Chapter 3

TRIGONOMETRIC FUNCTIONS

❖*A mathematician knows how to solve a problem, he can not solve it. – MILNE* ❖

3.1 Introduction

Arya Bhatt (476-550)

The word 'trigonometry' is derived from the Greek words '*trigon*' and '*metron*' and it means 'measuring the sides of a triangle'. The subject was originally developed to solve geometric problems involving triangles. It was studied by sea captains for navigation, surveyor to map out the new lands, by engineers and others. Currently, trigonometry is used in many areas such as the science of seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analysing a musical tone and in many other areas.

In earlier classes, we have studied the trigonometric ratios of acute angles as the ratio of the sides of a right angled triangle. We have also studied the trigonometric identities and application of trigonometric ratios in solving the problems related to heights and distances. In this Chapter, we will generalise the concept of trigonometric ratios to trigonometric functions and study their properties.

3.2 Angles

Angle is a measure of rotation of a given ray about its initial point. The original ray is

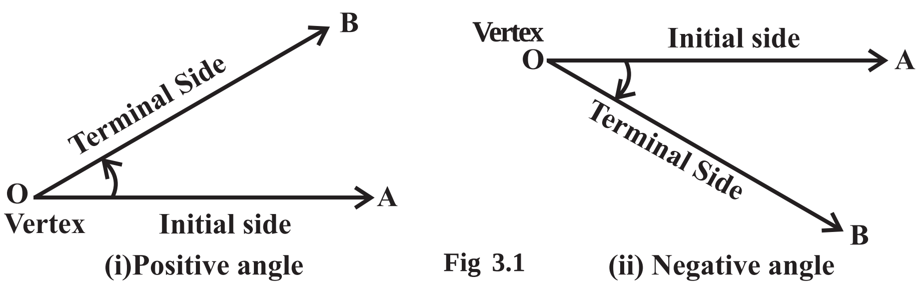


Fig 3.1

called the *initial side* and the final position of the ray after rotation is called the *terminal side* of the angle. The point of rotation is called the *vertex*. If the direction of rotation is anticlockwise, the angle is said to be positive and if the direction of rotation is clockwise, then the angle is *negative* (Fig 3.1).

The measure of an angle is the amount of rotation performed to get the terminal side from the initial side. There are several units for measuring angles. The definition of an angle suggests a unit, viz. *one complete revolution* from the position of the initial side as indicated in Fig 3.2.

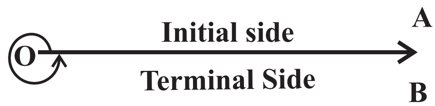


Fig 3.2

This is often convenient for large angles. For example, we can say that a rapidly spinning wheel is making an angle of say 15 revolution per second. We shall describe two other units of measurement of an angle which are most commonly used, viz. degree measure and radian measure.

3.2.1 *Degree measure*

If a rotation from the initial side to terminal side is $\left(\frac{1}{360}\right)^{th}$ of a revolution, the angle is said to have a measure of one *degree*, written as 1°. A degree is divided into 60 minutes, and a minute is divided into 60 seconds . One sixtieth of a degree is called a *minute*, written as 1′, and one sixtieth of a minute is called a *second*, written as 1″.

Thus, $1° = 60′$, $1′ = 60″$

Some of the angles whose measures are 360°,180°, 270°, 420°, – 30°, – 420° are shown in Fig 3.3.

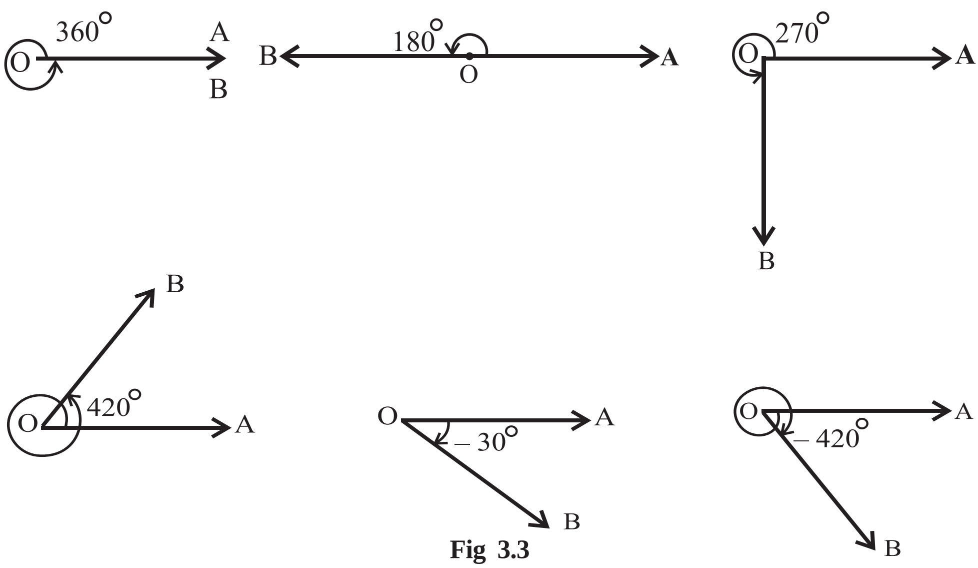


Fig 3.3

3.2.2 ***Radian measure*** There is another unit for measurement of an angle, called the *radian* measure. Angle subtended at the centre by an arc of length 1 unit in a unit circle (circle of radius 1 unit) is said to have a measure of 1 radian. In the Fig 3.4(i) to (iv), OA is the initial side and OB is the terminal side. The figures show the angles whose measures are 1 radian, –1 radian, $1\frac{1}{2}$ radian and $-1\frac{1}{2}$ radian.

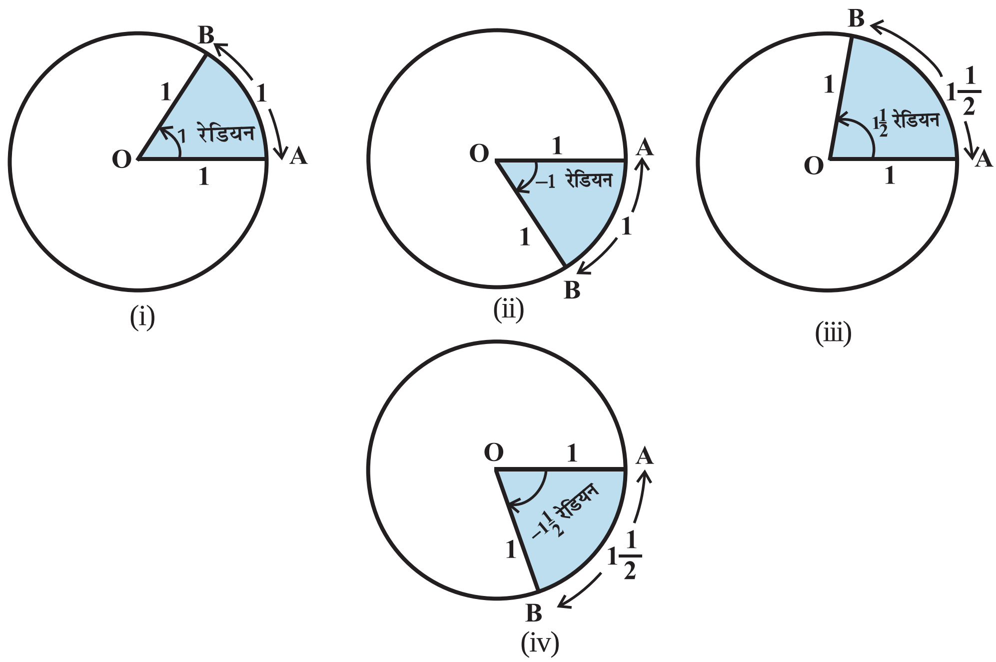


Fig 3.4 (i) to (iv)

We know that the circumference of a circle of radius 1 unit is 2π. Thus, one complete revolution of the initial side subtends an angle of 2π radian.

More generally, in a circle of radius r, an arc of length r will subtend an angle of 1 radian. It is well-known that equal arcs of a circle subtend equal angle at the centre. Since in a circle of radius r, an arc of length r subtends an angle whose measure is 1 radian, an arc of length l will subtend an angle whose measure is $\frac{l}{r}$ radian. Thus, if in a circle of radius r, an arc of length l subtends an angle θ radian at the centre, we have

$$\theta = \frac{l}{r} \text{ or } l = r\,\theta.$$

3.2.3 *Relation between radian and real numbers*

Consider the unit circle with centre O. Let A be any point on the circle. Consider OA as initial side of an angle. Then the length of an arc of the circle will give the radian measure of the angle which the arc will subtend at the centre of the circle. Consider the line PAQ which is tangent to the circle at A. Let the point A represent the real number zero, AP represents positive real number and AQ represents negative real numbers (Fig 3.5). If we rope the line AP in the anticlockwise direction along the circle, and AQ in the clockwise direction, then every real number will correspond to a radian measure and conversely. Thus, radian measures and real numbers can be considered as one and the same.

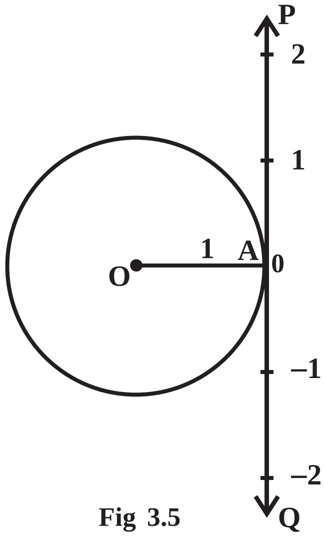


Fig 3.5

3.2.4 *Relation between degree and radian*

Since a circle subtends at the centre an angle whose radian measure is 2π and its degree measure is 360°, it follows that

$$2\pi \text{ radian} = 360^\circ \quad \text{or} \quad \pi \text{ radian} = 180^\circ$$

The above relation enables us to express a radian measure in terms of degree measure and a degree measure in terms of radian measure. Using approximate value of π as $\frac{22}{7}$, we have

$$1 \text{ radian} = \frac{180^\circ}{\pi} = 57^\circ\, 16' \text{ approximately.}$$

Also $\quad 1^\circ = \frac{\pi}{180}$ radian $= 0.01746$ radian approximately.

The relation between degree measures and radian measure of some common angles are given in the following table:

Degree	30°	45°	60°	90°	180°	270°	360°
Radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Notational Convention

Since angles are measured either in degrees or in radians, we adopt the convention that whenever we write angle $\theta°$, we mean the angle whose degree measure is θ and whenever we write angle β, we mean the angle whose radian measure is β.

Note that when an angle is expressed in radians, the word 'radian' is frequently omitted. Thus, $\pi = 180°$ and $\frac{\pi}{4} = 45°$ are written with the understanding that π and $\frac{\pi}{4}$ are radian measures. Thus, we can say that

$$\text{Radian measure} = \frac{\pi}{180} \times \text{Degree measure}$$

$$\text{Degree measure} = \frac{180}{\pi} \times \text{Radian measure}$$

Example 1 Convert $40°\,20'$ into radian measure.

Solution We know that $180° = \pi$ radian.

Hence $40°\,20' = 40\frac{1}{3}$ degree $= \frac{\pi}{180} \times \frac{121}{3}$ radian $= \frac{121\pi}{540}$ radian.

Therefore $40°\,20' = \frac{121\pi}{540}$ radian.

Example 2 Convert 6 radians into degree measure.

Solution We know that π radian = 180°.

Hence 6 radians $= \frac{180}{\pi} \times 6$ degree $= \frac{1080 \times 7}{22}$ degree

$= 343\frac{7}{11}$ degree $= 343° + \frac{7 \times 60}{11}$ minute [as $1° = 60'$]

$= 343° + 38' + \frac{2}{11}$ minute [as $1' = 60''$]

$= 343° + 38' + 10.9''$ $= 343°38'\,11''$ approximately.

Hence 6 radians = $343°\,38'\,11''$ approximately.

Example 3 Find the radius of the circle in which a central angle of 60° intercepts an arc of length 37.4 cm (use $\pi = \frac{22}{7}$).

Solution Here $l = 37.4$ cm and $\theta = 60° = \frac{60\pi}{180}$ radian $= \frac{\pi}{3}$

Hence, by $r = \frac{l}{\theta}$, we have

$$r = \frac{37.4 \times 3}{\pi} = \frac{37.4 \times 3 \times 7}{22} = 35.7 \text{ cm}$$

Example 4 The minute hand of a watch is 1.5 cm long. How far does its tip move in 40 minutes? (Use $\pi = 3.14$).

Solution In 60 minutes, the minute hand of a watch completes one revolution. Therefore, in 40 minutes, the minute hand turns through $\frac{2}{3}$ of a revolution. Therefore, $\theta = \frac{2}{3} \times 360°$ or $\frac{4\pi}{3}$ radian. Hence, the required distance travelled is given by

$$l = r\,\theta = 1.5 \times \frac{4\pi}{3} \text{ cm} = 2\pi \text{ cm} = 2 \times 3.14 \text{ cm} = 6.28 \text{ cm}.$$

Example 5 If the arcs of the same lengths in two circles subtend angles 65°and 110° at the centre, find the ratio of their radii.

Solution Let r_1 and r_2 be the radii of the two circles. Given that

$$\theta_1 = 65° = \frac{\pi}{180} \times 65 = \frac{13\pi}{36} \text{ radian}$$

and $$\theta_2 = 110° = \frac{\pi}{180} \times 110 = \frac{22\pi}{36} \text{ radian}$$

Let l be the length of each of the arc. Then $l = r_1\theta_1 = r_2\theta_2$, which gives

$$\frac{13\pi}{36} \times r_1 = \frac{22\pi}{36} \times r_2, \text{ i.e., } \frac{r_1}{r_2} = \frac{22}{13}$$

Hence $r_1 : r_2 = 22 : 13$.

EXERCISE 3.1

1. Find the radian measures corresponding to the following degree measures:

(i) 25° (ii) – 47°30′ (iii) 240° (iv) 520°

2. Find the degree measures corresponding to the following radian measures (Use $\pi = \frac{22}{7}$).

(i) $\frac{11}{16}$ (ii) -4 (iii) $\frac{5\pi}{3}$ (iv) $\frac{7\pi}{6}$

3. A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second?

4. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm (Use $\pi = \frac{22}{7}$).

5. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc of the chord.

6. If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii.

7. Find the angle in radian through which a pendulum swings if its length is 75 cm and th e tip describes an arc of length

(i) 10 cm (ii) 15 cm (iii) 21 cm

3.3 Trigonometric Functions

In earlier classes, we have studied trigonometric ratios for acute angles as the ratio of sides of a right angled triangle. We will now extend the definition of trigonometric ratios to any angle in terms of radian measure and study them as trigonometric functions.

Consider a unit circle with centre at origin of the coordinate axes. Let P (a, b) be any point on the circle with angle AOP = x radian, i.e., length of arc AP = x (Fig 3.6).

We define $\cos x = a$ and $\sin x = b$ Since ΔOMP is a right triangle, we have

$$OM^2 + MP^2 = OP^2 \text{ or } a^2 + b^2 = 1$$

Thus, for every point on the unit circle, we have

$$a^2 + b^2 = 1 \text{ or } \cos^2 x + \sin^2 x = 1$$

Since one complete revolution subtends an angle of 2π radian at the centre of the circle, $\angle AOB = \frac{\pi}{2}$,

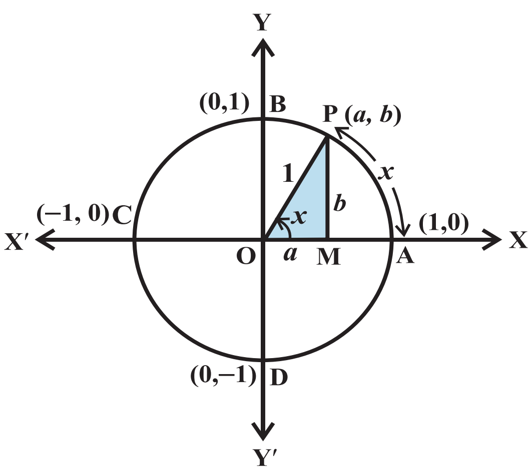


Fig 3.6

$\angle AOC = \pi$ and $\angle AOD = \frac{3\pi}{2}$. All angles which are integral multiples of $\frac{\pi}{2}$ are called *quadrantal angles*. The coordinates of the points A, B, C and D are, respectively, (1, 0), (0, 1), (–1, 0) and (0, –1). Therefore, for quadrantal angles, we have

$$\cos 0° = 1 \qquad \sin 0° = 0,$$

$$\cos \frac{\pi}{2} = 0 \qquad \sin \frac{\pi}{2} = 1$$

$$\cos\pi = -1 \qquad \sin\pi = 0$$

$$\cos \frac{3\pi}{2} = 0 \qquad \sin \frac{3\pi}{2} = -1$$

$$\cos 2\pi = 1 \qquad \sin 2\pi = 0$$

Now, if we take one complete revolution from the point P, we again come back to same point P. Thus, we also observe that if x increases (or decreases) by any integral multiple of 2π, the values of sine and cosine functions do not change. Thus,

$$\sin (2n\pi + x) = \sin x,\ n \in \mathbf{Z},\quad \cos (2n\pi + x) = \cos x,\ n \in \mathbf{Z}$$

Further, $\sin x = 0$, if $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, ...$, i.e., when x is an integral multiple of π and $\cos x = 0$, if $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, ...$ i.e., $\cos x$ vanishes when x is an odd multiple of $\frac{\pi}{2}$. Thus

$\sin x = 0$ implies $x = n\pi$, where n is any integer

$\cos x = 0$ implies $x = (2n + 1)\frac{\pi}{2}$, where n is any integer.

We now define other trigonometric functions in terms of sine and cosine functions:

$$\text{cosec } x = \frac{1}{\sin x},\ x \neq n\pi, \text{ where } n \text{ is any integer.}$$

$$\sec x = \frac{1}{\cos x},\ x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$

$$\tan x = \frac{\sin x}{\cos x},\ x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$

$$\cot x = \frac{\cos x}{\sin x},\ x \neq n\pi, \text{ where } n \text{ is any integer.}$$

We have shown that for all real x, $\sin^2 x + \cos^2 x = 1$

It follows that

$$1 + \tan^2 x = \sec^2 x \qquad \text{(why?)}$$

$$1 + \cot^2 x = \text{cosec}^2 x \qquad \text{(why?)}$$

In earlier classes, we have discussed the values of trigonometric ratios for 0°, 30°, 45°, 60° and 90°. The values of trigonometric functions for these angles are same as that of trigonometric ratios studied in earlier classes. Thus, we have the following table:

	0°	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	– 1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	– 1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0

The values of cosec x, sec x and cot x are the reciprocal of the values of sin x, cos x and tan x, respectively.

3.3.1 *Sign of trigonometric functions*

Let P (a, b) be a point on the unit circle with centre at the origin such that $\angle AOP = x$. If $\angle AOQ = -x$, then the coordinates of the point Q will be $(a, -b)$ (Fig 3.7). Therefore

$$\cos(-x) = \cos x$$

and $$\sin(-x) = -\sin x$$

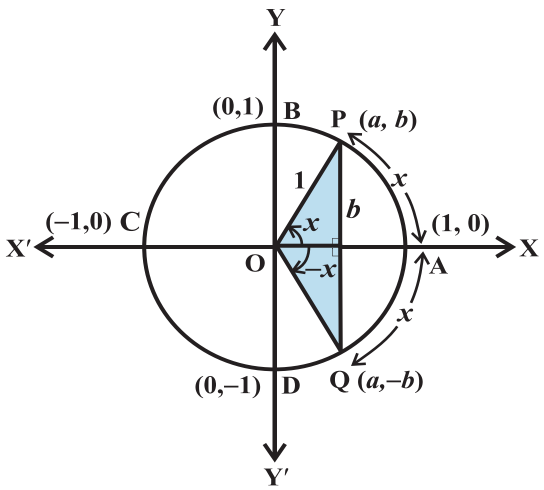


Fig 3.7

Since for every point P (a, b) on the unit circle, $-1 \le a \le 1$ and

$-1 \leq b \leq 1$, we have $-1 \leq \cos x \leq 1$ and $-1 \leq \sin x \leq 1$ for all x. We have learnt in previous classes that in the first quadrant ($0 < x < \frac{\pi}{2}$) a and b are both positive, in the second quadrant ($\frac{\pi}{2} < x < \pi$) a is negative and b is positive, in the third quadrant ($\pi < x < \frac{3\pi}{2}$) a and b are both negative and in the fourth quadrant ($\frac{3\pi}{2} < x < 2\pi$) a is positive and b is negative. Therefore, sin x is positive for $0 < x < \pi$, and negative for $\pi < x < 2\pi$. Similarly, cos x is positive for $0 < x < \frac{\pi}{2}$, negative for $\frac{\pi}{2} < x < \frac{3\pi}{2}$ and also positive for $\frac{3\pi}{2} < x < 2\pi$. Likewise, we can find the signs of other trigonometric functions in different quadrants. In fact, we have the following table.

	I	II	III	IV	
sin x	+	+		–	
cos x	+		–	– +	
tan x	+		– +		–
cosec x	+	+		–	–
sec x	+		–	– +	
cot x	+		– +		–

3.3.2 ***Domain and range of trigonometric functions*** From the definition of sine and cosine functions, we observe that they are defined for all real numbers. Further, we observe that for each real number x,

$$-1 \leq \sin x \leq 1 \text{ and } -1 \leq \cos x \leq 1$$

Thus, domain of $y = \sin x$ and $y = \cos x$ is the set of all real numbers and range is the interval $[-1, 1]$, i.e., $-1 \leq y \leq 1$.

Since cosec $x = \frac{1}{\sin x}$, the domain of y = cosec x is the set $\{ x : x \in \mathbf{R}$ and $x \neq n\pi, n \in \mathbf{Z}\}$ and range is the set $\{y : y \in \mathbf{R}, y \geq 1$ or $y \leq -1\}$. Similarly, the domain of y = sec x is the set $\{x : x \in \mathbf{R}$ and $x \neq (2n + 1)\frac{\pi}{2}, n \in \mathbf{Z}\}$ and range is the set $\{y : y \in \mathbf{R}, y \leq -1$ or $y \geq 1\}$. The domain of y = tan x is the set $\{x : x \in \mathbf{R}$ and $x \neq (2n + 1)\frac{\pi}{2}, n \in \mathbf{Z}\}$ and range is the set of all real numbers. The domain of y = cot x is the set $\{x : x \in \mathbf{R}$ and $x \neq n\pi, n \in \mathbf{Z}\}$ and the range is the set of all real numbers.

We further observe that in the first quadrant, as x increases from 0 to $\frac{\pi}{2}$, sin x increases from 0 to 1, as x increases from $\frac{\pi}{2}$ to π, sin x decreases from 1 to 0. In the third quadrant, as x increases from π to $\frac{3\pi}{2}$, sin x decreases from 0 to –1and finally, in the fourth quadrant, sin x increases from –1 to 0 as x increases from $\frac{3\pi}{2}$ to 2π. Similarly, we can discuss the behaviour of other trigonometric functions. In fact, we have the following table:

	I quadrant	II quadrant	III quadrant	IV quadrant
sin	increases from 0 to 1	decreases from 1 to 0	decreases from 0 to –1	increases from –1 to 0
cos	decreases from 1 to 0	decreases from 0 to – 1	increases from –1 to 0	increases from 0 to 1
tan	increases from 0 to ∞	increases from –∞to 0	increases from 0 to ∞	increases from –∞to 0
cot	decreases from ∞ to 0	decreases from 0 to–∞	decreases from ∞ to 0	decreases from 0to –∞
sec	increases from 1 to ∞	increases from –∞to–1	decreases from –1to–∞	decreases from ∞ to 1
cosec	decreases from ∞ to 1	increases from 1 to ∞	increases from –∞to–1	decreases from–1to–∞

Remark In the above table, the statement tan x increases from 0 to ∞ (infinity) for $0 < x < \frac{\pi}{2}$ simply means that tan x increases as x increases for $0 < x < \frac{\pi}{2}$ and assumes arbitraily large positive values as x approaches to $\frac{\pi}{2}$. Similarly, to say that cosec x decreases from –1 to $-\infty$ (minus infinity) in the fourth quadrant means that cosec x decreases for $x \in (\frac{3\pi}{2}, 2\pi)$ and assumes arbitrarily large negative values as x approaches to 2π. The symbols ∞ and $-\infty$ simply specify certain types of behaviour of functions and variables.

We have already seen that values of sin x and cos x repeats after an interval of 2π. Hence, values of cosec x and sec x will also repeat after an interval of 2π. We

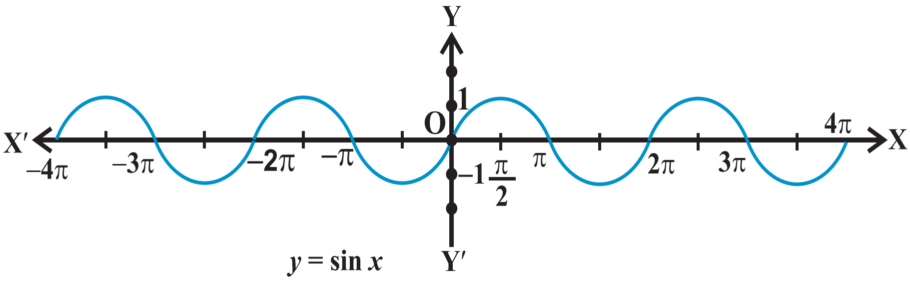


Fig 3.8

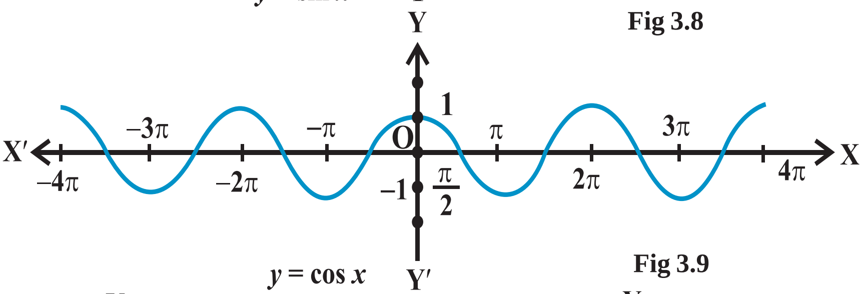


Fig 3.9

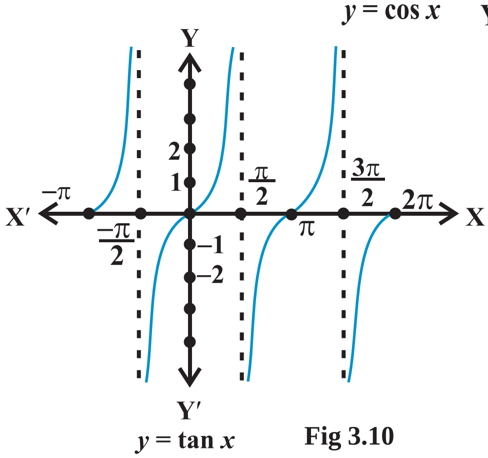


Fig 3.10

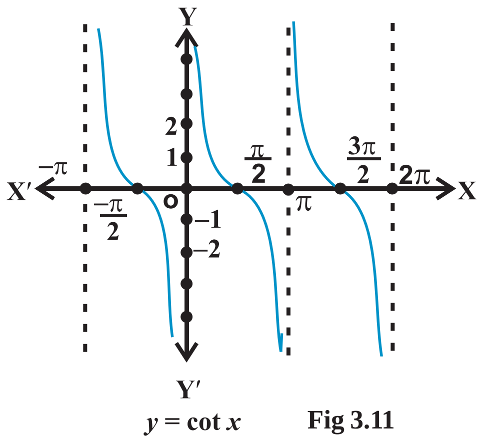
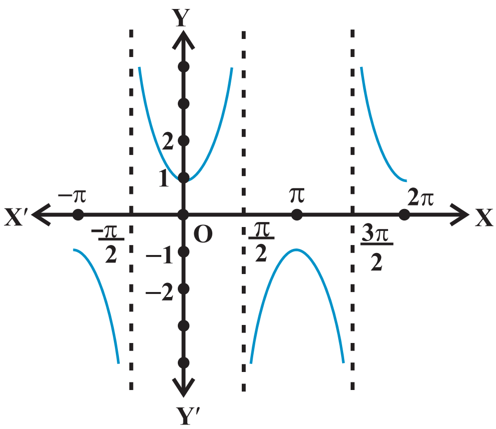
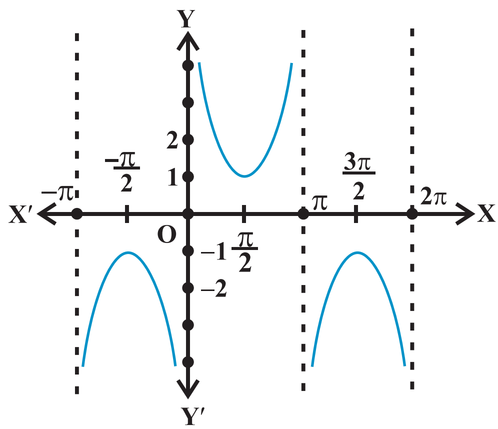


Fig 3.11



$y = \sec x$

Fig 3.12



$y = \text{cosec}\, x$

Fig 3.13

shall see in the next section that tan $(\pi + x) = \tan x$. Hence, values of tan x will repeat after an interval of π. Since cot x is reciprocal of tan x, its values will also repeat after an interval of π. Using this knowledge and behaviour of trigonometric functions, we can sketch the graph of these functions. The graph of these functions are given above:

Example 6 If $\cos x = -\frac{3}{5}$, x lies in the third quadrant, find the values of other five trigonometric functions.

Solution Since $\cos x = -\frac{3}{5}$, we have $\sec x = -\frac{5}{3}$

Now $\quad \sin^2 x + \cos^2 x = 1$, i.e., $\sin^2 x = 1 - \cos^2 x$

or $\quad \sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

Hence $\quad \sin x = \pm \frac{4}{5}$

Since x lies in third quadrant, sin x is negative. Therefore

$$\sin x = -\frac{4}{5}$$

which also gives

$$\text{cosec}\, x = -\frac{5}{4}$$

Further, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{4}{3} \quad \text{and} \quad \cot x = \frac{\cos x}{\sin x} = \frac{3}{4}.$$

Example 7 If $\cot x = -\frac{5}{12}$, x lies in second quadrant, find the values of other five trigonometric functions.

Solution Since $\cot x = -\frac{5}{12}$, we have $\tan x = -\frac{12}{5}$

Now $\sec^2 x = 1 + \tan^2 x = 1 + \frac{144}{25} = \frac{169}{25}$

Hence $\sec x = \pm \frac{13}{5}$

Since x lies in second quadrant, $\sec x$ will be negative. Therefore

$$\sec x = -\frac{13}{5},$$

which also gives

$$\cos x = -\frac{5}{13}$$

Further, we have

$$\sin x = \tan x \cos x = \left(-\frac{12}{5}\right) \times \left(-\frac{5}{13}\right) = \frac{12}{13}$$

and $\operatorname{cosec} x = \frac{1}{\sin x} = \frac{13}{12}.$

Example 8 Find the value of $\sin \frac{31\pi}{3}$.

Solution We know that values of $\sin x$ repeats after an interval of 2π. Therefore

$$\sin \frac{31\pi}{3} = \sin \left(10\pi + \frac{\pi}{3}\right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Example 9 Find the value of cos (–1710°).

Solution We know that values of cos x repeats after an interval of 2π or 360°. Therefore, $\cos(-1710°) = \cos(-1710° + 5 \times 360°)$

$$= \cos(-1710° + 1800°) = \cos 90° = 0.$$

EXERCISE 3.2

Find the values of other five trigonometric functions in Exercises 1 to 5.

1. $\cos x = -\frac{1}{2}$, x lies in third quadrant.

2. $\sin x = \frac{3}{5}$, x lies in second quadrant.

3. $\cot x = \frac{3}{4}$, x lies in third quadrant.

4. $\sec x = \frac{13}{5}$, x lies in fourth quadrant.

5. $\tan x = -\frac{5}{12}$, x lies in second quadrant.

Find the values of the trigonometric functions in Exercises 6 to 10.

6. sin 765°

7. cosec (– 1410°)

8. $\tan \frac{19\pi}{3}$

9. $\sin\left(-\frac{11\pi}{3}\right)$

10. $\cot\left(-\frac{15\pi}{4}\right)$

3.4 Trigonometric Functions of Sum and Difference of Two Angles

In this Section, we shall derive expressions for trigonometric functions of the sum and difference of two numbers (angles) and related expressions. The basic results in this connection are called *trigonometric identities*. We have seen that

1. $\mathbf{\sin(-x) = -\sin x}$

2. $\mathbf{\cos(-x) = \cos x}$

We shall now prove some more results:

3. $\cos(x + y) = \cos x \cos y - \sin x \sin y$

Consider the unit circle with centre at the origin. Let x be the angle P_4OP_1 and y be the angle P_1OP_2. Then $(x + y)$ is the angle P_4OP_2. Also let $(-y)$ be the angle P_4OP_3. Therefore, P_1, P_2, P_3 and P_4 will have the coordinates $P_1(\cos x, \sin x)$, $P_2[\cos(x + y), \sin(x + y)]$, $P_3[\cos(-y), \sin(-y)]$ and $P_4(1, 0)$ (Fig 3.14).

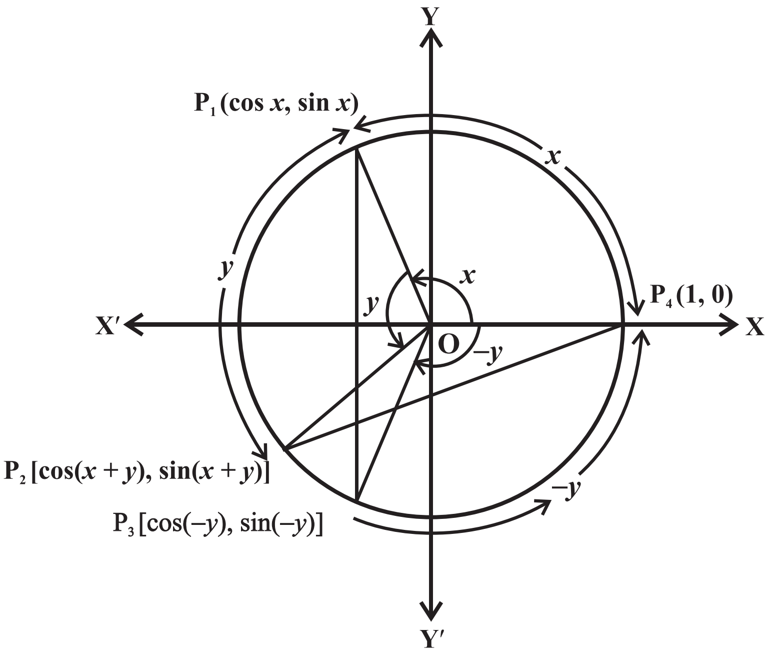


Fig 3.14

Consider the triangles P_1OP_3 and P_2OP_4. They are congruent (Why?). Therefore, P_1P_3 and P_2P_4 are equal. By using distance formula, we get

$$P_1P_3^2 = [\cos x - \cos(-y)]^2 + [\sin x - \sin(-y)]^2$$

$$= (\cos x - \cos y)^2 + (\sin x + \sin y)^2$$

$$= \cos^2 x + \cos^2 y - 2\cos x \cos y + \sin^2 x + \sin^2 y + 2\sin x \sin y$$

$$= 2 - 2(\cos x \cos y - \sin x \sin y) \qquad \text{(Why?)}$$

Also, $$P_2P_4^2 = [1 - \cos(x + y)]^2 + [0 - \sin(x + y)]^2$$

$$= 1 - 2\cos(x + y) + \cos^2(x + y) + \sin^2(x + y)$$

$$= 2 - 2\cos(x + y)$$

Since $P_1P_3 = P_2P_4$, we have $P_1P_3^2 = P_2P_4^2$.

Therefore, $2 - 2 (\cos x \cos y - \sin x \sin y) = 2 - 2 \cos (x + y)$.

Hence $\cos (x + y) = \cos x \cos y - \sin x \sin y$

4. $\cos (x - y) = \cos x \cos y + \sin x \sin y$

Replacing y by $-y$ in identity 3, we get

$$\cos (x + (-y)) = \cos x \cos (-y) - \sin x \sin (-y)$$

or $$\cos (x - y) = \cos x \cos y + \sin x \sin y$$

5. $\cos \left(\frac{\pi}{2} - x\right) = \sin x$

If we replace x by $\frac{\pi}{2}$ and y by x in Identity (4), we get

$$\cos \left(\frac{\pi}{2} - x\right) = \cos \frac{\pi}{2} \cos x + \sin \frac{\pi}{2} \sin x = \sin x.$$

6. $\sin \left(\frac{\pi}{2} - x\right) = \cos x$

Using the Identity 5, we have

$$\sin \left(\frac{\pi}{2} - x\right) = \cos \left[\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right] = \cos x.$$

7. $\sin (x + y) = \sin x \cos y + \cos x \sin y$

We know that

$$\sin (x + y) = \cos \left(\frac{\pi}{2} - (x + y)\right) = \cos \left(\left(\frac{\pi}{2} - x\right) - y\right)$$

$$= \cos \left(\frac{\pi}{2} - x\right) \cos y + \sin \left(\frac{\pi}{2} - x\right) \sin y$$

$$= \sin x \cos y + \cos x \sin y$$

8. $\sin (x - y) = \sin x \cos y - \cos x \sin y$

If we replace y by $-y$, in the Identity 7, we get the result.

9. By taking suitable values of x and y in the identities 3, 4, 7 and 8, we get the following results:

$$\cos \left(\frac{\pi}{2} + x\right) = -\sin x \qquad \sin \left(\frac{\pi}{2} + x\right) = \cos x$$

$$\cos (\pi - x) = -\cos x \qquad \sin (\pi - x) = \sin x$$

$\cos(\pi + x) = -\cos x$ $\sin(\pi + x) = -\sin x$

$\cos(2\pi - x) = \cos x$ $\sin(2\pi - x) = -\sin x$

Similar results for tan x, cot x, sec x and cosec x can be obtained from the results of sin x and cos x.

10. If none of the angles x, y and $(x + y)$ is an odd multiple of $\frac{\pi}{2}$, then

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Since none of the x, y and $(x + y)$ is an odd multiple of $\frac{\pi}{2}$, it follows that cos x, cos y and cos $(x + y)$ are non-zero. Now

$$\tan(x + y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

Dividing numerator and denominator by cos x cos y, we have

$$\tan(x + y) = \frac{\frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

11. $$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

If we replace y by $-y$ in Identity 10, we get

$$\tan(x - y) = \tan[x + (-y)]$$

$$= \frac{\tan x + \tan(-y)}{1 - \tan x \tan(-y)} = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

12. If none of the angles x, y and $(x + y)$ is a multiple of π, then

$$\cot(x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

Since, none of the x, y and $(x + y)$ is multiple of π, we find that $\sin x$ $\sin y$ and $\sin (x + y)$ are non-zero. Now,

$$\cot (x + y) = \frac{\cos (x + y)}{\sin (x + y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y}$$

Dividing numerator and denominator by $\sin x \sin y$, we have

$$\cot (x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

13. $\cot (x - y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$ if none of angles x, y and $x - y$ is a multiple of π

If we replace y by $-y$ in identity 12, we get the result

14. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

We know that

$$\cos (x + y) = \cos x \cos y - \sin x \sin y$$

Replacing y by x, we get

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1$$

Again, $\cos 2x = \cos^2 x - \sin^2 x$

$$= 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x.$$

We have $\cos 2x = \cos^2 x - \sin^2 x = \dfrac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x}$

Dividing numerator and denominator by $\cos^2 x$, we get

$$\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}, \quad x \neq n\pi + \frac{\pi}{2}, \text{ where n is an integer}$$

15. $\sin 2x = 2 \sin x \cos x = \dfrac{2\tan x}{1 + \tan^2 x}$ $x \neq n\pi + \dfrac{\pi}{2}$, where n is an integer

We have

$$\sin (x + y) = \sin x \cos y + \cos x \sin y$$

Replacing y by x, we get $\sin 2x = 2 \sin x \cos x$.

Again $\sin 2x = \dfrac{2\sin x \cos x}{\cos^2 x + \sin^2 x}$

Dividing each term by $\cos^2 x$, we get

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

16. $\mathbf{\tan 2x = \frac{2\tan x}{1-\tan^2 x}}$ if $2x \neq n\pi + \frac{\pi}{2}$, where n is an integer

We know that

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Replacing y by x , we get $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

17. $\mathbf{\sin 3x = 3\sin x - 4\sin^3 x}$

We have,

$$\begin{aligned}\sin 3x &= \sin(2x + x)\\ &= \sin 2x \cos x + \cos 2x \sin x\\ &= 2\sin x\cos x\cos x + (1 - 2\sin^2 x)\sin x\\ &= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x\\ &= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x\\ &= 3\sin x - 4\sin^3 x\end{aligned}$$

18. $\mathbf{\cos 3x = 4\cos^3 x - 3\cos x}$

We have,

$$\begin{aligned}\cos 3x &= \cos(2x + x)\\ &= \cos 2x \cos x - \sin 2x \sin x\\ &= (2\cos^2 x - 1)\cos x - 2\sin x\cos x\sin x\\ &= (2\cos^2 x - 1)\cos x - 2\cos x(1 - \cos^2 x)\\ &= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x\\ &= 4\cos^3 x - 3\cos x.\end{aligned}$$

19. $\mathbf{\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}}$ if $3x \neq n\pi + \frac{\pi}{2}$, where n is an integer

We have $\tan 3x = \tan(2x + x)$

$$= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} = \frac{\dfrac{2\tan x}{1-\tan^2 x} + \tan x}{1 - \dfrac{2\tan x \cdot \tan x}{1-\tan^2 x}}$$

$$= \frac{2\tan x + \tan x - \tan^3 x}{1 - \tan^2 x - 2\tan^2 x} = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

20. **(i)** $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$

(ii) $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

(iii) $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$

(iv) $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

We know that

$\cos(x+y) = \cos x \cos y - \sin x \sin y$ ... (1)

and $\cos(x-y) = \cos x \cos y + \sin x \sin y$ ... (2)

Adding and subtracting (1) and (2), we get

$\cos(x+y) + \cos(x-y) = 2\cos x \cos y$ ... (3)

and $\cos(x+y) - \cos(x-y) = -2\sin x \sin y$ ... (4)

Further $\sin(x+y) = \sin x \cos y + \cos x \sin y$ ... (5)

and $\sin(x-y) = \sin x \cos y - \cos x \sin y$ ... (6)

Adding and subtracting (5) and (6), we get

$\sin(x+y) + \sin(x-y) = 2\sin x \cos y$ ... (7)

$\sin(x+y) - \sin(x-y) = 2\cos x \sin y$ ... (8)

Let $x + y = \theta$ and $x - y = \phi$. Therefore

$$x = \left(\frac{\theta+\phi}{2}\right) \text{ and } y = \left(\frac{\theta-\phi}{2}\right)$$

Substituting the values of x and y in (3), (4), (7) and (8), we get

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

Since θ and ϕ can take any real values, we can replace θ by x and ϕ by y.
Thus, we get

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}\ ;\ \cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2},$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}\ ;\ \sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}.$$

Remark As a part of identities given in 20, we can prove the following results:

21. **(i)** $2\cos x\cos y = \cos(x+y) + \cos(x-y)$

(ii) $-2\sin x\sin y = \cos(x+y) - \cos(x-y)$

(iii) $2\sin x\cos y = \sin(x+y) + \sin(x-y)$

(iv) $2\cos x\sin y = \sin(x+y) - \sin(x-y)$.

Example 10 Prove that

$$3\sin\frac{\pi}{6}\sec\frac{\pi}{3} - 4\sin\frac{5\pi}{6}\cot\frac{\pi}{4} = 1$$

Solution We have

$$\text{L.H.S.} = 3\sin\frac{\pi}{6}\sec\frac{\pi}{3} - 4\sin\frac{5\pi}{6}\cot\frac{\pi}{4}$$

$$= 3 \times \frac{1}{2} \times 2 - 4\sin\left(\pi - \frac{\pi}{6}\right) \times 1 = 3 - 4\sin\frac{\pi}{6}$$

$$= 3 - 4 \times \frac{1}{2} = 1 = \text{R.H.S.}$$

Example 11 Find the value of sin 15°.

Solution We have

$$\sin 15° = \sin(45° - 30°)$$

$$= \sin 45°\cos 30° - \cos 45°\sin 30°$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Example 12 Find the value of $\tan\frac{13\pi}{12}$.

Solution We have

$$\tan\frac{13\pi}{12} = \tan\left(\pi+\frac{\pi}{12}\right) = \tan\frac{\pi}{12} = \tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$$

$$= \frac{\tan\frac{\pi}{4}-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$$

Example 13 Prove that

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x+\tan y}{\tan x-\tan y}.$$

Solution We have

$$\text{L.H.S.} \quad = \frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x\cos y+\cos x\sin y}{\sin x\cos y-\cos x\sin y}$$

Dividing the numerator and denominator by cos x cos y, we get

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x+\tan y}{\tan x-\tan y}.$$

Example 14 Show that

$$\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$$

Solution We know that $3x = 2x + x$

Therefore, $\tan 3x = \tan (2x + x)$

or $$\tan 3x = \frac{\tan 2x+\tan x}{1-\tan 2x\tan x}$$

or $\tan 3x - \tan 3x \tan 2x \tan x = \tan 2x + \tan x$

or $\tan 3x - \tan 2x - \tan x = \tan 3x \tan 2x \tan x$

or $\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$.

Example 15 Prove that

$$\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$$

Solution Using the Identity 20(i), we have

$$\text{L.H.S.} = \cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)$$

$$= 2\cos\left(\frac{\frac{\pi}{4}+x+\frac{\pi}{4}-x}{2}\right)\cos\left(\frac{\frac{\pi}{4}+x-(\frac{\pi}{4}-x)}{2}\right)$$

$$= 2\cos\frac{\pi}{4}\cos x = 2\times\frac{1}{\sqrt{2}}\cos x = \sqrt{2}\cos x = \text{R.H.S.}$$

Example 16 Prove that $\dfrac{\cos 7x+\cos 5x}{\sin 7x-\sin 5x}=\cot x$

Solution Using the Identities 20 (i) and 20 (iv), we get

$$\text{L.H.S.} = \frac{2\cos\frac{7x+5x}{2}\cos\frac{7x-5x}{2}}{2\cos\frac{7x+5x}{2}\sin\frac{7x-5x}{2}} = \frac{\cos x}{\sin x} = \cot x = \text{R.H.S.}$$

Example 17 Prove that $=\dfrac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x}=\tan x$

Solution We have

$$\text{L.H.S.} = \frac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x} = \frac{\sin 5x+\sin x-2\sin 3x}{\cos 5x-\cos x}$$

$$= \frac{2\sin 3x\cos 2x-2\sin 3x}{-2\sin 3x\sin 2x} = -\frac{\sin 3x\,(\cos 2x-1)}{\sin 3x\sin 2x}$$

$$= \frac{1-\cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x\cos x} = \tan x = \text{R.H.S.}$$

EXERCISE 3.3

Prove that:

1. $\sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{3} - \tan^2 \frac{\pi}{4} = -\frac{1}{2}$

2. $2\sin^2 \frac{\pi}{6} + \text{cosec}^2 \frac{7\pi}{6} \cos^2 \frac{\pi}{3} = \frac{3}{2}$

3. $\cot^2 \frac{\pi}{6} + \text{cosec} \frac{5\pi}{6} + 3\tan^2 \frac{\pi}{6} = 6$

4. $2\sin^2 \frac{3\pi}{4} + 2\cos^2 \frac{\pi}{4} + 2\sec^2 \frac{\pi}{3} = 10$

5. Find the value of:

(i) sin 75° (ii) tan 15°

Prove the following:

6. $\cos\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{4} - y\right) - \sin\left(\frac{\pi}{4} - x\right)\sin\left(\frac{\pi}{4} - y\right) = \sin(x + y)$

7. $\dfrac{\tan\left(\frac{\pi}{4} + x\right)}{\tan\left(\frac{\pi}{4} - x\right)} = \left(\dfrac{1 + \tan x}{1 - \tan x}\right)^2$

8. $\dfrac{\cos(\pi + x)\cos(-x)}{\sin(\pi - x)\cos\left(\frac{\pi}{2} + x\right)} = \cot^2 x$

9. $\cos\left(\frac{3\pi}{2} + x\right)\cos(2\pi + x)\left[\cot\left(\frac{3\pi}{2} - x\right) + \cot(2\pi + x)\right] = 1$

10. $\sin(n + 1)x \sin(n + 2)x + \cos(n + 1)x \cos(n + 2)x = \cos x$

11. $\cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) = -\sqrt{2}\sin x$

12. $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$

13. $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$

14. $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \sin 4x$

15. $\cot 4x (\sin 5x + \sin 3x) = \cot x (\sin 5x - \sin 3x)$

16. $\dfrac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\dfrac{\sin 2x}{\cos 10x}$

17. $\dfrac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$

18. $\dfrac{\sin x - \sin y}{\cos x + \cos y} = \tan \dfrac{x - y}{2}$

19. $\dfrac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$

20. $\dfrac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2\sin x$

21. $\dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$

22. $\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$

23. $\tan 4x = \dfrac{4\tan x\,(1-\tan^2 x)}{1-6\tan^2 x+\tan^4 x}$

24. $\cos 4x = 1 - 8\sin^2 x \cos^2 x$

25. $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$

3.5 Trigonometric Equations

Equations involving trigonometric functions of a variable are called *trigonometric equations*. In this Section, we shall find the solutions of such equations. We have already learnt that the values of $\sin x$ and $\cos x$ repeat after an interval of 2π and the values of $\tan x$ repeat after an interval of π. The solutions of a trigonometric equation for which $0 \le x < 2\pi$ are called *principal solutions*. The expression involving integer 'n' which gives all solutions of a trigonometric equation is called the *general solution***.** We shall use '**Z**' to denote the set of integers.

The following examples will be helpful in solving trigonometric equations:

Example 18 Find the principal solutions of the equation $\sin x = \dfrac{\sqrt{3}}{2}$.

Solution We know that, $\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$ and $\sin\dfrac{2\pi}{3} = \sin\left(\pi - \dfrac{\pi}{3}\right) = \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$.

Therefore, principal solutions are $x = \dfrac{\pi}{3}$ and $\dfrac{2\pi}{3}$.

Example 19 Find the principal solutions of the equation $\tan x = -\dfrac{1}{\sqrt{3}}$.

Solution We know that, $\tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$. Thus, $\tan\left(\pi - \dfrac{\pi}{6}\right) = -\tan\dfrac{\pi}{6} = -\dfrac{1}{\sqrt{3}}$

and $\tan\left(2\pi - \dfrac{\pi}{6}\right) = -\tan\dfrac{\pi}{6} = -\dfrac{1}{\sqrt{3}}$

Thus $\tan\dfrac{5\pi}{6} = \tan\dfrac{11\pi}{6} = -\dfrac{1}{\sqrt{3}}$.

Therefore, principal solutions are $\dfrac{5\pi}{6}$ and $\dfrac{11\pi}{6}$.

We will now find the general solutions of trigonometric equations. We have already seen that:

$\sin x = 0$ gives $x = n\pi$, where $n \in \mathbf{Z}$

$\cos x = 0$ gives $x = (2n + 1)\frac{\pi}{2}$, where $n \in \mathbf{Z}$.

We shall now prove the following results:

Theorem 1 For any real numbers x and y,

$$\sin x = \sin y \text{ implies } x = n\pi + (-1)^n y, \text{ where } n \in \mathbf{Z}$$

Proof If $\sin x = \sin y$, then

$$\sin x - \sin y = 0 \text{ or } 2\cos\frac{x+y}{2}\sin\frac{x-y}{2} = 0$$

which gives $\cos\frac{x+y}{2} = 0$ or $\sin\frac{x-y}{2} = 0$

Therefore $\frac{x+y}{2} = (2n+1)\frac{\pi}{2}$ or $\frac{x-y}{2} = n\pi$, where $n \in \mathbf{Z}$

i.e. $x = (2n + 1)\pi - y$ or $x = 2n\pi + y$, where $n \in \mathbf{Z}$

Hence $x = (2n + 1)\pi + (-1)^{2n+1} y$ or $x = 2n\pi + (-1)^{2n} y$, where $n \in \mathbf{Z}$.

Combining these two results, we get

$$x = n\pi + (-1)^n y, \text{ where } n \in \mathbf{Z}.$$

Theorem 2 For any real numbers x and y, $\cos x = \cos y$, implies $x = 2n\pi \pm y$, where $n \in \mathbf{Z}$

Proof If $\cos x = \cos y$, then

$$\cos x - \cos y = 0 \quad \text{i.e.,} \quad -2\sin\frac{x+y}{2}\sin\frac{x-y}{2} = 0$$

Thus $\sin\frac{x+y}{2} = 0$ or $\sin\frac{x-y}{2} = 0$

Therefore $\frac{x+y}{2} = n\pi$ or $\frac{x-y}{2} = n\pi$, where $n \in \mathbf{Z}$

i.e. $x = 2n\pi - y$ or $x = 2n\pi + y$, where $n \in \mathbf{Z}$

Hence $x = 2n\pi \pm y$, where $n \in \mathbf{Z}$

Theorem 3 Prove that if x and y are not odd mulitple of $\frac{\pi}{2}$, then

$$\tan x = \tan y \text{ implies } x = n\pi + y, \text{ where } n \in \mathbf{Z}$$

Proof If $\tan x = \tan y$, then $\tan x - \tan y = 0$

or $$\frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y} = 0$$

which gives $\sin(x - y) = 0$ (Why?)

Therefore $x - y = n\pi$, i.e., $x = n\pi + y$, where $n \in \mathbf{Z}$

Example 20 Find the solution of $\sin x = -\frac{\sqrt{3}}{2}$.

Solution We have $\sin x = -\frac{\sqrt{3}}{2} = -\sin\frac{\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = \sin\frac{4\pi}{3}$

Hence $\sin x = \sin\frac{4\pi}{3}$, which gives

$$x = n\pi + (-1)^n \frac{4\pi}{3}, \text{ where } n \in \mathbf{Z}.$$

> **Note** $\frac{4\pi}{3}$ is one such value of x for which $\sin x = -\frac{\sqrt{3}}{2}$. One may take any other value of x for which $\sin x = -\frac{\sqrt{3}}{2}$. The solutions obtained will be the same although these may apparently look different.

Example 21 Solve $\cos x = \frac{1}{2}$.

Solution We have, $\cos x = \frac{1}{2} = \cos\frac{\pi}{3}$

Therefore $x = 2n\pi \pm \frac{\pi}{3}$, where $n \in \mathbf{Z}$.

Example 22 Solve $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right)$.

Solution We have, $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right) = \tan\left(\frac{\pi}{2} + x + \frac{\pi}{3}\right)$

or $\tan 2x = \tan\left(x + \frac{5\pi}{6}\right)$

Therefore $2x = n\pi + x + \frac{5\pi}{6}$, where $n \in \mathbf{Z}$

or $x = n\pi + \frac{5\pi}{6}$, where $n \in \mathbf{Z}$.

Example 23 Solve sin $2x$ – sin $4x$ + sin $6x$ = 0.

Solution The equation can be written as

$$\sin 6x + \sin 2x - \sin 4x = 0$$

or $2\sin 4x \cos 2x - \sin 4x = 0$

i.e. $\sin 4x(2\cos 2x - 1) = 0$

Therefore $\sin 4x = 0$ or $\cos 2x = \frac{1}{2}$

i.e. $\sin 4x = 0$ or $\cos 2x = \cos\frac{\pi}{3}$

Hence $4x = n\pi$ or $2x = 2n\pi \pm \frac{\pi}{3}$, where $n \in \mathbf{Z}$

i.e. $x = \frac{n\pi}{4}$ or $x = n\pi \pm \frac{\pi}{6}$, where $n \in \mathbf{Z}$.

Example 24 Solve $2\cos^2 x + 3\sin x = 0$

Solution The equation can be written as

$$2\left(1 - \sin^2 x\right) + 3\sin x = 0$$

or $2\sin^2 x - 3\sin x - 2 = 0$

or $(2\sin x + 1)(\sin x - 2) = 0$

Hence $\sin x = -\frac{1}{2}$ or $\sin x = 2$

But $\sin x = 2$ is not possible (Why?)

Therefore $\sin x = -\frac{1}{2} = \sin\frac{7\pi}{6}$.

Hence, the solution is given by

$$x = n\pi + (-1)^n \frac{7\pi}{6}, \text{ where } n \in \mathbf{Z}.$$

EXERCISE 3.4

Find the principal and general solutions of the following equations:

1. $\tan x = \sqrt{3}$

2. $\sec x = 2$

3. $\cot x = -\sqrt{3}$

4. $\text{cosec } x = -2$

Find the general solution for each of the following equations:

5. $\cos 4x = \cos 2x$

6. $\cos 3x + \cos x - \cos 2x = 0$

7. $\sin 2x + \cos x = 0$

8. $\sec^2 2x = 1 - \tan 2x$

9. $\sin x + \sin 3x + \sin 5x = 0$

Miscellaneous Examples

Example 25 If $\sin x = \frac{3}{5}$, $\cos y = -\frac{12}{13}$, where x and y both lie in second quadrant, find the value of $\sin (x + y)$.

Solution We know that

$$\sin (x + y) = \sin x \cos y + \cos x \sin y \qquad \dots (1)$$

Now $\cos^2 x = 1 - \sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

Therefore $\cos x = \pm \frac{4}{5}$.

Since x lies in second quadrant, $\cos x$ is negative.

Hence $\cos x = -\frac{4}{5}$

Now $\sin^2 y = 1 - \cos^2 y = 1 - \frac{144}{169} = \frac{25}{169}$

i.e. $\sin y = \pm \frac{5}{13}$.

Since y lies in second quadrant, hence $\sin y$ is positive. Therefore, $\sin y = \frac{5}{13}$. Substituting the values of $\sin x$, $\sin y$, $\cos x$ and $\cos y$ in (1), we get

$$\sin(x+y) = \frac{3}{5}\times\left(-\frac{12}{13}\right)+\left(-\frac{4}{5}\right)\times\frac{5}{13} = -\frac{36}{65}-\frac{20}{65}=-\frac{56}{65}.$$

Example 26 Prove that

$$\cos 2x\ \cos\frac{x}{2}-\cos 3x\ \cos\frac{9x}{2} = \sin 5x\ \sin\frac{5x}{2}.$$

Solution We have

$$\text{L.H.S.} = \frac{1}{2}\left[2\cos 2x\cos\frac{x}{2}-2\cos\frac{9x}{2}\cos 3x\right]$$

$$= \frac{1}{2}\left[\cos\left(2x+\frac{x}{2}\right)+\cos\left(2x-\frac{x}{2}\right)-\cos\left(\frac{9x}{2}+3x\right)-\cos\left(\frac{9x}{2}-3x\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{5x}{2}+\cos\frac{3x}{2}-\cos\frac{15x}{2}-\cos\frac{3x}{2}\right] = \frac{1}{2}\left[\cos\frac{5x}{2}-\cos\frac{15x}{2}\right]$$

$$= \frac{1}{2}\left[-2\sin\left\{\frac{\frac{5x}{2}+\frac{15x}{2}}{2}\right\}\sin\left\{\frac{\frac{5x}{2}-\frac{15x}{2}}{2}\right\}\right]$$

$$= -\sin 5x\ \sin\left(-\frac{5x}{2}\right) = \sin 5x\ \sin\frac{5x}{2} = \text{R.H.S.}$$

Example 27 Find the value of $\tan\frac{\pi}{8}$.

Solution Let $x=\frac{\pi}{8}$. Then $2x=\frac{\pi}{4}$.

Now $$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

or $$\tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$$

Let $y=\tan\frac{\pi}{8}$. Then $1=\frac{2y}{1-y^2}$

or $$y^2 + 2y - 1 = 0$$

Therefore $$y = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

Since $\frac{\pi}{8}$ lies in the first quadrant, $y = \tan\frac{\pi}{8}$ is positve. Hence

$$\tan\frac{\pi}{8} = \sqrt{2} - 1.$$

Example 28 If $\tan x = \frac{3}{4}, \pi < x < \frac{3\pi}{2}$, find the value of $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$.

Solution Since $\pi < x < \frac{3\pi}{2}$, $\cos x$ is negative.

Also $$\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}.$$

Therefore, $\sin\frac{x}{2}$ is positive and $\cos\frac{x}{2}$ is negative.

Now $$\sec^2 x = 1 + \tan^2 x = 1 + \frac{9}{16} = \frac{25}{16}$$

Therefore $$\cos^2 x = \frac{16}{25} \text{ or } \cos x = -\frac{4}{5} \quad \text{(Why?)}$$

Now $$2\sin^2\frac{x}{2} = 1 - \cos x = 1 + \frac{4}{5} = \frac{9}{5}.$$

Therefore $$\sin^2\frac{x}{2} = \frac{9}{10}$$

or $$\sin\frac{x}{2} = \frac{3}{\sqrt{10}} \quad \text{(Why?)}$$

Again $$2\cos^2\frac{x}{2} = 1 + \cos x = 1 - \frac{4}{5} = \frac{1}{5}$$

Therefore $$\cos^2\frac{x}{2} = \frac{1}{10}$$

or $\cos\frac{x}{2} = -\frac{1}{\sqrt{10}}$ (Why?)

Hence $\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{3}{\sqrt{10}} \times \left(\frac{-\sqrt{10}}{1}\right) = -3.$

Example 29 Prove that $\cos^2 x + \cos^2\left(x+\frac{\pi}{3}\right) + \cos^2\left(x-\frac{\pi}{3}\right) = \frac{3}{2}$.

Solution We have

$$\text{L.H.S.} = \frac{1+\cos 2x}{2} + \frac{1+\cos\left(2x+\frac{2\pi}{3}\right)}{2} + \frac{1+\cos\left(2x-\frac{2\pi}{3}\right)}{2}.$$

$$= \frac{1}{2}\left[3+\cos 2x+\cos\left(2x+\frac{2\pi}{3}\right)+\cos\left(2x-\frac{2\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\frac{2\pi}{3}\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\left(\pi-\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x-2\cos 2x\cos\frac{\pi}{3}\right]$$

$$= \frac{1}{2}\left[3+\cos 2x-\cos 2x\right] = \frac{3}{2} = \text{R.H.S.}$$

Miscellaneous Exercise on Chapter 3

Prove that:

1. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$

2. $(\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x = 0$

3. $(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4\cos^2 \frac{x+y}{2}$

4. $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4\sin^2 \frac{x-y}{2}$

5. $\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x \cos 2x \sin 4x$

6. $\frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$

7. $\sin 3x + \sin 2x - \sin x = 4\sin x \cos \frac{x}{2} \cos \frac{3x}{2}$

Find $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ in each of the following :

8. $\tan x = -\frac{4}{3}$, x in quadrant II

9. $\cos x = -\frac{1}{3}$, x in quadrant III

10. $\sin x = \frac{1}{4}$, x in quadrant II

Summary

- If in a circle of radius r, an arc of length l subtends an angle of θ radians, then $l = r\,\theta$
- Radian measure $= \frac{\pi}{180} \times$ Degree measure
- Degree measure $= \frac{180}{\pi} \times$ Radian measure
- $\cos^2 x + \sin^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \operatorname{cosec}^2 x$
- $\cos (2n\pi + x) = \cos x$
- $\sin (2n\pi + x) = \sin x$
- $\sin (-x) = -\sin x$
- $\cos (-x) = \cos x$

- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
- $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$ $\quad$ $\sin\left(\frac{\pi}{2} + x\right) = \cos x$

  $\cos(\pi - x) = -\cos x$ $\quad$ $\sin(\pi - x) = \sin x$

  $\cos(\pi + x) = -\cos x$ $\quad$ $\sin(\pi + x) = -\sin x$

  $\cos(2\pi - x) = \cos x$ $\quad$ $\sin(2\pi - x) = -\sin x$

- If none of the angles x, y and $(x \pm y)$ is an odd multiple of $\frac{\pi}{2}$, then

  $$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

- $$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

- If none of the angles x, y and $(x \pm y)$ is a multiple of π, then

  $$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

- $$\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$$

- $$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$

- $\sin 2x = 2 \sin x \cos x = \dfrac{2 \tan x}{1 + \tan^2 x}$
- $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$
- $\sin 3x = 3 \sin x - 4 \sin^3 x$
- $\cos 3x = 4 \cos^3 x - 3 \cos x$
- $\tan 3x = \dfrac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$
- (i) $\cos x + \cos y = 2 \cos \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$

  (ii) $\cos x - \cos y = -2 \sin \dfrac{x+y}{2} \sin \dfrac{x-y}{2}$

  (iii) $\sin x + \sin y = 2 \sin \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$

  (iv) $\sin x - \sin y = 2 \cos \dfrac{x+y}{2} \sin \dfrac{x-y}{2}$
- (i) $2 \cos x \cos y = \cos (x + y) + \cos (x - y)$

  (ii) $-2 \sin x \sin y = \cos (x + y) - \cos (x - y)$

  (iii) $2 \sin x \cos y = \sin (x + y) + \sin (x - y)$

  (iv) $2 \cos x \sin y = \sin (x + y) - \sin (x - y)$.
- $\sin x = 0$ gives $x = n\pi$, where $n \in \mathbf{Z}$.
- $\cos x = 0$ gives $x = (2n + 1) \dfrac{\pi}{2}$, where $n \in \mathbf{Z}$.
- $\sin x = \sin y$ implies $x = n\pi + (-1)^n y$, where $n \in \mathbf{Z}$.
- $\cos x = \cos y$, implies $x = 2n\pi \pm y$, where $n \in \mathbf{Z}$.
- $\tan x = \tan y$ implies $x = n\pi + y$, where $n \in \mathbf{Z}$.

Historical Note

The study of trigonometry was first started in India. The ancient Indian Mathematicians, Aryabhatta (476), Brahmagupta (598), Bhaskara I (600) and Bhaskara II (1114) got important results. All this knowledge first went from India to middle-east and from there to Europe. The Greeks had also started the study of trigonometry but their approach was so clumsy that when the Indian approach became known, it was immediately adopted throughout the world.

In India, the predecessor of the modern trigonometric functions, known as the sine of an angle, and the introduction of the sine function represents the main contribution of the *siddhantas* (Sanskrit astronomical works) to the history of mathematics.

Bhaskara I (about 600) gave formulae to find the values of sine functions for angles more than 90°. A sixteenth century Malayalam work *Yuktibhasa* (period) contains a proof for the expansion of sin (A + B). Exact expression for sines or cosines of 18°, 36°, 54°, 72°, etc., are given by Bhaskara II.

The symbols $\sin^{-1} x$, $\cos^{-1} x$, etc., for arc sin x, arc cos x, etc., were suggested by the astronomer Sir John F.W. Hersehel (1813) The names of Thales (about 600 B.C.) is invariably associated with height and distance problems. He is credited with the determination of the height of a great pyramid in Egypt by measuring shadows of the pyramid and an auxiliary staff (or gnomon) of known height, and comparing the ratios:

$$\frac{\text{H}}{\text{S}} = \frac{h}{s} = \tan \text{(sun's altitude)}$$

Thales is also said to have calculated the distance of a ship at sea through the proportionality of sides of similar triangles. Problems on height and distance using the similarity property are also found in ancient Indian works.

— ❖ —