesses an element of springiness and an element of inertia. The former stores its potential energy and the latter stores its kinetic energy.

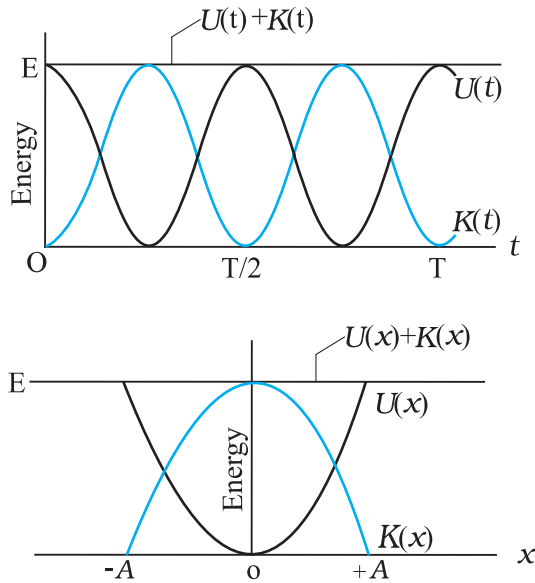


Fig. 14.17 (a) Potential energy $U(t)$, kinetic energy $K(t)$ and the total energy E as functions of time t for a linear harmonic oscillator. All energies are positive and the potential and kinetic energies peak twice in every period of the oscillator. (b) Potential energy $U(x)$, kinetic energy $K(x)$ and the total energy E as functions of position x for a linear harmonic oscillator with amplitude A . For $x = 0$, the energy is all kinetic and for $x = \pm A$ it is all potential.

▶ Example 14.7 A block whose mass is 1 kg is fastened to a spring. The spring has a spring constant of 50 N m^{-1} . The block is pulled to a distance $x = 10 \text{ cm}$ from its equilibrium position at $x = 0$ on a frictionless surface from rest at $t = 0$. Calculate the kinetic, potential and total energies of the block when it is 5 cm away from the mean position.

Answer The block executes SHM, its angular frequency, as given by Eq. (14.14b), is

$$\begin{aligned} \omega &= \sqrt{\frac{k}{m}} \\ &= \sqrt{\frac{50 \text{ N m}^{-1}}{1 \text{ kg}}} \\ &= 7.07 \text{ rad s}^{-1} \end{aligned}$$

Its displacement at any time t is then given by,

$$x(t) = 0.1 \cos(7.07t)$$

Therefore, when the particle is 5 cm away from the mean position, we have

$$0.05 = 0.1 \cos(7.07t)$$

Or $\cos(7.07t) = 0.5$ and hence

$$\sin(7.07t) = \frac{\sqrt{3}}{2} = 0.866,$$

Then the velocity of the block at $x = 5 \text{ cm}$ is

$$\begin{aligned} &= 0.1 \times 7.07 \times 0.866 \text{ m s}^{-1} \\ &= 0.61 \text{ m s}^{-1} \end{aligned}$$

Hence the K.E. of the block,

$$\begin{aligned} &= \frac{1}{2} m v^2 \\ &= \frac{1}{2} [1 \text{ kg} \times (0.6123 \text{ m s}^{-1})^2] \\ &= 0.19 \text{ J} \end{aligned}$$

The P.E. of the block,

$$\begin{aligned} &= \frac{1}{2} k x^2 \\ &= \frac{1}{2} (50 \text{ N m}^{-1} \times 0.05 \text{ m} \times 0.05 \text{ m}) \\ &= 0.0625 \text{ J} \end{aligned}$$

The total energy of the block at $x = 5 \text{ cm}$,

$$\begin{aligned} &= \text{K.E.} + \text{P.E.} \\ &= 0.25 \text{ J} \end{aligned}$$

we also know that at maximum displacement, K.E. is zero and hence the total energy of the system is equal to the P.E. Therefore, the total energy of the system,

$$\begin{aligned} &= \frac{1}{2} (50 \text{ N m}^{-1} \times 0.1 \text{ m} \times 0.1 \text{ m}) \\ &= 0.25 \text{ J} \end{aligned}$$

which is same as the sum of the two energies at a displacement of 5 cm. This is in conformity with the principle of conservation of energy. ◀

14.8 SOME SYSTEMS EXECUTING SIMPLE HARMONIC MOTION

There are no physical examples of absolutely pure **simple harmonic motion**. In practice we come across systems that execute simple harmonic motion approximately under certain conditions. In the subsequent part of this section, we discuss the motion executed by some such systems.

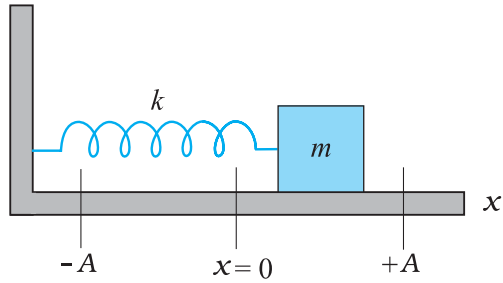


Fig. 14.18 A linear simple harmonic oscillator consisting of a block of mass m attached to a spring. The block moves over a frictionless surface. Once pulled to the side and released, it executes simple harmonic motion.

14.8.1 Oscillations due to a Spring

The simplest observable example of simple harmonic motion is the small oscillations of a block of mass m fixed to a spring, which in turn is fixed to a rigid wall as shown in Fig. 14.18. The block is placed on a frictionless horizontal surface. If the block is pulled on one side and is released, it then executes a to and fro motion about a mean position. Let $x = 0$, indicate the position of the centre of the block when the spring is in equilibrium. The positions marked as $-A$ and $+A$ indicate the maximum displacements to the left and the right of the mean position. We have already learnt that springs have special properties, which were first discovered by the English physicist Robert Hooke. He had shown that such a system when deformed, is subject to a restoring force, the magnitude of which is proportional to the deformation or the displacement and acts in opposite direction. This is known as Hooke's law (Chapter 9). It holds good for displacements small in comparison to the length of the spring. At any time t , if the displacement of the block from its mean position is x , the restoring force F acting on the block is,

$$F(x) = -kx \quad (14.19)$$

The constant of proportionality, k , is called the spring constant, its value is governed by the elastic properties of the spring. A stiff spring has large k and a soft spring has small k . Equation (14.19) is same as the force law for SHM and therefore the system executes a simple harmonic motion. From Eq. (14.14) we have,

$$\omega = \sqrt{\frac{k}{m}} \quad (14.20)$$

and the period, T , of the oscillator is given by,

$$T = 2\pi \sqrt{\frac{m}{k}} \quad (14.21)$$

Equations (14.20) and (14.21) tell us that a large angular frequency and hence a small period is associated with a stiff spring (high k) and a light block (small m).

Example 14.8 A 5 kg collar is attached to a spring of spring constant 500 N m^{-1} . It slides without friction over a horizontal rod. The collar is displaced from its equilibrium position by 10.0 cm and released. Calculate (a) the period of oscillation, (b) the maximum speed and (c) maximum acceleration of the collar.

Answer (a) The period of oscillation as given by Eq. (14.21) is,

$$\begin{aligned} T &= 2\pi \sqrt{\frac{m}{k}} \\ &= 2\pi \sqrt{\frac{5.0 \text{ kg}}{500 \text{ N m}^{-1}}} \\ &= (2\pi/10) \text{ s} \\ &= 0.63 \text{ s} \end{aligned}$$

(b) The velocity of the collar executing SHM is given by,

$$v(t) = -A\omega \sin(\omega t + \phi)$$

The maximum speed is given by,

$$\begin{aligned} v_m &= A\omega \\ &= 0.1 \times \sqrt{\frac{k}{m}} \\ &= 0.1 \times \sqrt{\frac{500 \text{ N m}^{-1}}{5 \text{ kg}}} \\ &= 1 \text{ m s}^{-1} \end{aligned}$$

and it occurs at $x = 0$

(c) The acceleration of the collar at the displacement $x(t)$ from the equilibrium is given by,

$$\begin{aligned} a(t) &= -\omega^2 x(t) \\ &= -\frac{k}{m} x(t) \end{aligned}$$

Therefore the maximum acceleration is,

$$a_{\text{max}} = \omega^2 A$$

$$= \frac{500 \text{ N m}^{-1}}{5 \text{ kg}} \times 0.1 \text{ m}$$

$$= 10 \text{ m s}^{-2}$$

and it occurs at the extremities. ◀

14.8.2 The Simple Pendulum

It is said that Galileo measured the periods of a swinging chandelier in a church by his pulse beats. He observed that the motion of the chandelier was periodic. The system is a kind of pendulum. You can also make your own pendulum by tying a piece of stone to a long unstretchable thread, approximately 100 cm long. Suspend your pendulum from a suitable support so that it is free to oscillate. Displace the stone to one side by a small distance and let it go. The stone executes a to and fro motion, it is periodic with a period of about two seconds. Is this motion simple harmonic? To answer this question, we consider a **simple pendulum**, which consists of a particle of mass m (called the bob of the pendulum) suspended from one end of an unstretchable, massless string of length L fixed at the other end as shown in Fig. 14.19(a). The bob is free to swing to and fro in the plane of the page, to the left and right of a vertical line through the pivot point.

The forces acting on the bob are the force \mathbf{T} , tension in the string and the gravitational force $\mathbf{F}_g (= m\mathbf{g})$, as shown in Fig. 14.19(b). The string makes an angle θ with the vertical. We resolve the force \mathbf{F}_g into a radial component $F_g \cos \theta$ and a tangential component $F_g \sin \theta$. The radial component is cancelled by the tension, since there is no motion along the length of the string. The tangential component produces a restoring torque about the pendulum's pivot point. This

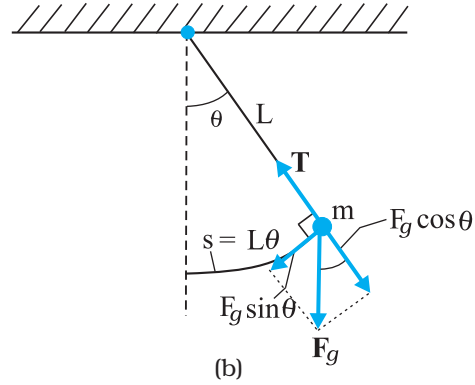
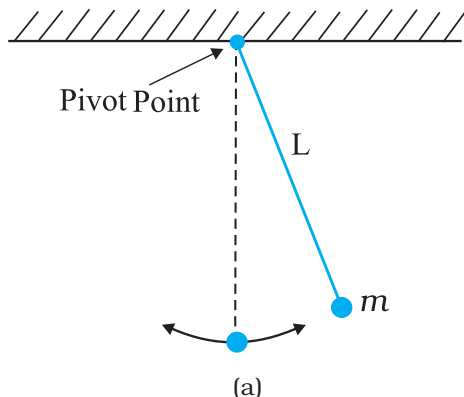


Fig. 14.19 (a) A simple pendulum. (b) The forces acting on the bob are the force due to gravity, $F_g (= m g)$, and the tension T in the string. (b) The tangential component $F_g \sin \theta$ of the gravitational force is a restoring force that tends to bring the pendulum back to the central position.

torque always acts opposite to the displacement of the bob so as to bring it back towards its central location. The central location is called the **equilibrium position** ($\theta = 0$), because at this position the pendulum would be at rest if it were not swinging.

The restoring torque τ is given by,

$$\tau = -L(F_g \sin \theta) \tag{14.22}$$

where the negative sign indicates that the torque acts to reduce θ , and L is the length of the moment arm of the force $F_g \sin \theta$ about the pivot point. For rotational motion we have,

$$\tau = I \alpha \tag{14.23}$$

where I is the pendulum's rotational inertia about the pivot point and α is its angular acceleration about that point. From Eqs. (14.22) and (14.23) we have,

$$-L(F_g \sin \theta) = I \alpha \tag{14.24}$$

Substituting the magnitude of F_g , i.e. mg , we have,

$$-L m g \sin \theta = I \alpha$$

Or,

$$\alpha = \frac{m g L}{I} \sin \theta \tag{14.25}$$

We can simplify Eq. (14.25) if we assume that the displacement θ is small. We know that $\sin \theta$ can be expressed as,

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \pm \dots \tag{14.26}$$

where θ is in radians.

Now if θ is small, $\sin \theta$ can be approximated by θ and Eq. (14.25) can then be written as,

$$\alpha = -\frac{mgL}{I}\theta \quad (14.27)$$

In Table 14.1, we have listed the angle θ in degrees, its equivalent in radians, and the value of the function $\sin \theta$. From this table it can be seen that for θ as large as 20 degrees, $\sin \theta$ is nearly the same as θ **expressed in radians**.

Table 14.1 $\sin \theta$ as a function of angle θ

θ (degrees)	θ (radians)	$\sin \theta$
0	0	0
5	0.087	0.087
10	0.174	0.174
15	0.262	0.256
20	0.349	0.342

Equation (14.27) is the angular analogue of Eq. (14.11) and tells us that the angular acceleration of the pendulum is proportional to the angular displacement θ but opposite in sign. Thus as the pendulum moves to the right, its pull to the left increases until it stops and begins to return to the left. Similarly, when it moves towards left, its acceleration to the right tends to return it to the right and so on, as it swings to and fro in SHM. Thus the motion of a **simple pendulum swinging through small angles is approximately SHM**.

Comparing Eq. (14.27) with Eq. (14.11), we see that the angular frequency of the pendulum is,

$$\omega = \sqrt{\frac{mgL}{I}}$$

and the period of the pendulum, T , is given by,

$$T = 2\pi \sqrt{\frac{I}{mgL}} \quad (14.28)$$

All the mass of a simple pendulum is centred in the mass m of the bob, which is at a radius of L from the pivot point. Therefore, for this system, we can write $I = mL^2$ and substituting this in Eq. (14.28) we get,

SHM - how small should the amplitude be?

When you perform the experiment to determine the time period of a simple pendulum, your teacher tells you to keep the amplitude small. But have you ever asked how small is small? Should the amplitude be 5° , 2° , 1° , or 0.5° ? Or could it be 10° , 20° , or 30° ?

To appreciate this, it would be better to measure the time period for different amplitudes, up to large amplitudes. Of course, for large oscillations, you will have to take care that the pendulum oscillates in a vertical plane. Let us denote the time period for small-amplitude oscillations as $T(0)$ and write the time period for amplitude θ_0 as $T(\theta_0) = cT(0)$, where c is the multiplying factor. If you plot a graph of c versus θ_0 , you will get values somewhat like this:

θ_0	:	20°	45°	50°	70°	90°
c	:	1.02	1.04	1.05	1.10	1.18

This means that the error in the time period is about 2% at an amplitude of 20° , 5% at an amplitude of 50° , and 10% at an amplitude of 70° and 18% at an amplitude of 90° .

In the experiment, you will never be able to measure $T(0)$ because this means there are no oscillations. Even theoretically, $\sin \theta$ is exactly equal to θ only for $\theta = 0$. There will be some inaccuracy for all other values of θ . The difference increases with increasing θ . Therefore we have to decide how much error we can tolerate. No measurement is ever perfectly accurate. You must also consider questions like these: What is the accuracy of the stopwatch? What is your own accuracy in starting and stopping the stopwatch? You will realise that the accuracy in your measurements at this level is never better than 5% or 10%. Since the above table shows that the time period of the pendulum increases hardly by 5% at an amplitude of 50° over its low amplitude value, you could very well keep the amplitude to be 50° in your experiments.

$$T = 2\pi \sqrt{\frac{L}{g}} \tag{14.29}$$

Equation (14.29) represents a simple expression for the time period of a simple pendulum.

► **Example 14.9** What is the length of a simple pendulum, which ticks seconds ?

Answer From Eq. (14.29), the time period of a simple pendulum is given by,

$$T = 2\pi \sqrt{\frac{L}{g}}$$

From this relation one gets,

$$L = \frac{gT^2}{4\pi^2}$$

The time period of a simple pendulum, which ticks seconds, is 2 s. Therefore, for $g = 9.8 \text{ m s}^{-2}$ and $T = 2 \text{ s}$, L is

$$L = \frac{9.8(\text{m s}^{-2}) \cdot 4(\text{s}^2)}{4\pi^2} = 1 \text{ m}$$

14.9 DAMPED SIMPLE HARMONIC MOTION

We know that the motion of a simple pendulum, swinging in air, dies out eventually. Why does it happen ? This is because the air drag and the friction at the support oppose the motion of the pendulum and dissipate its energy gradually. The pendulum is said to execute **damped oscillations**. In damped oscillations, although the energy of the system is continuously dissipated, the oscillations remain apparently periodic. The dissipating forces are generally the frictional forces. To understand the effect of such external forces on the motion of an oscillator, let us consider a system as shown in Fig. 14.20. Here a block of mass m oscillates vertically on a spring with spring constant k . The block is connected to a vane through a rod (the vane and the rod are considered to be massless). The vane is submerged in a liquid. As the block oscillates up and down, the vane also moves along with it in the liquid. The up and down motion of the vane displaces the liquid, which in

turn, exerts an inhibiting drag force (viscous drag) on it and thus on the entire oscillating system. With time, the mechanical energy of the block-spring system decreases, as energy is transferred to the thermal energy of the liquid and vane.

Let the damping force exerted by the liquid on the system be \mathbf{F}_d . Its magnitude is proportional to the velocity \mathbf{v} of the vane or the

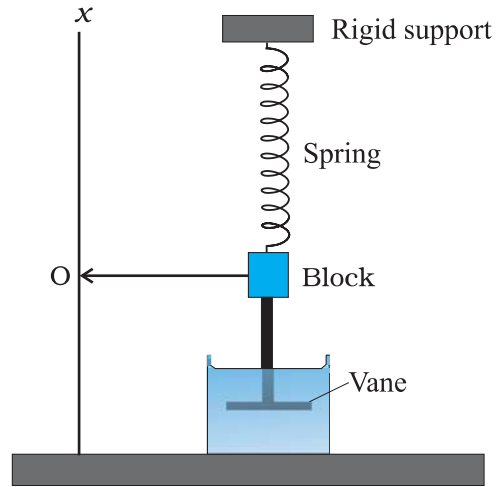


Fig. 14.20 A damped simple harmonic oscillator. The vane immersed in a liquid exerts a damping force on the block as it oscillates up and down.

block. The force acts in a direction opposite to the direction of \mathbf{v} . This assumption is valid only when the vane moves slowly. Then for the motion along the x -axis (vertical direction as shown in Fig. 14.20), we have

$$\mathbf{F}_d = -b \mathbf{v} \tag{14.30}$$

where b is a **damping constant** that depends on the characteristics of the liquid and the vane. The negative sign makes it clear that the force is opposite to the velocity at every moment.

When the mass m is attached to the spring and released, the spring will elongate a little and the mass will settle at some height. This position, shown by O in Fig 14.20, is the equilibrium position of the mass. If the mass is pulled down or pushed up a little, the restoring force on the block due to the spring is $\mathbf{F}_s = -k\mathbf{x}$, where \mathbf{x} is the displacement of the mass from its equilibrium position. Thus the total force acting

* Under gravity, the block will be at a certain equilibrium position O on the spring; x here represents the displacement from that position.

on the mass at any time t is $\mathbf{F} = -k\mathbf{x} - b\mathbf{v}$. If $\mathbf{a}(t)$ is the acceleration of the mass at time t , then by Newton's second law of motion for force components along the x -axis, we have

$$m a(t) = -k x(t) - b v(t) \quad (14.31)$$

Here we have dropped the vector notation because we are discussing one-dimensional motion. Substituting dx/dt for $v(t)$ and d^2x/dt^2 for the acceleration $a(t)$ and rearranging gives us the differential equation,

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0 \quad (14.32)$$

The solution of Eq. (14.32) describes the motion of the block under the influence of a damping force which is proportional to velocity. The solution is found to be of the form

$$x(t) = A e^{-bt/2m} \cos(\omega' t + \phi) \quad (14.33)$$

where A is the amplitude and ω' is the angular frequency of the damped oscillator given by,

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \quad (14.34)$$

In this function, the cosine function has a period $2\pi/\omega'$ but the function $x(t)$ is not strictly periodic because of the factor $e^{-bt/2m}$ which decreases continuously with time. However, if the decrease is small in one time period T , the motion represented by Eq. (14.33) is approximately periodic.

The solution, Eq. (14.33), can be graphically represented as shown in Fig. 14.21. We can

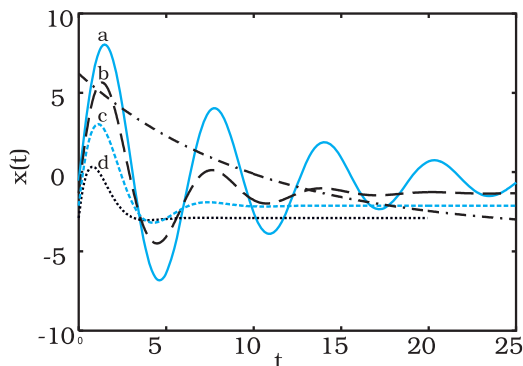


Fig. 14.21 Displacement as a function of time in damped harmonic oscillations. Damping goes on increasing successively from curve **a** to **d**.

regard it as a cosine function whose amplitude, which is $Ae^{-bt/2m}$, gradually decreases with time.

If $b = 0$ (there is no damping), then Eqs. (14.33) and (14.34) reduce to Eqs. (14.4) and (14.14b), expressions for the displacement and angular frequency of an undamped oscillator. We have seen that the mechanical energy of an undamped oscillator is constant and is given by Eq. (14.18) ($E = 1/2 k A^2$). If the oscillator is damped, the mechanical energy is not constant but decreases with time. If the damping is small, we can find $E(t)$ by replacing A in Eq. (14.18) by $Ae^{-bt/2m}$, the amplitude of the damped oscillations. Thus we find,

$$E(t) = \frac{1}{2} k A^2 e^{-bt/m} \quad (14.35)$$

Equation (14.35) shows that the total energy of the system decreases exponentially with time. Note that small damping means that the

dimensionless ratio $\left(\frac{b}{\sqrt{km}}\right)$ is much less than 1.

▶ Example 14.10 For the damped oscillator shown in Fig. 14.20, the mass m of the block is 200 g, $k = 90 \text{ N m}^{-1}$ and the damping constant b is 40 g s⁻¹. Calculate (a) the period of oscillation, (b) time taken for its amplitude of vibrations to drop to half of its initial value and (c) the time taken for its mechanical energy to drop to half its initial value.

Answer (a) We see that $km = 90 \times 0.2 = 18 \text{ kg m}^{-1} = \text{kg}^2 \text{ s}^{-2}$; therefore $\sqrt{km} = 4.243 \text{ kg s}^{-1}$, and $b = 0.04 \text{ kg s}^{-1}$. Therefore b is much less than \sqrt{km} . Hence the time period T from Eq. (14.34) is given by

$$\begin{aligned} T &= 2\pi \sqrt{\frac{m}{k}} \\ &= 2\pi \sqrt{\frac{0.2 \text{ kg}}{90 \text{ N m}^{-1}}} \\ &= 0.3 \text{ s} \end{aligned}$$

(b) Now, from Eq. (14.33), the time, $T_{1/2}$, for the amplitude to drop to half of its initial value is given by,

$$T_{1/2} = \frac{\ln(1/2)}{b/2m}$$

$$\frac{0.693}{40} \times 2 \times 200 \text{ s}$$

$$= 6.93 \text{ s}$$

(c) For calculating the time, $t_{1/2}$, for its mechanical energy to drop to half its initial value we make use of Eq. (14.35). From this equation we have,

$$E(t_{1/2})/E(0) = \exp(-bt_{1/2}/m)$$

Or $\frac{1}{2} = \exp(-bt_{1/2}/m)$

$$\ln(1/2) = -(bt_{1/2}/m)$$

Or $t_{1/2} = \frac{0.693}{40 \text{ g s}^{-1}} \times 200 \text{ g}$

$$= 3.46 \text{ s}$$

This is just half of the decay period for amplitude. This is not surprising, because, according to Eqs. (14.33) and (14.35), energy depends on the square of the amplitude. Notice that there is a factor of 2 in the exponents of the two exponentials. ◀

14.10 FORCED OSCILLATIONS AND RESONANCE

A person swinging in a swing without anyone pushing it or a simple pendulum, displaced and released, are examples of free oscillations. In both the cases, the amplitude of swing will gradually decrease and the system would, ultimately, come to a halt. Because of the ever-present dissipative forces, the free oscillations cannot be sustained in practice. They get damped as seen in section 14.9. However, while swinging in a swing if you apply a push periodically by pressing your feet against the ground, you find that not only the oscillations can now be maintained but the amplitude can also be increased. Under this condition the swing has **forced**, or **driven**, **oscillations**. In case of a system executing driven oscillations under the action of a harmonic force, two angular frequencies are important : (1) **the natural** angular frequency ω of the system, which is the angular frequency at which it will oscillate if it were displaced from equilibrium position and then left to oscillate freely, and (2)

the angular frequency ω_d of the external force causing the driven oscillations.

Suppose an external force $F(t)$ of amplitude F_o that varies periodically with time is applied to a damped oscillator. Such a force can be represented as,

$$F(t) = F_o \cos \omega_d t \tag{14.36}$$

The motion of a particle under the combined action of a linear restoring force, damping force and a time dependent driving force represented by Eq. (14.36) is given by,

$$m a(t) = -kx(t) - bv(t) + F_o \cos \omega_d t \tag{14.37a}$$

Substituting d^2x/dt^2 for acceleration in Eq. (14.37a) and rearranging it, we get

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = F_o \cos \omega_d t \tag{14.37b}$$

This is the equation of an oscillator of mass m on which a periodic force of (angular) frequency ω_d is applied. The oscillator initially oscillates with its natural frequency ω . When we apply the external periodic force, the oscillations with the natural frequency die out, and then the body oscillates with the (angular) frequency of the external periodic force. Its displacement, after the natural oscillations die out, is given by

$$x(t) = A \cos (\omega_d t + \phi) \tag{14.38}$$

where t is the time measured from the moment when we apply the periodic force.

The amplitude A is a function of the forced frequency ω_d and the natural frequency ω . Analysis shows that it is given by

$$A = \frac{F_o}{m \sqrt{(\omega_d^2 - \omega^2)^2 + b^2 \omega_d^2}} \tag{14.39a}$$

and $\tan \phi = \frac{-v_o}{\omega_d x_o}$ (14.39b)

where m is the mass of the particle and v_o and x_o are the velocity and the displacement of the particle at time $t = 0$, which is the moment when we apply the periodic force. Equation (14.39) shows that the amplitude of the forced oscillator depends on the (angular) frequency of the driving force. We can see a different behaviour of the oscillator when ω_d is far from ω and when it is close to ω . We consider these two cases.

(a) **Small Damping, Driving Frequency far from Natural Frequency :** In this case, $\omega_d b$ will be much smaller than $m(\omega^2 - \omega_d^2)$, and we can neglect that term. Then Eq. (14.39) reduces to

$$A = \frac{F}{m \omega_d^2} \quad (14.40)$$

Figure 14.22 shows the dependence of the displacement amplitude of an oscillator on the angular frequency of the driving force for different amounts of damping present in the system. It may be noted that in all the cases the amplitude is greatest when $\omega_d / \omega = 1$. The curves in this figure show that smaller the damping, the taller and narrower is the resonance peak.

If we go on changing the driving frequency, the amplitude tends to infinity when it equals the natural frequency. But this is the ideal case of zero damping, a case which never arises in a real system as the damping is never perfectly zero. You must have experienced in a swing that when the timing of your push exactly matches with the time period of the swing, your swing gets the maximum amplitude. This amplitude is large, but not infinity, because there is always some damping in your swing. This will become clear in the (b).

(b) **Driving Frequency Close to Natural Frequency :** If ω_d is very close to ω , $m(\omega^2 - \omega_d^2)$

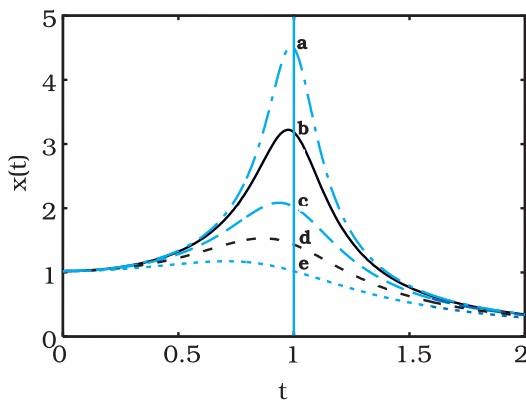


Fig. 14.22 The amplitude of a forced oscillator as a function of the angular frequency of the driving force. The amplitude is greatest near $\omega_d / \omega = 1$. The five curves correspond to different extents of damping present in the system. Curve **a** corresponds to the least damping, and damping goes on increasing successively in curves **b, c, d, e**. Notice that the peak shifts to the left with increasing **b**.

would be much less than $\omega_d b$, for any reasonable value of b , then Eq. (14.39) reduces to

$$A = \frac{F_0}{\omega_d b} \quad (14.41)$$

This makes it clear that the maximum possible amplitude for a given driving frequency is governed by the driving frequency and the damping, and is never infinity. The phenomenon of increase in amplitude when the driving force is close to the natural frequency of the oscillator is called **resonance**.

In our daily life we encounter phenomena which involve resonance. Your experience with swings is a good example of resonance. You might have realised that the skill in swinging to greater heights lies in the synchronisation of the rhythm of pushing against the ground with the natural frequency of the swing.

To illustrate this point further, let us consider a set of five simple pendulums of assorted lengths suspended from a common rope as shown in Fig. 14.23. The pendulums 1 and 4 have the same lengths and the others have different lengths. Now let us set pendulum 1 into motion. The energy from this pendulum gets transferred to other pendulums through the connecting rope and they start oscillating. The driving force is provided through the connecting rope. The frequency of this force is the frequency with which pendulum 1 oscillates. If we observe the response of pendulums 2, 3 and 5, they first start oscillating with their natural frequencies

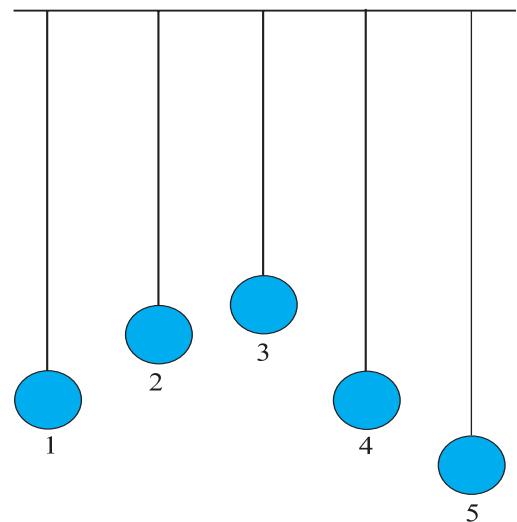


Fig. 14.23 A system of five simple pendulums suspended from a common rope.

of oscillations and different amplitudes, but this motion is gradually damped and not sustained. Their frequencies of oscillation gradually change and ultimately they oscillate with the frequency of pendulum 1, i.e. the frequency of the driving force but with different amplitudes. They oscillate with small amplitudes. The response of pendulum 4 is in contrast to this set of pendulums. It oscillates with the same frequency as that of pendulum 1 and its amplitude gradually picks up and becomes very large. A resonance-like response is seen. This happens because in this the condition for resonance is satisfied, i.e. the natural frequency of the system coincides with that of the driving force.

All mechanical structures have one or more natural frequencies, and if a structure is subjected to a strong external periodic driving force that matches one of these frequencies, the

resulting oscillations of the structure may rupture it. The Tacoma Narrows Bridge at Puget Sound, Washington, USA was opened on July 1, 1940. Four months later winds produced a pulsating resultant force in resonance with the natural frequency of the structure. This caused a steady increase in the amplitude of oscillations until the bridge collapsed. It is for the same reason the marching soldiers break steps while crossing a bridge. Aircraft designers make sure that none of the natural frequencies at which a wing can oscillate match the frequency of the engines in flight. Earthquakes cause vast devastation. It is interesting to note that sometimes, in an earthquake, short and tall structures remain unaffected while the medium height structures fall down. This happens because the natural frequencies of the short structures happen to be higher and those of taller structures lower than the frequency of the seismic waves.

SUMMARY

1. The motions which repeat themselves are called *periodic motions*.
2. The *period* T is the time required for one complete oscillation, or cycle. It is related to the frequency ν by,

$$T = \frac{1}{\nu}$$

The *frequency* ν of periodic or oscillatory motion is the number of oscillations per unit time. In the SI, it is measured in hertz :

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ oscillation per second} = 1 \text{ s}^{-1}$$

3. In *simple harmonic motion* (SHM), the displacement $x(t)$ of a particle from its equilibrium position is given by,

$$x(t) = A \cos(\omega t + \phi) \quad (\text{displacement}),$$

in which A is the *amplitude* of the displacement, the quantity $(\omega t + \phi)$ is the phase of the motion, and ϕ is the *phase constant*. The *angular frequency* ω is related to the period and frequency of the motion by,

$$\frac{2\pi}{T} = \omega \quad (\text{angular frequency}).$$

4. Simple harmonic motion is the projection of uniform circular motion on the diameter of the circle in which the latter motion occurs.
5. The particle velocity and acceleration during SHM as functions of time are given by,

$$v(t) = -\omega A \sin(\omega t + \phi) \quad (\text{velocity}),$$

$$a(t) = -\omega^2 A \cos(\omega t + \phi)$$

$$= -\omega^2 x(t) \quad (\text{acceleration}),$$

Thus we see that both velocity and acceleration of a body executing simple harmonic motion are periodic functions, having the velocity *amplitude* $v_m = \omega A$ and *acceleration amplitude* $a_m = \omega^2 A$, respectively.

6. The force acting simple harmonic motion is proportional to the displacement and is always directed towards the centre of motion.
7. A particle executing simple harmonic motion has, at any time, kinetic energy $K = \frac{1}{2} mv^2$ and potential energy $U = \frac{1}{2} kx^2$. If no friction is present the mechanical energy of the system, $E = K + U$ always remains constant even though K and U change with time
8. A particle of mass m oscillating under the influence of a Hooke's law restoring force given by $F = -kx$ exhibits simple harmonic motion with

$$\omega = \sqrt{\frac{k}{m}} \quad (\text{angular frequency})$$

$$T = 2\pi \sqrt{\frac{m}{k}} \quad (\text{period})$$

Such a system is also called a linear oscillator.

9. The motion of a simple pendulum swinging through small angles is approximately simple harmonic. The period of oscillation is given by,

$$T = 2\pi \sqrt{\frac{L}{g}}$$

10. The mechanical energy in a real oscillating system decreases during oscillations because external forces, such as drag, inhibit the oscillations and transfer mechanical energy to thermal energy. The real oscillator and its motion are then said to be *damped*. If the *damping force* is given by $F_d = -bv$, where v is the velocity of the oscillator and b is a *damping constant*, then the displacement of the oscillator is given by,

$$x(t) = A e^{-bt/2m} \cos(\omega' t + \phi)$$

where ω' , the angular frequency of the damped oscillator, is given by

$$\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

If the damping constant is small then $\omega' \approx \omega$ where ω is the angular frequency of the undamped oscillator. The mechanical energy E of the damped oscillator is given by

$$E(t) = \frac{1}{2} kA^2 e^{-bt/m}$$

11. If an external force with angular frequency ω_d acts on an oscillating system with natural angular frequency ω the system oscillates with angular frequency ω_d . The amplitude of oscillations is the greatest when

$$\omega_d = \omega$$

a condition called *resonance*.

Physical quantity	Symbol	Dimensions	Unit	Remarks
Period	T	[T]	s	The least time for motion to repeat itself
Frequency	ν (or f)	[T ⁻¹]	s ⁻¹	$\nu = \frac{1}{T}$
Angular frequency	ω	[T ⁻¹]	s ⁻¹	$\omega = 2\pi\nu$
Phase constant	ϕ	Dimensionless	rad	Initial value of phase of displacement in SHM
Force constant	k	[MT ⁻²]	N m ⁻¹	Simple harmonic motion $F = -kx$

POINTS TO PONDER

- The period T is the *least time* after which motion repeats itself. Thus, motion repeats itself after nT where n is an integer.
- Every periodic motion is not simple harmonic motion. Only that periodic motion governed by the force law $F = -kx$ is simple harmonic.
- Circular motion can arise due to an inverse-square law force (as in planetary motion) as well as due to simple harmonic force in two dimensions equal to: $-m\omega^2 r$. In the latter case, the phases of motion, in two perpendicular directions (x and y) must differ by $\omega/2$. Thus, a particle subject to a force $-m\omega^2 r$ with initial position $(0, A)$ and velocity $(\omega A, 0)$ will move uniformly in a circle of radius A .
- For linear simple harmonic motion with a given ω two arbitrary initial conditions are necessary and sufficient to determine the motion completely. The initial condition may be (i) initial position and initial velocity or (ii) amplitude and phase or (iii) energy and phase.
- From point 4 above, given amplitude or energy, phase of motion is determined by the initial position or initial velocity.
- A combination of two simple harmonic motions with arbitrary amplitudes and phases is not necessarily periodic. It is periodic only if frequency of one motion is an integral multiple of the other's frequency. However, a periodic motion can always be expressed as a sum of infinite number of harmonic motions with appropriate amplitudes.
- The period of SHM does not depend on amplitude or energy or the phase constant. Contrast this with the periods of planetary orbits under gravitation (Kepler's third law).
- The motion of a simple pendulum is simple harmonic for small angular displacement.
- For motion of a particle to be simple harmonic, its displacement x must be expressible in either of the following forms :

$$x = A \cos \omega t + B \sin \omega t$$

$$x = A \cos (\omega t + \alpha), \quad x = B \sin (\omega t + \beta)$$

The three forms are completely equivalent (any one can be expressed in terms of any other two forms).

Thus, damped simple harmonic motion [Eq. (14.31)] is not strictly simple harmonic. It is approximately so only for time intervals much less than $2m/b$ where b is the damping constant.

- In forced oscillations, the steady state motion of the particle (after the force oscillations die out) is simple harmonic motion whose frequency is the frequency of the driving frequency ω_d , not the natural frequency ω of the particle.

11. In the ideal case of zero damping, the amplitude of simple harmonic motion at resonance is infinite. This is no problem since all real systems have some damping, however, small.
12. Under forced oscillation, the phase of harmonic motion of the particle differs from the phase of the driving force.

Exercises

- 14.1** Which of the following examples represent periodic motion?
- (a) A swimmer completing one (return) trip from one bank of a river to the other and back.
 - (b) A freely suspended bar magnet displaced from its N-S direction and released.
 - (c) A hydrogen molecule rotating about its center of mass.
 - (d) An arrow released from a bow.
- 14.2** Which of the following examples represent (nearly) simple harmonic motion and which represent periodic but not simple harmonic motion?
- (a) the rotation of earth about its axis.
 - (b) motion of an oscillating mercury column in a U-tube.
 - (c) motion of a ball bearing inside a smooth curved bowl, when released from a point slightly above the lower most point.
 - (d) general vibrations of a polyatomic molecule about its equilibrium position.
- 14.3** Figure 14.27 depicts four $x-t$ plots for linear motion of a particle. Which of the plots represent periodic motion? What is the period of motion (in case of periodic motion) ?

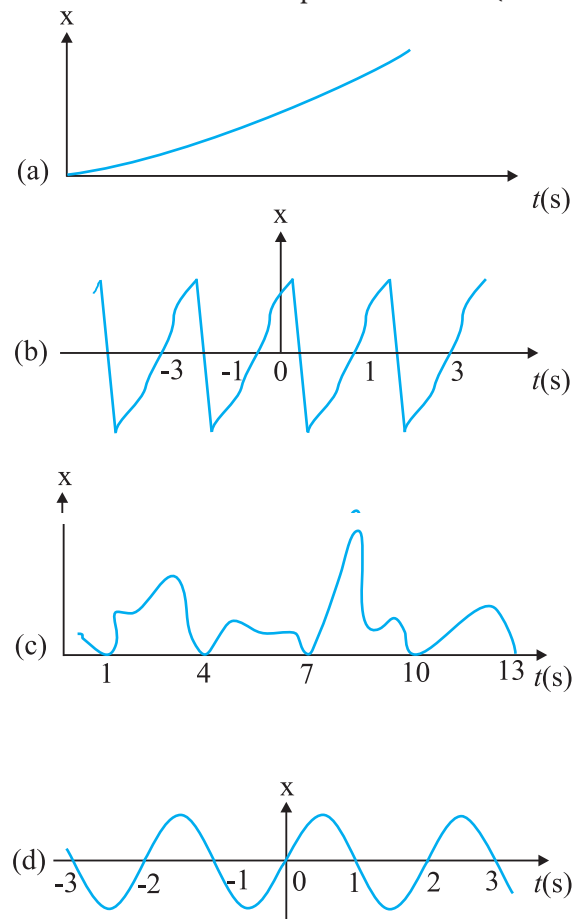


Fig. 14.27

- 14.4** Which of the following functions of time represent (a) simple harmonic, (b) periodic but not simple harmonic, and (c) non-periodic motion? Give period for each case of periodic motion (ω is any positive constant):
- (a) $\sin at - \cos at$
 - (b) $\sin^3 at$
 - (c) $3 \cos (\pi/4 - 2at)$
 - (d) $\cos at + \cos 3at + \cos 5at$
 - (e) $\exp (-\omega^2 t^2)$
 - (f) $1 + at + \omega^2 t^2$
- 14.5** A particle is in linear simple harmonic motion between two points, A and B, 10 cm apart. Take the direction from A to B as the positive direction and give the signs of velocity, acceleration and force on the particle when it is
- (a) at the end A,
 - (b) at the end B,
 - (c) at the mid-point of AB going towards A,
 - (d) at 2 cm away from B going towards A,
 - (e) at 3 cm away from A going towards B, and
 - (f) at 4 cm away from B going towards A.
- 14.6** Which of the following relationships between the acceleration a and the displacement x of a particle involve simple harmonic motion?
- (a) $a = 0.7x$
 - (b) $a = -200x^2$
 - (c) $a = -10x$
 - (d) $a = 100x^3$
- 14.7** The motion of a particle executing simple harmonic motion is described by the displacement function,
- $$x(t) = A \cos (\omega t + \phi).$$
- If the initial ($t = 0$) position of the particle is 1 cm and its initial velocity is ω cm/s, what are its amplitude and initial phase angle? The angular frequency of the particle is $\pi \text{ s}^{-1}$. If instead of the cosine function, we choose the sine function to describe the SHM : $x = B \sin (\omega t + \alpha)$, what are the amplitude and initial phase of the particle with the above initial conditions.
- 14.8** A spring balance has a scale that reads from 0 to 50 kg. The length of the scale is 20 cm. A body suspended from this balance, when displaced and released, oscillates with a period of 0.6 s. What is the weight of the body ?
- 14.9** A spring having with a spring constant 1200 N m^{-1} is mounted on a horizontal table as shown in Fig. 14.28. A mass of 3 kg is attached to the free end of the spring. The mass is then pulled sideways to a distance of 2.0 cm and released.

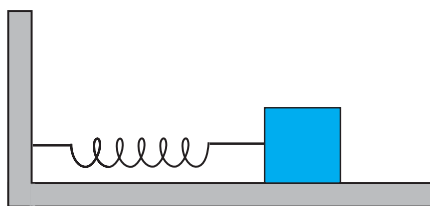


Fig. 14.28

- Determine (i) the frequency of oscillations, (ii) maximum acceleration of the mass, and (iii) the maximum speed of the mass.
- 14.10** In Exercise 14.9, let us take the position of mass when the spring is unstretched as $x = 0$, and the direction from left to right as the positive direction of x -axis. Give x as a function of time t for the oscillating mass if at the moment we start the stopwatch ($t = 0$), the mass is

- (a) at the mean position,
 (b) at the maximum stretched position, and
 (c) at the maximum compressed position.
 In what way do these functions for SHM differ from each other, in frequency, in amplitude or the initial phase?

14.11 Figures 14.29 correspond to two circular motions. The radius of the circle, the period of revolution, the initial position, and the sense of revolution (i.e. clockwise or anti-clockwise) are indicated on each figure.

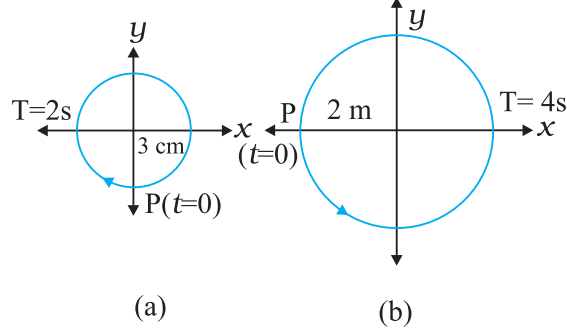


Fig. 14.29

Obtain the corresponding simple harmonic motions of the x -projection of the radius vector of the revolving particle P , in each case.

14.12 Plot the corresponding reference circle for each of the following simple harmonic motions. Indicate the initial ($t=0$) position of the particle, the radius of the circle, and the angular speed of the rotating particle. For simplicity, the sense of rotation may be fixed to be anticlockwise in every case: (x is in cm and t is in s).

- (a) $x = -2 \sin(3t + \pi/3)$
 (b) $x = \cos(\pi/6 - t)$
 (c) $x = 3 \sin(2\pi t + \pi/4)$
 (d) $x = 2 \cos \pi t$

14.13 Figure 14.30 (a) shows a spring of force constant k clamped rigidly at one end and a mass m attached to its free end. A force \mathbf{F} applied at the free end stretches the spring. Figure 14.30 (b) shows the same spring with both ends free and attached to a mass m at either end. Each end of the spring in Fig. 14.30(b) is stretched by the same force \mathbf{F} .

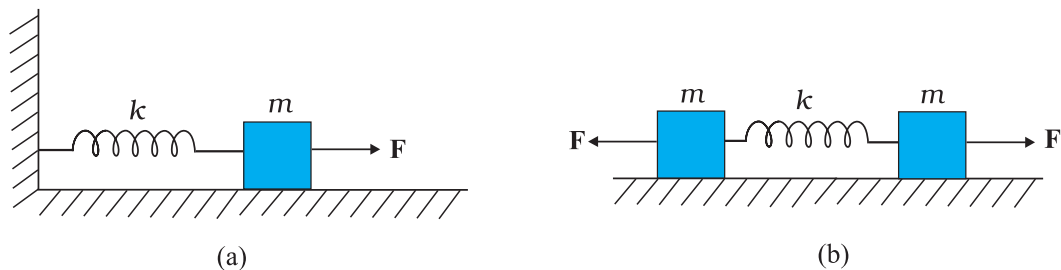


Fig. 14.30

- (a) What is the maximum extension of the spring in the two cases?
 (b) If the mass in Fig. (a) and the two masses in Fig. (b) are released, what is the period of oscillation in each case?

- 14.14** The piston in the cylinder head of a locomotive has a stroke (twice the amplitude) of 1.0 m. If the piston moves with simple harmonic motion with an angular frequency of 200 rad/min, what is its maximum speed ?
- 14.15** The acceleration due to gravity on the surface of moon is 1.7 m s^{-2} . What is the time period of a simple pendulum on the surface of moon if its time period on the surface of earth is 3.5 s ? (g on the surface of earth is 9.8 m s^{-2})

14.16 Answer the following questions :

- (a) Time period of a particle in SHM depends on the force constant k and mass m of the particle:

$T = 2\pi\sqrt{\frac{m}{k}}$. A simple pendulum executes SHM approximately. Why then is the time period of a pendulum independent of the mass of the pendulum?

- (b) The motion of a simple pendulum is approximately simple harmonic for small angle oscillations. For larger angles of oscillation, a more involved analysis

shows that T is greater than $2\pi\sqrt{\frac{l}{g}}$. Think of a qualitative argument to appreciate this result.

- (c) A man with a wristwatch on his hand falls from the top of a tower. Does the watch give correct time during the free fall ?
- (d) What is the frequency of oscillation of a simple pendulum mounted in a cabin that is freely falling under gravity ?

14.17 A simple pendulum of length l and having a bob of mass M is suspended in a car. The car is moving on a circular track of radius R with a uniform speed v . If the pendulum makes small oscillations in a radial direction about its equilibrium position, what will be its time period ?

14.18 A cylindrical piece of cork of density of base area A and height h floats in a liquid of density ρ_l . The cork is depressed slightly and then released. Show that the cork oscillates up and down simple harmonically with a period

$$T = 2\pi\sqrt{\frac{h\rho}{\rho_l g}}$$

where ρ is the density of cork. (Ignore damping due to viscosity of the liquid).

14.19 One end of a U-tube containing mercury is connected to a suction pump and the other end to atmosphere. A small pressure difference is maintained between the two columns. Show that, when the suction pump is removed, the column of mercury in the U-tube executes simple harmonic motion.

Additional Exercises

14.20 An air chamber of volume V has a neck area of cross section a into which a ball of mass m just fits and can move up and down without any friction (Fig. 14.33). Show that when the ball is pressed down a little and released, it executes SHM. Obtain an expression for the time period of oscillations assuming pressure-volume variations of air to be isothermal [see Fig. 14.33].

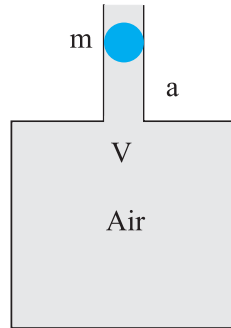


Fig.14.33

- 14.21** You are riding in an automobile of mass 3000 kg. Assuming that you are examining the oscillation characteristics of its suspension system. The suspension sags 15 cm when the entire automobile is placed on it. Also, the amplitude of oscillation decreases by 50% during one complete oscillation. Estimate the values of (a) the spring constant k and (b) the damping constant b for the spring and shock absorber system of one wheel, assuming that each wheel supports 750 kg.
- 14.22** Show that for a particle in linear SHM the average kinetic energy over a period of oscillation equals the average potential energy over the same period.
- 14.23** A circular disc of mass 10 kg is suspended by a wire attached to its centre. The wire is twisted by rotating the disc and released. The period of torsional oscillations is found to be 1.5 s. The radius of the disc is 15 cm. Determine the torsional spring constant of the wire. (Torsional spring constant α is defined by the relation $J = -\alpha \theta$, where J is the restoring couple and θ the angle of twist).
- 14.24** A body describes simple harmonic motion with an amplitude of 5 cm and a period of 0.2 s. Find the acceleration and velocity of the body when the displacement is (a) 5 cm, (b) 3 cm, (c) 0 cm.
- 14.25** A mass attached to a spring is free to oscillate, with angular velocity ω in a horizontal plane without friction or damping. It is pulled to a distance x_0 and pushed towards the centre with a velocity v_0 at time $t = 0$. Determine the amplitude of the resulting oscillations in terms of the parameters ω , x_0 and v_0 . [Hint : Start with the equation $x = a \cos(\omega t + \theta)$ and note that the initial velocity is negative.]