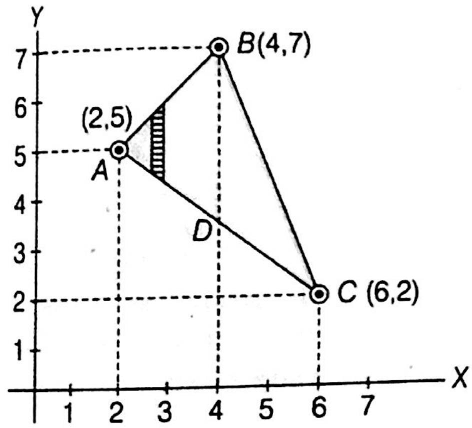
Area by integration-CBSE(solutions)

Solutions

1. Given, the vertices of ΔABC are $A(2, 5)$, $B(4, 7)$ and $C(6, 2)$.



(1)

By plotting these points on the graph, we find the required region.

Equation of the line AB is given by

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1} \Rightarrow \frac{y-5}{7-5}=\frac{x-2}{4-2}$$

$$\Rightarrow \quad y = x + 3 \qquad \text{...(i)} \quad \textbf{(1)}$$

Similarly, equation of the line BC is

$$\frac{y-7}{2-7}=\frac{x-4}{6-4} \Rightarrow y=\frac{1}{2}(-5x+34)$$

$$\Rightarrow \quad y=-\frac{5x}{2}+17 \qquad \text{...(ii)} \quad \textbf{(1)}$$

Equation of line AC is

$$\frac{y-5}{2-5}=\frac{x-2}{6-2}$$

$$y=\frac{1}{4}(-3x+26)$$

$$\Rightarrow \quad y=\frac{-3}{4}x+\frac{13}{2} \qquad \text{...(iii)} \quad \textbf{(1)}$$

Now, required area

= (Area under line segment AB)

+ (Area under line segment BC)

− (Area under line segment AC) **(1)**

$$=\int_2^4 (x+3)\,dx+\int_4^6\left(-\frac{5x}{2}+17\right)dx-\int_2^6\left(-\frac{3}{4}x+\frac{13}{2}\right)dx$$

$$=\left[\frac{x^2}{2}+3x\right]_2^4+\left[-\frac{5x^2}{4}+17x\right]_4^6-\left[-\frac{3x^2}{8}+\frac{13}{2}x\right]_2^6$$

$$=12+9-14=7 \text{ sq units} \qquad \textbf{(1)}$$

2. Do same as Q. No. 1. $\left[\textbf{Ans. } \frac{3}{2} \text{sq units}\right]$

3. The equation of circle is

$$x^2+y^2=8x \qquad \text{...(i)}$$

and the equation of parabola is

$$y^2=4x \qquad \text{...(ii)}$$

Eq. (i) can be written as

$$(x^2-8x)+y^2=0$$

$$\Rightarrow \quad (x^2-8x+16)+y^2=16$$

$$\Rightarrow \quad (x-4)^2+y^2=(4)^2 \qquad \text{...(iii)} \quad \textbf{(1)}$$

which is a circle with centre $C(4, 0)$ and radius = 4.

From Eqs. (i) and (ii), we get

$$x^2+4x=8x$$

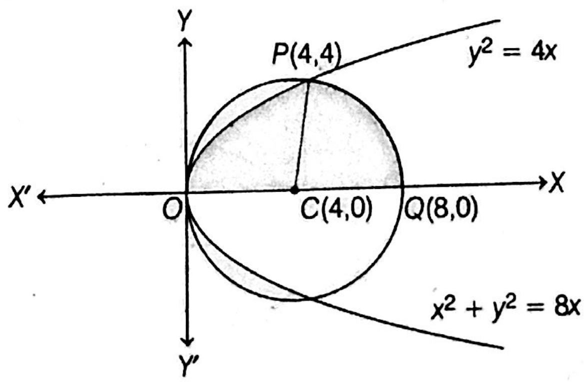
$$\Rightarrow \quad x^2-4x=0$$

$$\Rightarrow \quad x(x-4)=0 \Rightarrow x=0, 4 \qquad \textbf{(1)}$$

Now, from Eq. (ii), we get

$$y=0, 4$$

∴ Points of intersection of circle (i) and parabola (ii), above the X-axis, are $O(0, 0)$ and $P(4, 4)$.



(1)

Now, required area = area of region $OPQCO$

= (area of region $OCPO$) + (area of region $PCQP$)

$$= \int_0^4 y(\text{Parabola})\, dx + \int_4^8 y(\text{Circle})\, dx$$

$$= 2\int_0^4 \sqrt{x}\, dx + \int_4^8 \sqrt{(4)^2 - (x-4)^2}\, dx$$

[from Eqs. (ii), (iii)] **(1)**

$$= 2\left[\frac{x^{3/2}}{3/2}\right]_0^4 + \left[\frac{(x-4)}{2}\sqrt{(4)^2-(x-4)^2} + \frac{(4)^2}{2}\cdot\sin^{-1}\frac{x-4}{4}\right]_4^8$$

$$\left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C\right]$$

$$= \frac{4}{3}[x^{3/2}]_0^4 + \left\{\left[\left(\frac{8-4}{2}\right)\sqrt{16-16} + 8\sin^{-1}(1)\right] - [0 + 8\sin^{-1} 0]\right\}$$

(1)

$$= \frac{4}{3}[(4)^{3/2} - 0] + \left[0 + 8\times\frac{\pi}{2}\right] - [0+0]$$

$$= \frac{4}{3}\times 8 + 4\pi = \frac{32}{3} + 4\pi = \frac{4}{3}(8+3\pi) \text{ sq units}$$

(1)

4. First, we draw a square formed by the lines $x=0$, $x=4$, $y=4$ and $y=0$ and after that, we draw given parabolas which intersect each other on the square such that the whole region divided into three parts. Now, we find separately area of each part and show that area of each part is equal.

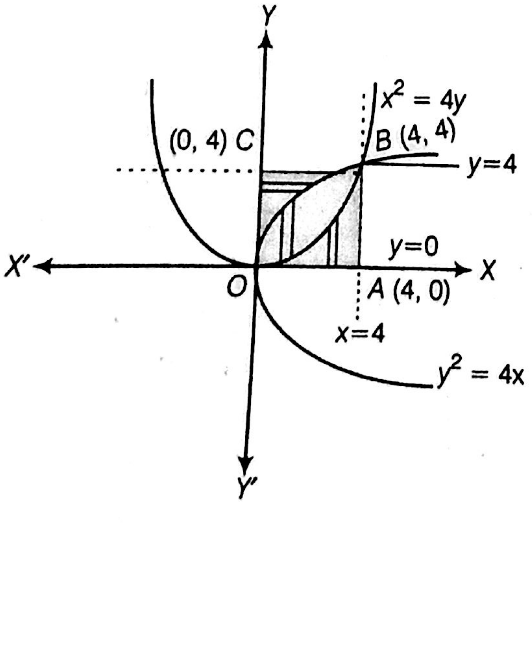
Let $OABC$ be the square whose sides are represented by following equations

Equation of OA is $y = 0$

Equation of AB is $x = 4$

Equation of BC is $y = 4$

Equation of CO is $x = 0$



(1)

On solving equations $y^2 = 4x$ and $x^2 = 4y$, we get $A(0, 0)$ and $B(4, 4)$ as their points of intersection. Now, area bounded by these curves is given by $\int_0^4 [y_{(\text{parabola } y^2=4x)} - y_{(\text{parabola } x^2=4y)}]\, dx$

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \left[2\cdot\frac{2}{3}x^{3/2} - \frac{x^3}{12}\right]_0^4$$

$$= \left[\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right]_0^4 = \frac{4}{3}\cdot(4)^{3/2} - \frac{64}{12}$$

$$= \frac{4}{3}\cdot(2^2)^{3/2} - \frac{64}{12} = \frac{4}{3}\cdot(2)^3 - \frac{64}{12}$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{ sq units}$$

Hence, area bounded by curves $y^2 = 4x$ and $x^2 = 4y$ is $\frac{16}{3}$ sq units ...(i) **(1½)**

Now, area bounded by curve $x^2 = 4y$ and the lines $x = 0$, $x = 4$ and X-axis

$$= \int_0^4 y_{(\text{parabola } x^2=4y)}\, dx = \int_0^4 \frac{x^2}{4}\, dx$$

$$= \left[\frac{x^3}{12}\right]_0^4 = \frac{64}{12} = \frac{16}{3} \text{ sq units} \quad \text{...(ii)}$$ **(1½)**

Similarly, the area bounded by curve $y^2 = 4x$, the lines $y = 0$, $y = 4$ and Y-axis

$$= \int_0^4 x_{(\text{parabola } y^2=4x)}\, dy = \int_0^4 \frac{y^2}{4}\, dy$$

$$= \left[\frac{y^3}{12}\right]_0^4 = \frac{64}{12} = \frac{16}{3} \text{ sq units} \quad \text{...(iii)}$$ **(1½)**

From Eqs. (i), (ii) and (iii), it is clear that area bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$ divides the area of square into three equal parts.

Hence proved. **(1/2)**

5. Do same as Q. No. 1. $\left[\textbf{Ans. } \frac{3}{2}\text{sq units}\right]$

6. First, find the intersecting points of two circles and then draw a rough sketch of these two circles. The common shaded region is symmetrical about X-axis. So, we find area of one part only, i.e. upper part of X-axis. After that required area is twice of that area.

Given circles are

$$x^2 + y^2 = 4 \quad \text{...(i)}$$

and $$(x-2)^2 + y^2 = 4 \quad \text{...(ii)}$$

Eq. (i) is a circle with centre origin and radius 2, Eq. (ii) is a circle with centre $C(2, 0)$ and radius 2.

On solving Eqs. (i) and (ii), we get

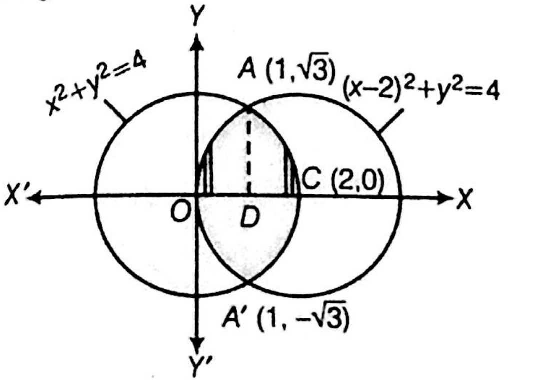
$$(x-2)^2+y^2=x^2+y^2$$

$$\Rightarrow \quad x^2-4x+4+y^2=x^2+y^2 \Rightarrow x=1$$

On putting $x=1$ in Eq. (i), we get

$$y=\pm\sqrt{3}$$ **(1½)**

Thus, the points of intersection of the given circles are $A(1, \sqrt{3})$ and $A'(1, -\sqrt{3})$ as shown in the figure given below:



(1)

Clearly, required area = Area of the enclosed region $OACA'O$ between circles

$=2$[Area of the region $ODCAO$]

$=2$[Area of the region $ODAO$ + Area of the region $DCAD$] **(1)**

$$=2\left[\int_0^1 y_2\,dx+\int_1^2 y_1\,dx\right]$$

[where, $y_1=\sqrt{4-x^2}$ and $y_2=\sqrt{4-(x-2)^2}$]

$$=2\left[\int_0^1\sqrt{4-(x-2)^2}\,dx+\int_1^2\sqrt{4-x^2}\,dx\right]$$

$$=2\left[\frac{1}{2}(x-2)\sqrt{4-(x-2)^2}+\frac{1}{2}\times4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1+2\left[\frac{1}{2}x\sqrt{4-x^2}+\frac{1}{2}\times4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$\left[\because \int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$=\left[(x-2)\sqrt{4-(x-2)^2}+4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1+\left[x\sqrt{4-x^2}+4\sin^{-1}\frac{x}{2}\right]_1^2$$ **(1)**

$$=\left[\left\{-\sqrt{3}+4\sin^{-1}\left(\frac{-1}{2}\right)\right\}-0-4\sin^{-1}(-1)\right]+\left[0+4\sin^{-1}1-\sqrt{3}-4\sin^{-1}\frac{1}{2}\right]$$

$$=\left[\left(-\sqrt{3}-4\times\frac{\pi}{6}\right)+4\times\frac{\pi}{2}\right]+\left[4\times\frac{\pi}{2}-\sqrt{3}-4\times\frac{\pi}{6}\right]$$

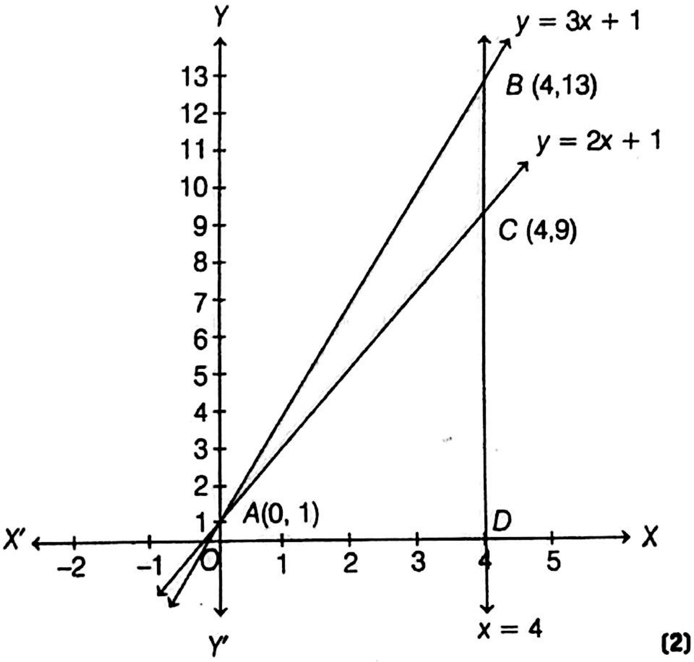
$$=\left(-\sqrt{3}-\frac{2\pi}{3}+2\pi\right)+\left(2\pi-\sqrt{3}-\frac{2\pi}{3}\right)$$

$$=\frac{8\pi}{3}-2\sqrt{3}\text{ sq units}$$ **(1½)**

7. Given, equation of sides are

$$y=2x+1,\ y=3x+1 \text{ and } x=4$$

On drawing the graph of these equations, we get the following triangular region



(2)

By solving these equations we get the vertices of triangle as $A(0, 1)$, $B(4, 13)$ and $C(4, 9)$.

$\therefore$ Required area = Area $(OABDO)$ − area $(OACDO)$

$$=\int_0^4(3x+1)\,dx-\int_0^4(2x+1)\,dx$$ **(2)**

$$=\left[\frac{3x^2}{2}+x\right]_0^4-[x^2+x]_0^4$$

$$=\frac{3\times4^2}{2}+4-0-(4^2+4-0)$$

$$=24+4-20$$

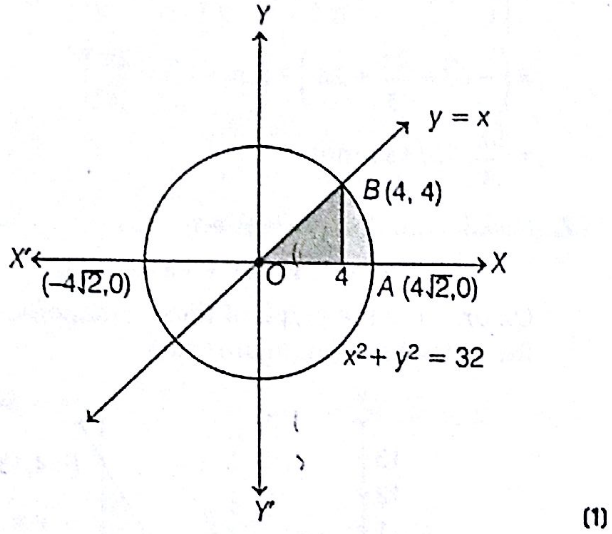
$$=8\text{ sq units}$$ **(2)**

8. Given the circle $x^2 + y^2 = 32$ ...(i)

having centre (0, 0) and radius $4\sqrt{2}$

and the line $y = x$ ...(ii)

Let us find the point of intersection of Eqs. (i) and (ii).



(1)

On substituting $y = x$ in Eq. (i), we get

$$x^2 + x^2 = 32$$

$$\Rightarrow \quad 2x^2 = 32$$

$$\Rightarrow \quad x^2 = 16 \Rightarrow x = \pm 4$$ (1)

Thus, the points of intersection are (4, 4) and (−4,−4). $[\because y = x]$

Clearly, the required area

= Area of shaded region $OABO$

$$= \int_0^4 y(\text{line})\, dx + \int_4^{4\sqrt{2}} y(\text{circle})\, dx$$ (1)

$$= \int_0^4 x\, dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\, dx$$

$[\because x^2 + y^2 = 32 \Rightarrow y = \pm\sqrt{32 - x^2}$ and $y > 0]$

$$= \left[\frac{x^2}{2}\right]_0^4 + \int_4^{4\sqrt{2}} \sqrt{(4\sqrt{2})^2 - x^2}\, dx$$

$$= \frac{1}{2}[16 - 0] + \frac{1}{2}\left[x\sqrt{(4\sqrt{2})^2 - x^2} + (4\sqrt{2})^2 \sin^{-1}\left(\frac{x}{4\sqrt{2}}\right)\right]_4^{4\sqrt{2}}$$ (1)

$$= 8 + \frac{1}{2}\left[(0 + 32\sin^{-1}(1)) - \left(4\sqrt{32-16} + 32\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)\right]$$

$$= 8 + \frac{1}{2}\left[32\sin^{-1}(1) - 16 - 32\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$= 8 + \frac{1}{2}\left[32\cdot\frac{\pi}{2} - 16 - 32\cdot\frac{\pi}{4}\right]$$

$$= 8 + \frac{1}{2}[16\pi - 16 - 8\pi]$$

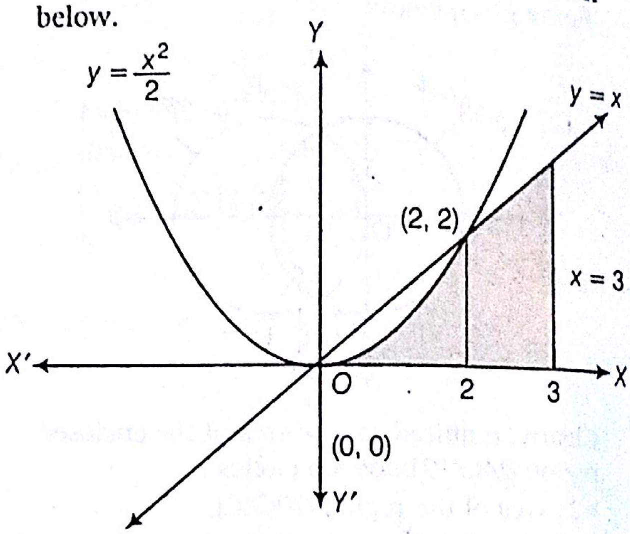
$$= 8 + \frac{1}{2}[8\pi - 16] = 8 + 4\pi - 8$$

$= 4\pi$ sq units (2)

9. Given region is

$\{(x, y): 0 \le 2y \le x^2, 0 \le y \le x, 0 \le x \le 3\}$

which can be represent graphically as shown below.



(2)

Now, let us find the point of intersection of $y = x$ and $y = \frac{x^2}{2}$.

For this consider,

$$x = \frac{x^2}{2}$$

$$\Rightarrow \quad x^2 - 2x = 0$$

$$\Rightarrow \quad x(x-2) = 0$$

$$\Rightarrow \quad x = 0 \text{ or } 2$$

Clearly, when $x = 0$, then $y = 0$

and when $x = 2$, then $y = 2$

Thus, the points of intersection are (0, 0) and (2, 2). (2)

$\therefore$ Required area $= \int_0^2 \frac{x^2}{2} dx + \int_2^3 x\, dx$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 + \frac{1}{2}[x^2]_2^3$$

$$= \frac{1}{6}[8 - 0] + \frac{1}{2}[9 - 4]$$

$$= \frac{8}{6} + \frac{5}{2} = \frac{23}{6} \text{ sq units}$$ (2)

10. Do same as Q. No.1. **[Ans. 4 sq units]**

11. Given equations of circle is $x^2 + y^2 = 16$ and $x = \sqrt{3}y$.

$\Rightarrow y = \frac{1}{\sqrt{3}}x$ represents a line through the origin.

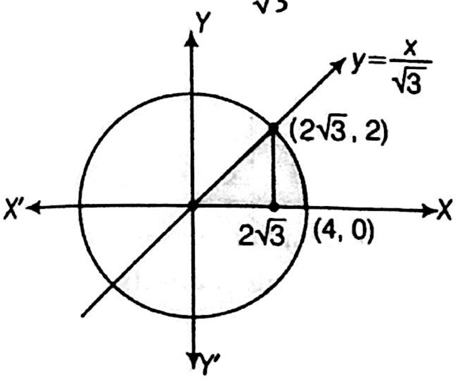
The line $y = \frac{1}{\sqrt{3}}x$ intersects the circle

$$\therefore \quad x^2 + \frac{x^2}{3} = 16 \qquad \textbf{(1)}$$

$$\Rightarrow \quad \frac{3x^2 + x^2}{3} = 16$$

$$\Rightarrow \quad 4x^2 = 48 \Rightarrow x^2 = 12 \Rightarrow x = \pm 2\sqrt{3} \quad \textbf{(1)}$$

when $x = 2\sqrt{3}$, then $y = \frac{2\sqrt{3}}{\sqrt{3}} = 2$



Required area (shaded region in first quadrant)

= (Area under the line $y = \frac{1}{\sqrt{3}}x$ from $x = 0$ to $2\sqrt{3}$)

+ (Area under the circle from $x = 2\sqrt{3}$ to $x = 4$) **(1)**

$$= \int_0^{2\sqrt{3}} \frac{1}{\sqrt{3}} x\,dx + \int_{2\sqrt{3}}^{4} \sqrt{16 - x^2}\,dx$$

$$= \frac{1}{\sqrt{3}}\left[\frac{x^2}{2}\right]_0^{2\sqrt{3}} + \left[\frac{x}{2}\sqrt{16 - x^2} + \frac{(4)^2}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_{2\sqrt{3}}^{4} \quad \textbf{(1)}$$

$$= \frac{1}{2\sqrt{3}}[(2\sqrt{3})^2 - 0] + [0 + 8\sin^{-1}(1)$$

$$- \frac{2\sqrt{3}}{2}\sqrt{16 - 12} - 8\sin^{-1}\left(\frac{2\sqrt{3}}{4}\right)\Bigg]$$

$$= 2\sqrt{3} + 8\left(\frac{\pi}{2}\right) - \frac{2\sqrt{3}}{2} \times 2 - 8\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \quad \textbf{(1)}$$

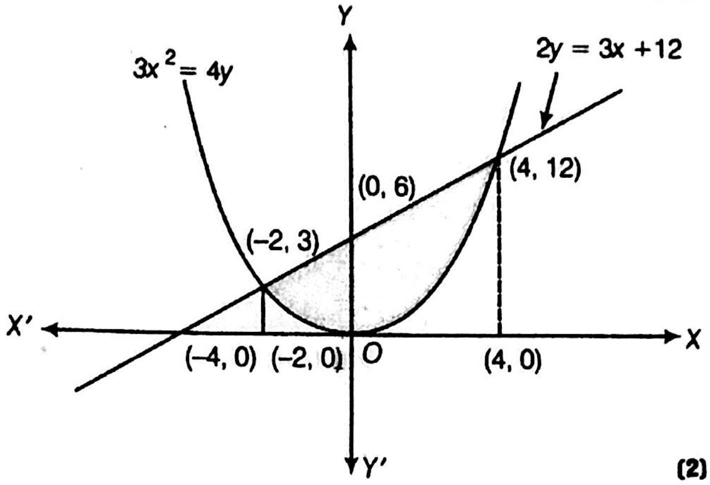
$$= 2\sqrt{3} + 4\pi - 2\sqrt{3} - 8\left(\frac{\pi}{3}\right)$$

$$= 4\pi - \frac{8\pi}{3} = \frac{12\pi - 8\pi}{3} = \frac{4\pi}{3} \text{ sq units} \quad \textbf{(1)}$$

2. Do same as Q. No. 1. **[Ans. 7 sq units]**

13. Given parabola is $4y = 3x^2$ ...(i)

represents an upward parabola with vertex (0, 0) and equation of line is $2y = 3x + 12$ ...(ii)



(2)

From Eqs. (i) and (ii), we get

$$2(3x + 12) = 3x^2$$

$$\Rightarrow \quad 3x^2 - 6x - 24 = 0$$

$$\Rightarrow \quad x^2 - 2x - 8 = 0$$

$$\Rightarrow \quad (x - 4)(x + 2) = 0 \Rightarrow x = 4, -2 \quad \textbf{(1)}$$

when $x = 4$, then $y = \frac{3 \times 4 + 12}{2} = 12$ [from Eq. (ii)]

when $x = -2$, then $y = \frac{3 \times (-2) + 12}{2} = 3$

Thus, intersection points are (–2, 3) and (4, 12). **(1)**

Required area $= \int_{-2}^{4}\left(\frac{3x + 12}{2} - \frac{3x^2}{4}\right)dx$

$$= \left(\frac{3x^2}{4} + 6x - \frac{x^3}{4}\right)_{-2}^{4} \quad \textbf{(1)}$$

$$= \left[\frac{3 \times 4^2}{4} + 6 \times 4 - \frac{4^3}{4}\right] - \left[\frac{3 \times 4}{4} - 12 + \frac{8}{4}\right]$$

$$= 12 + 24 - 16 - 3 + 12 - 2$$

$$= 27 \text{ sq units} \quad \textbf{(1)}$$

14. Given region is

$$\{(x, y) : x^2 + y^2 \le 2ax,\ y^2 \ge ax;\ x, y \ge 0\}$$

Now, we have $x^2 + y^2 \le 2ax$

$$\Rightarrow \quad x^2 + y^2 - 2ax \le 0$$

$$\Rightarrow \quad x^2 - 2ax + a^2 + y^2 \le a^2$$

[adding a^2 both sides of inequality]

$$\Rightarrow \quad (x - a)^2 + y^2 \le a^2$$

which is the interior of the circle having centre $(a, 0)$ and radius a. **(1)**

Also, we have $\quad y^2 \geq ax$

which is the exterior of the parabola having vertex (0, 0) and axis is X-axis.

Now, let us find the intersection point of circle $(x-a)^2 + y^2 = a^2$ and parabola $y^2 = ax$.

On substituting $y^2 = ax$ in the given circle, we get

$$(x-a)^2 + ax = a^2$$

$$\Rightarrow \quad x^2 + a^2 - 2ax + ax = a^2$$

$$\Rightarrow \quad x^2 - ax = 0$$

$$\Rightarrow \quad x(x-a) = 0$$

$$\Rightarrow \quad x = 0, a \qquad \textbf{(1/2)}$$

When $x = 0$, then

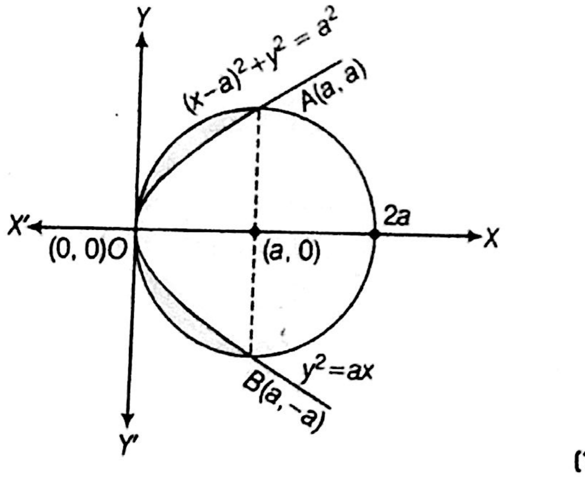
$$y^2 = 0 \Rightarrow y = 0$$

When $x = a$, then

$$y^2 = a^2 \Rightarrow y = \pm a$$

So, the points of intersection are $O(0, 0)$, $A(a, a)$ and $B(a, -a)$. **(1)**

Now, draw the graph of given curve as shown below:



(1)

Clearly, the required area of region will lie in first quadrant as $x, y \geq 0$.

$\therefore$ Required area = Area of shaded region

$$= \int_0^a y_{\text{(circle)}}\, dx - \int_0^a y_{\text{(parabola)}}\, dx \qquad \textbf{(1/2)}$$

$$= \int_0^a \sqrt{a^2 - (x-a)^2}\, dx - \int_0^a \sqrt{ax}\, dx$$

$$= \left[\frac{(x-a)}{2}\sqrt{a^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)\right]_0^a - \left[\sqrt{a}\frac{x^{3/2}}{3/2}\right]_0^a \qquad \textbf{(1)}$$

$$= \left[0 + \frac{a^2}{2}\sin^{-1}(0) - \left\{0 + \frac{a^2}{2}\sin^{-1}(-1)\right\}\right] - \frac{2}{3}\sqrt{a}\left(a^{3/2} - 0\right)$$

$$= \left[0 - \left\{-\frac{a^2}{2} \times \frac{\pi}{2}\right\}\right] - \frac{2}{3}a^2$$

$$[\because \sin^{-1}(-\theta) = -\sin^{-1}\theta]$$

$$= \frac{a^2\pi}{4} - \frac{2}{3}a^2 = a^2\left(\frac{\pi}{4} - \frac{2}{3}\right) \text{ sq units} \qquad \textbf{(1)}$$

15. Do same as Q. No. 1. $\left[\textbf{Ans. } \frac{13}{2} \text{ sq units}\right]$

16. Given equations of curves are

$$y = \sqrt{4 - x^2} \qquad \text{...(i)}$$

and $\quad x^2 + y^2 - 4x = 0 \qquad$ ...(ii)

Consider the curve

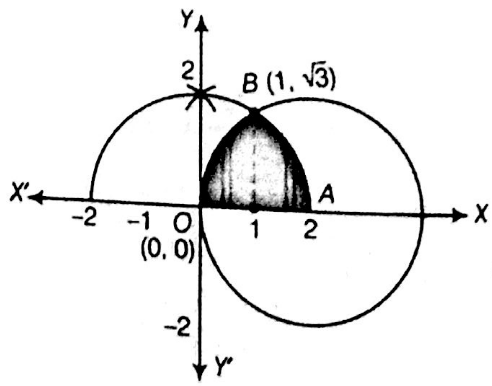
$$y = \sqrt{4-x^2} \Rightarrow y^2 = 4 - x^2$$

$\Rightarrow x^2 + y^2 = 4$, which represents a circle with centre (0, 0) and radius 2units.

Now, consider the curve $x^2 + y^2 - 4x = 0$

$\Rightarrow (x-2)^2 + y^2 = 4$, which also represents a circle with centre (2, 0) and radius 2 units. **(1)**

Now, let us sketch the graph of given curves and find their points of intersection.



(1)

On substituting the value of y from Eq. (i) in Eq. (ii), we get

$$x^2 + (4 - x^2) - 4x = 0 \Rightarrow 4 - 4x = 0 \Rightarrow x = 1$$

On substituting $x = 1$ in Eq. (i), we get $y = \sqrt{3}$

Thus, the point of intersection is $(1, \sqrt{3})$. **(1)**

Clearly, required area

= Area of shaded region $OABO$

$$= \int_0^1 y_{\text{(second circle)}}\, dx + \int_1^2 y_{\text{(first circle)}}\, dx \qquad \textbf{(1)}$$

$$= \int_0^1 \sqrt{4x - x^2}\, dx + \int_1^2 \sqrt{4 - x^2}\, dx \quad \textbf{(1/2)}$$

$$= \int_0^1 \sqrt{-(x^2 - 4x)}\, dx + \int_1^2 \sqrt{2^2 - x^2}\, dx$$

$$= \int_0^1 \sqrt{-[x^2 - 2(2)(x) + 4 - 4]}\, dx + \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_1^2 \quad \textbf{(1/2)}$$

$$= \int_0^1 \sqrt{4 - (x - 2)^2}\, dx + \left[2\sin^{-1}(1) - \left\{\frac{1}{2}\sqrt{3} + 2\sin^{-1}\left(\frac{1}{2}\right)\right\}\right]$$

$$= \left[\frac{(x - 2)}{2}\sqrt{4x - x^2} + 2\sin^{-1}\left(\frac{x - 2}{2}\right)\right]_0^1 + \left[2\cdot\frac{\pi}{2} - \frac{\sqrt{3}}{2} - 2\cdot\frac{\pi}{6}\right]$$

$$= \left[\left\{\frac{-\sqrt{3}}{2} + 2\sin^{-1}\left(\frac{-1}{2}\right)\right\} - \{2\sin^{-1}(-1)\}\right] + \left(\pi - \frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$= -\frac{\sqrt{3}}{2} - 2\sin^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}(1) + \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

$[\because \sin^{-1}(-x) = -\sin^{-1} x]$

$$= 2\cdot\frac{\pi}{2} - 2\cdot\frac{\pi}{6} + \frac{2\pi}{3} - \sqrt{3} = \pi - \frac{\pi}{3} + \frac{2\pi}{3} - \sqrt{3}$$

$$= \pi + \frac{\pi}{3} - \sqrt{3} = \left(\frac{4\pi}{3} - \sqrt{3}\right) \text{ sq units} \quad \textbf{(1)}$$

17. Given, equation of circle is

$$x^2 + y^2 = 32 \quad \ldots\text{(i)}$$

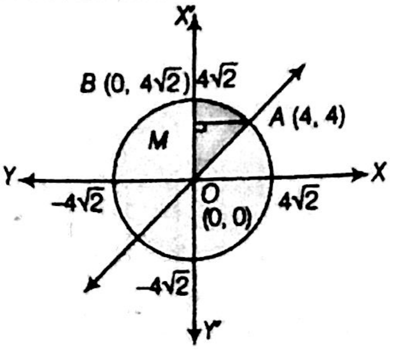
and equation of line is

$$y = x \quad \ldots\text{(ii)}$$

Consider $x^2 + y^2 = 32 \Rightarrow x^2 + y^2 = (4\sqrt{2})^2$

So, given circle has centre (0, 0) and radius $4\sqrt{2}$ units. **(1/2)**

Now, let us sketch the graph of given curves and find their points of intersection.



(1)

On substituting $y = x$ in Eq. (i), we get

$$2x^2 = 32 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$$

Thus, the points of intersection are (4, 4) and (−4, −4). **(1)**

So, given line and the circle intersect in the first quadrant at point $A(4, 4)$ and the circle cut the Y-axis at point $B\ (0, 4\sqrt{2})$.

Let us draw AM perpendicular to Y-axis.

Clearly, required area

= Area of shaded region $OABO$ **(1/2)**

$$= \int_0^4 x_{(\text{line})}\, dy + \int_4^{4\sqrt{2}} x_{(\text{circle})}\, dy \quad \textbf{(1)}$$

$$= \int_0^4 y\,dy + \int_4^{4\sqrt{2}} \sqrt{32 - y^2}\,dy$$

$[\because x^2 + y^2 = 32 \Rightarrow x = \pm\sqrt{32 - y^2}$, but we need area of region enclosed in the first quadrant only, so $x = \sqrt{32 - y^2}]$

$$= \left[\frac{y^2}{2}\right]_0^4 + \int_4^{4\sqrt{2}} \sqrt{(4\sqrt{2})^2 - y^2}\, dy$$

$$= \frac{1}{2}(16 - 0) + \left[\frac{y}{2}\sqrt{32 - y^2} + \frac{32}{2}\sin^{-1}\left(\frac{y}{4\sqrt{2}}\right)\right]_4^{4\sqrt{2}} \quad \textbf{(1)}$$

$$= 8 + \left[16\sin^{-1}(1) - \left\{2 \times 4 + 16\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right\}\right]$$

$$= 8 + \left[16\cdot\frac{\pi}{2} - 8 - 16\cdot\frac{\pi}{4}\right] = 16\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= 16\cdot\frac{\pi}{4} = 4\pi \text{ sq units} \quad \textbf{(1)}$$

18. First, differentiate the given curve w.r.t. x and determine the value of $\frac{dy}{dx}$ at $(1, \sqrt{3})$.

Find the equations of tangent and normal at point $(1, \sqrt{3})$ by using formula $y - y_1 = \frac{dy}{dx}(x - x_1)$ and $y - y_1 = -\frac{1}{dy/dx}(x - x_1)$.

Further, plot the above lines on a graph paper and find the area by using integration.

Given equation of circle is

$$x^2 + y^2 = 4 \quad \ldots\text{(i)}$$

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$\Rightarrow \quad x + y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$

$\Rightarrow \quad \left(\dfrac{dy}{dx}\right)_{(1,\sqrt{3})} = -\dfrac{1}{\sqrt{3}}$ **(1)**

Now, equation of tangent at point $(1, \sqrt{3})$ is

$(y - \sqrt{3}) = -\dfrac{1}{\sqrt{3}}(x - 1) \quad [\because y - y_1 = m(x - x_1)]$

$\Rightarrow \sqrt{3}y - 3 = -x + 1$

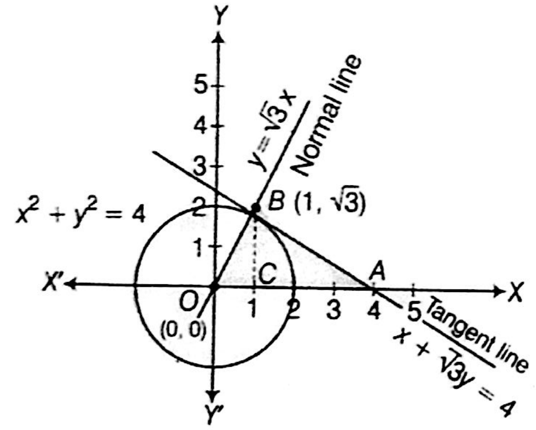
$\Rightarrow x + \sqrt{3}y = 4$ ...(ii) **(1)**

and equation of normal is $(y - \sqrt{3}) = \sqrt{3}(x - 1)$

$$\left[\because \text{slope of normal} = -\dfrac{1}{\left(\dfrac{dy}{dx}\right)_{(1,\sqrt{3})}}\right]$$

$\Rightarrow \quad y - \sqrt{3} = \sqrt{3}x - \sqrt{3} \Rightarrow y = \sqrt{3}x$ ...(iii) **(1)**

Now, the Eqs. (ii) and (iii) can be represented in graph as shown below:



(1)

On putting $y = 0$ in Eq. (ii), we get

$$x + 0 = 4 \Rightarrow x = 4$$

$\therefore$ the tangent line $x + \sqrt{3}y = 4$ cuts the X-axis at $A(4, 0)$.

$\therefore$ Required area = Area of shaded region OAB

$= \int_0^1 y_{\text{(equation of normal)}}\,dx + \int_1^4 y_{\text{(equation of tangent)}}\,dx$ **(1)**

$= \int_0^1 \sqrt{3}x\,dx + \int_1^4 \left(\dfrac{4 - x}{\sqrt{3}}\right)dx$

$= \sqrt{3}\left[\dfrac{x^2}{2}\right]_0^1 + \dfrac{1}{\sqrt{3}}\left[4x - \dfrac{x^2}{2}\right]_1^4$

$= \dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{3}}\left[16 - \dfrac{16}{2} - 4 + \dfrac{1}{2}\right]$

$= \dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{3}}\left[12 - \dfrac{15}{2}\right] = \dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{3}}\left[\dfrac{9}{2}\right]$

$= \dfrac{\sqrt{3}}{2} + \dfrac{3\sqrt{3}}{2} = \dfrac{4\sqrt{3}}{2} = 2\sqrt{3}$ sq units **(1)**

19. Given curves are

$$x - y + 2 = 0 \quad \text{...(i)}$$

and $$x = \sqrt{y} \quad \text{...(ii)}$$

Consider $x = \sqrt{y} \Rightarrow x^2 = y$, which represents the parabola whose vertex is $(0, 0)$ and axis is Y-axis.

Now, the point of intersection of Eqs.(i) and (ii) is given by $x = \sqrt{x + 2}$ **(1)**

$\Rightarrow \quad x^2 = x + 2$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad (x - 2)(x + 1) = 0$

$\Rightarrow \quad x = -1, 2$

But $x = -1$ does not satisfy the Eq. (ii).

$\therefore \quad x = 2$

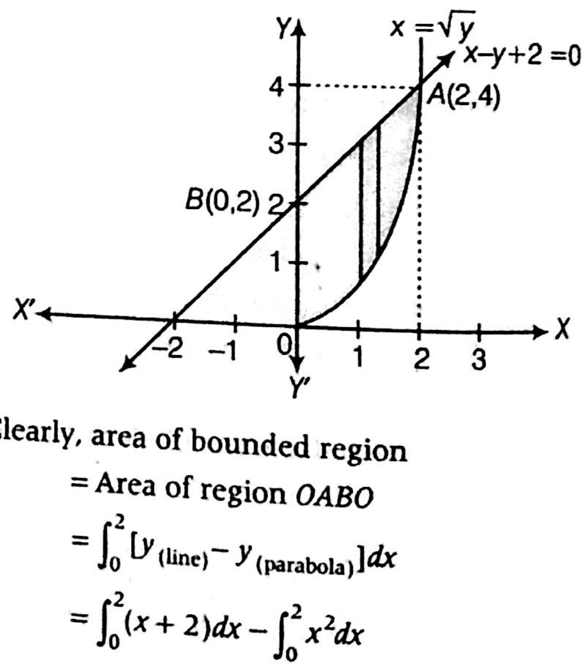
Now, putting $x = 2$ in Eq. (ii), we get

$$2 = \sqrt{y} \Rightarrow y = 4$$ **(1)**

Hence, the point of intersection is $(2, 4)$.

But actual equation of given parabola is $x = \sqrt{y}$, it means a semi-parabola which is on right side of Y-axis.

The graph of given curves are shown below:



(2)

Clearly, area of bounded region

= Area of region $OABO$

$= \int_0^2 [y_{\text{(line)}} - y_{\text{(parabola)}}]dx$ **(1)**

$= \int_0^2 (x + 2)dx - \int_0^2 x^2 dx$

$= \left[\dfrac{x^2}{2} + 2x\right]_0^2 - \left[\dfrac{x^3}{3}\right]_0^2 = \left[\dfrac{4}{2} + 4 - 0\right] - \left[\dfrac{8}{3} - 0\right]$

$= 6 - \dfrac{8}{3} = \dfrac{18 - 8}{3} = \dfrac{10}{3}$ sq units **(1)**

20. First, find the intersection points of given curves and then draw a rough diagram to represent the required area. If it is symmetrical about X-axis or Y-axis, then we first find area of only one portion from them and then required area is twice of that area.

Given curves are

$$y^2=4x \quad \text{...(i)}$$

and $4x^2+4y^2=9 \Rightarrow x^2+y^2=\frac{9}{4}$ ...(ii)

Eq. (i) represents a parabola having vertex (0,0) and axis is X-axis and Eq. (ii) represents a circle having centre (0, 0) and radius $\frac{3}{2}$. **(1)**

On substituting $y^2 = 4x$ in Eq. (ii), we get

$$x^2+4x=\frac{9}{4} \Rightarrow 4x^2+16x=9$$

$$\Rightarrow \quad 4x^2+18x-2x-9=0$$

$$\Rightarrow \quad 2x(2x+9)-1(2x+9)=0$$

$$\Rightarrow \quad (2x+9)(2x-1)=0$$

$$\Rightarrow \quad x=\frac{1}{2}, -\frac{9}{2}$$

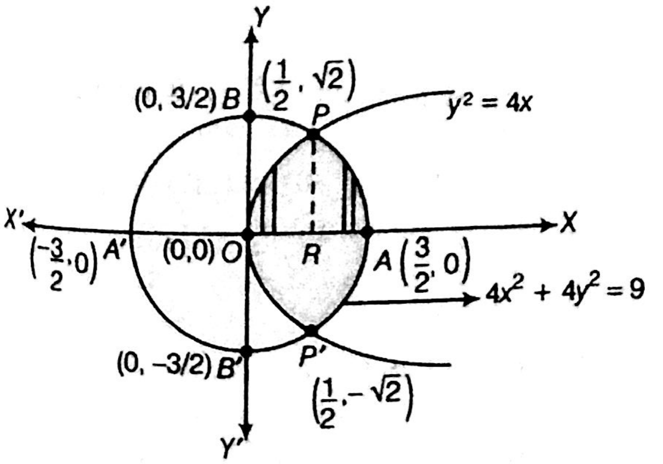
On putting $x=\frac{1}{2}$ in Eq. (i), we get

$$y=\pm\sqrt{2}$$

At $x=-\frac{9}{2}$, y have imaginary values.

So, intersection points are $P\left(\frac{1}{2},\sqrt{2}\right)$ and $P'\left(\frac{1}{2},-\sqrt{2}\right)$ **(1)**

Now, the shaded region represents the required region as shown below:



(1)

$\therefore$ Required area = 2 [Area of the region *ORPO* + Area of the region *RAPR*]

$$= 2\left[\int_0^{1/2} y_{\text{(parabola)}}\,dx + \int_{1/2}^{3/2} y_{\text{(circle)}}\,dx\right]$$

$$=2\left[\int_0^{1/2}\sqrt{4x}\,dx+\int_{1/2}^{3/2}\sqrt{\frac{9}{4}-x^2}\,dx\right] \quad \textbf{(1)}$$

$$=2\left\{\left[2\left(\frac{2}{3}x^{3/2}\right)\right]_0^{1/2} + \left[\frac{x}{2}\sqrt{\frac{9}{4}-x^2}+\frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right)\right]_{1/2}^{3/2}\right\}$$

$$\left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$=2\left[\frac{4}{3}\left\{\left(\frac{1}{2}\right)^{3/2}-0\right\} + \left\{0+\frac{9}{8}\sin^{-1}(1)-\frac{1}{4}\cdot\sqrt{2}-\frac{9}{8}\sin^{-1}\frac{1}{3}\right\}\right] \quad \textbf{(1)}$$

$$=2\left[\frac{4}{3}\times\frac{1}{2\sqrt{2}}+\frac{9}{8}\times\frac{\pi}{2}-\frac{\sqrt{2}}{4}-\frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$=2\left[\frac{2}{3\sqrt{2}}+\frac{9\pi}{16}-\frac{1}{2\sqrt{2}}-\frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$=2\left[\frac{4-3}{6\sqrt{2}}+\frac{9\pi}{16}-\frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$=2\left[\frac{1}{6\sqrt{2}}+\frac{9\pi}{16}-\frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$=\left[\frac{1}{3\sqrt{2}}+\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}\right] \text{sq units} \quad \textbf{(1)}$$

21. Do same as Q. No. 8. **[Ans.** $\frac{9\pi}{4}$ sq units**]**

22. Given curves are

$$y=|x+1|+1=\begin{cases}(x+1)+1, & \text{if } x+1\ge 0\\ -(x+1)+1, & \text{if } x+1<0\end{cases} \quad \text{...(i)}$$

$$=\begin{cases}x+2, & \text{if } x\ge -1\\ -x, & \text{if } x<-1\end{cases}$$

$$x=-3 \quad \text{...(ii)}$$

$$x=3 \quad \text{...(iii)}$$

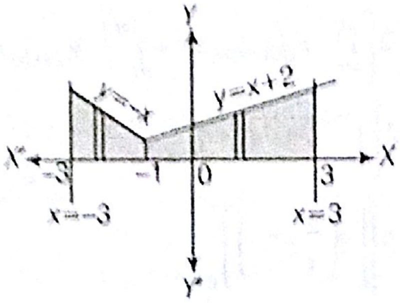
and $y=0$ ...(iv) **(1)**

Eq. (ii) represents the line parallel to Y-axis and passes through the point (– 3, 0).

Eq. (iii) represents the line parallel to Y-axis and passes through the point (3, 0). **(1)**

Eq. (iv) represent X-axis.

Now, Eqs. (i), (ii) (iii) and (iv) can be represented in graph as shown below:



(2)

Clearly, required area

$$= \int_{-3}^{-1} (-x)\, dx + \int_{-1}^{3} (x+2)\, dx \quad (1)$$

$$= -\left[\frac{x^2}{2}\right]_{-3}^{-1} + \left[\frac{x^2}{2} + 2x\right]_{-1}^{3}$$

$$= -\frac{1}{2}(1-9) + \left[\left(\frac{9}{2}+6\right) - \left(\frac{1}{2}-2\right)\right]$$

$$= 4 + \frac{21}{2} + \frac{3}{2} = 16 \text{ sq units}$$

Hence, the required area is 16 sq units. (1)

23. Do same as Q. No. 1. [**Ans.** 7 sq units]

24. Do same as Q. No. 1. [**Ans.** 4 sq units]

25. Given equation of ellipse is

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \quad \text{...(i)}$$

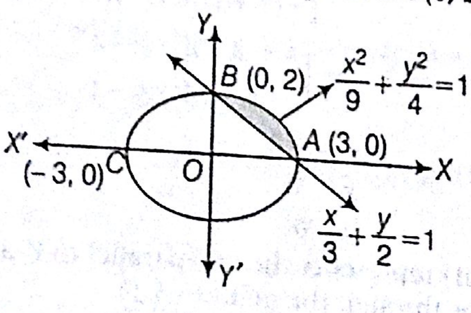
and equation of line is $\frac{x}{3} + \frac{y}{2} = 1$ ...(ii) (1)

For the points of intersection of ellipse and line, put the value of x from Eq. (ii) in Eq. (i), we get

$$\left(1 - \frac{y}{2}\right)^2 + \frac{y^2}{4} = 1 \Rightarrow 1 + \frac{y^2}{4} - y + \frac{y^2}{4} = 1$$

$$\Rightarrow \quad y^2 - 2y = 0 \Rightarrow y = 0, 2 \quad (1)$$

When $y = 0$, then $x = 3$ and point is $A(3, 0)$.
When $y = 2$, then $x = 0$ and point is $B(0, 2)$.



(1)

Clearly, required area $= \int_0^3 y_{(\text{ellipse})}\, dx - \int_0^3 y_{(\text{line})}\, dx$ (1)

$$= \int_0^3 2\sqrt{1 - \frac{x^2}{9}}\, dx - \int_0^3 2\left(1 - \frac{x}{3}\right) dx$$

$$= \frac{2}{3}\int_0^3 \sqrt{3^2 - x^2}\, dx - \frac{2}{3}\int_0^3 (3-x)\, dx$$

$$= \frac{2}{3}\left[\frac{x}{2}\sqrt{3^2 - x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3 - \frac{2}{3}\left[3x - \frac{x^2}{2}\right]_0^3 \quad (1)$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{2}{3}\left[0 + \frac{9}{2}\sin^{-1}(1) - 0\right] - \frac{2}{3}\left[9 - \frac{9}{2} - 0\right]$$

$$= \frac{2}{3} \times \frac{9}{2} \cdot \frac{\pi}{2} - \frac{2}{3} \times \frac{9}{2} = \frac{3}{2}(\pi - 2) \text{ sq units}$$

(1)

26. Given lines are

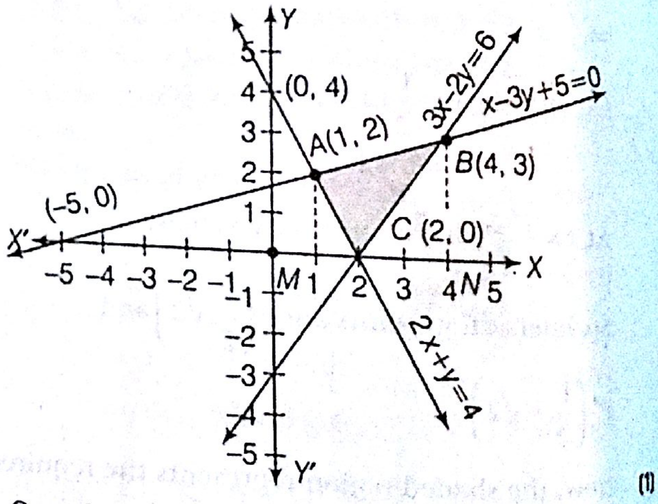
$$2x + y = 4 \quad \text{...(i)}$$

$$3x - 2y = 6 \quad \text{...(ii)}$$

and $$x - 3y = -5 \quad \text{...(iii)}$$

Clearly, the line $2x + y = 4$ passes through the points (2, 0) and (0, 4), the line $3x - 2y = 6$ passes through the points (2, 0) and (0, –3) and the line $x - 3y = -5$ passes through the points (–5, 0) and $\left(0, \frac{5}{3}\right)$. (1/2)

Now, the region bounded by these lines is shown below:



(1)

On solving Eqs. (i) and (ii), we get

$$x = 2 \text{ and } y = 0$$

So, lines $2x + y = 4$ and $3x - 2y = 6$ meet at the point $C(2, 0)$. (1)

Again, solving Eqs. (ii) and (iii), we get

$$x = 4 \text{ and } y = 3$$

So, lines $3x - 2y = 6$ and $x - 3y = -5$ meet at the point $B(4, 3)$. (1)

On solving Eqs. (iii) and (i), we get

$$x = 1 \text{ and } y = 2$$

So, lines $2x + y = 4$ and $x - 3y = -5$ meet at the point $A(1, 2)$. (1)

Now, required area of ΔABC

$= $ Area of region $ABNMA$

$- $(Area of ΔAMC + Area of ΔBCN)

$$= \int_1^4 y_{(\text{for line } AB)}\, dx - \int_1^2 y_{(\text{for line } AC)}\, dx - \int_2^4 y_{(\text{for line } BC)}\, dx \quad \textbf{(1/2)}$$

$$= \int_1^4 \left(\frac{x+5}{3}\right) dx - \int_1^2 (4-2x)\, dx - \int_2^4 \left(\frac{3x-6}{2}\right) dx$$

$$= \frac{1}{3}\left[\frac{x^2}{2} + 5x\right]_1^4 - [4x - x^2]_1^2 - \frac{1}{2}\left[\frac{3x^2}{2} - 6x\right]_2^4$$

$$= \frac{1}{3}\left[\left(\frac{16}{2} + 20\right) - \left(\frac{1}{2} + 5\right)\right] - [(8-4)-(4-1)] - \frac{1}{2}[(24-24)-(6-12)]$$

$$= \frac{1}{3}\left(28 - \frac{11}{2}\right) - 1 - \frac{1}{2}(0+6)$$

$$= \frac{1}{3} \times \frac{45}{2} - 1 - 3$$

$$= \frac{15}{2} - 4 = \frac{7}{2} \text{ sq units} \quad \textbf{(1)}$$

27. Given curves are

$$x^2 = 4y \quad \ldots\text{(i)}$$

and $$x = 4y - 2 \quad \ldots\text{(ii)}$$

Eq. (i) represents a parabola with vertex at origin and axis along positive direction of Y-axis. Eq. (ii) represents a straight line which meets the coordinate axes at $(-2, 0)$ and $\left(0, \frac{1}{2}\right)$ respectively. **(1)**

To find the points of intersection of the given parabola and the line, we solve Eqs. (i) and (ii), simultaneously.

On substituting $x = 4y - 2$ in Eq. (i), we get

$$(4y-2)^2 = 4y$$

$\Rightarrow$ $$16y^2 + 4 - 16y = 4y$$

$\Rightarrow$ $$16y^2 - 20y + 4 = 0$$

$\Rightarrow$ $$4y^2 - 5y + 1 = 0$$

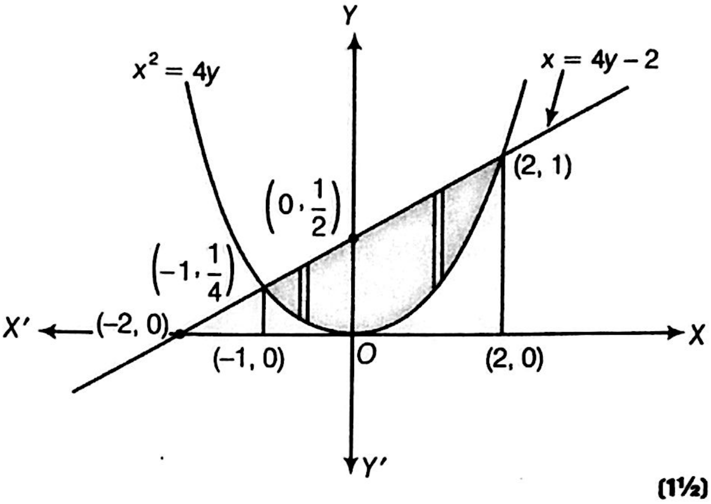
$\Rightarrow$ $$(4y-1)(y-1) = 0$$

$\Rightarrow$ $$y = 1, \frac{1}{4}$$

On putting the values of y in Eq. (ii), we get

$$x = 2, -1$$

So, the points of intersection of the given parabola and the line are $(2,1)$ and $(-1, 1/4)$. **(1½)**



(1½)

The region whose area is to be found out is shaded in figure.

$\therefore$ Required area, A is given by

$$A = \int_{-1}^{2} [y_{(\text{line})} - y_{(\text{parabola})}]dx \quad \textbf{(1)}$$

$$= \int_{-1}^{2} \left(\frac{x+2}{4} - \frac{x^2}{4}\right)dx = \left[\frac{x^2}{8} + \frac{1}{2}x - \frac{x^3}{12}\right]_{-1}^{2}$$

$$= \left[\frac{4}{8} + \frac{2}{2} - \frac{8}{12}\right] - \left[\frac{1}{8} - \frac{1}{2} + \frac{1}{12}\right]$$

$$= \left(\frac{12 + 24 - 16}{24}\right) - \left(\frac{3 - 12 + 2}{24}\right)$$

$$= \frac{27}{24} = \frac{9}{8} \text{ sq units} \quad \textbf{(1)}$$

28. Do same as Q.No. 27. $\left[\textbf{Ans. } \frac{1}{6} \text{ sq units}\right]$

29. Do same as Q. No. 20.

$\left[\textbf{Ans. } \left(\frac{4\sqrt{3}}{3}a^2 + \frac{16a^2\pi}{3}\right) \text{sq units}\right]$

30. Given curves are $x^2 = y$ ...(i)

and $y = |x|$ ...(ii)

From Eqs. (i) and (ii), we get

$$x^2 = |x| \quad \textbf{(1)}$$

***Case* I** When $x \le 0$

Then, $x^2 = -x \Rightarrow x(x+1) = 0$

$\therefore$ $x = 0, -1$

On putting the values of x in Eq. (i), we get

$$y = 0, 1 \quad \textbf{(1)}$$

Case **II** When $x \geq 0$

Then, $x^2 = x \Rightarrow x(x-1) = 0$

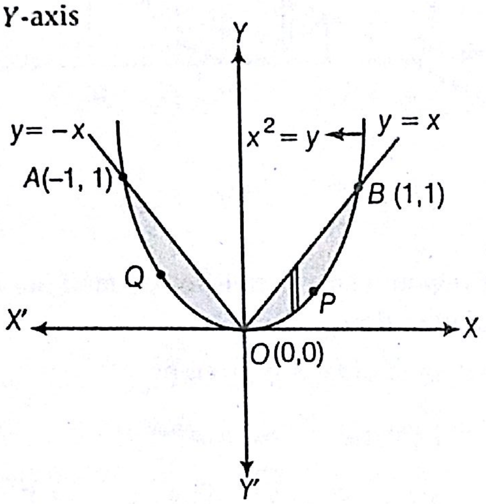
$\therefore \quad x = 0, 1$

On putting the values of x in Eq. (i), we get

$y = 0, 1$ **(1)**

So, both curves cut each other at points $A(-1, 1)$, $O(0, 0)$ and $B(1, 1)$.

The graphs of given curves is shown below, clearly the shaded region is symmetrical about the Y-axis



(1)

Now, area of region $OPBO$

$$= \int_0^1 [y_{(\text{line})} - y_{(\text{parabola})}]dx = \int_0^1 (x - x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \left[\frac{1}{2} - \frac{1}{3}\right] = \frac{3-2}{6} = \frac{1}{6}$$ **(1)**

Hence, required area $= 2 \times$ Area of region $OPBO$

[$\because$ region is symmetrical about Y-axis]

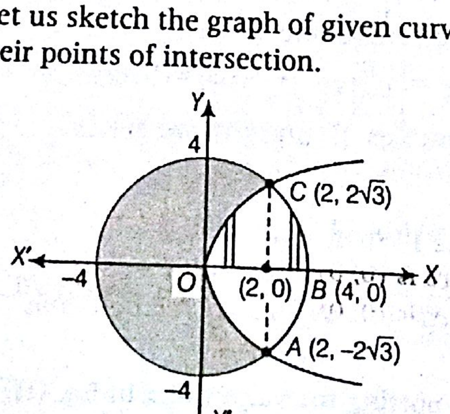
$$= 2 \times \frac{1}{6} = \frac{1}{3} \text{ sq unit}$$ **(1)**

31. Given, equation of circle is $x^2 + y^2 = 16$ ...(i)

and equation of parabola is $y^2 = 6x$ ...(ii)

Clearly, the given circle has centre $(0, 0)$ and radius 4 units and the given parabola has vertex $(0, 0)$ and axis parallel to X-axis. **(1/2)**

Now, let us sketch the graph of given curves and find their points of intersection.



(1)

On substituting $y^2 = 6x$ in Eq. (i), we get

$$x^2 + 6x - 16 = 0 \Rightarrow (x+8)(x-2) = 0$$

$$\Rightarrow \quad x = -8 \text{ or } x = 2$$

Clearly, from Eq. (ii), when $x = -8$, then $y^2 = -48$, which is not possible. So, $x \neq -8$

$\therefore \quad x = 2$ **(1/2)**

Now, on substituting $x = 2$ in Eq. (ii), we get

$$y^2 = 12 \Rightarrow y = \pm 2\sqrt{3}$$

Thus, the points of intersection are $(2, -2\sqrt{3})$ and $(2, 2\sqrt{3})$.

Clearly, required area = Area of shaded region **(1/2)**

= Area of circle − Area of region $OABCO$

$= \pi(4)^2 - 2$(Area of region $OBCO$) **(1)**

$$= 16\pi - 2\left[\int_0^2 y_{(\text{parabola})}\,dx + \int_2^4 y_{(\text{circle})}\,dx\right]$$

$$= 16\pi - 2\left[\int_0^2 \sqrt{6x}\,dx + \int_2^4 \sqrt{16 - x^2}\,dx\right]$$

$$\left[\begin{array}{l} \because x^2 + y^2 = 16 \Rightarrow y^2 = 16 - x^2 \\ \Rightarrow y = \pm\sqrt{16 - x^2} \text{ and } y^2 = 6x \\ \Rightarrow y = \pm\sqrt{6x} \text{ But in region } OBCO,\ y \geq 0 \end{array}\right]$$

$$= 16\pi - 2\left[\sqrt{6}\int_0^2 x^{1/2}\,dx + \int_2^4 \sqrt{4^2 - x^2}\,dx\right]$$

$$= 16\pi - 2\left\{\sqrt{6} \times \frac{2}{3}[x^{3/2}]_0^2 + \left[\frac{x}{2}\sqrt{4^2 - x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_2^4\right\}$$ **(1)**

$$= 16\pi - 2\left\{\frac{2\sqrt{6}}{3} \times 2\sqrt{2} + \left[8\sin^{-1}(1) - \left(\sqrt{4^2 - 2^2} + 8\sin^{-1}\left(\frac{1}{2}\right)\right)\right]\right\}$$

$$= 16\pi - 2\left\{\frac{8\sqrt{3}}{3} + \left[8\cdot\frac{\pi}{2} - \left(2\sqrt{3} + 8\cdot\frac{\pi}{6}\right)\right]\right\}$$ **(1/2)**

$$= 16\pi - \frac{16\sqrt{3}}{3} - 8\pi + 4\sqrt{3} + \frac{8\pi}{3}$$

$$= 8\pi + \frac{8\pi}{3} - \frac{4\sqrt{3}}{3}$$

$$= \frac{32\pi}{3} - \frac{4\sqrt{3}}{3}$$

$$= \frac{4}{3}(8\pi - \sqrt{3}) \text{ sq units}$$ **(1)**

32. Do same as Q. No. 26. $\left[\textbf{Ans. } \frac{13}{2} \text{ sq units}\right]$

33. Do same as Q. No. 26. **[Ans.** 10 sq units]

34. Do same as Q. No. 26. $\left[\textbf{Ans. } \frac{21}{2} \text{ sq units}\right]$

35. Given region is $\{(x, y): x^2 + y^2 \le 4, x + y \ge 2\}$.

The above region has a circle with equation

$$x^2 + y^2 = 4 \quad \ldots(i)$$

whose centre is (0, 0) and radius is 2,

and line with equation

$$x + y = 2 \quad \ldots(ii) \quad (1)$$

Point of intersection is calculated as follows

$$x^2 + y^2 = 4$$

$\Rightarrow \quad x^2 + (2-x)^2 = 4$ [from Eq. (ii)]

$\Rightarrow \quad x^2 + 4 + x^2 - 4x = 4$

$\Rightarrow \quad 2x^2 - 4x = 0$

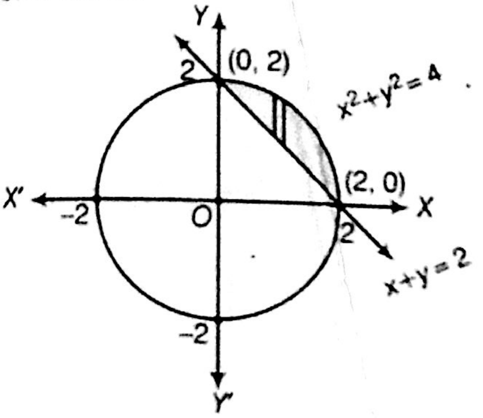
$\Rightarrow \quad 2x(x-2) = 0 \Rightarrow x = 0$ or 2

When $x = 0$, then $y = 2 - 0 = 2$

and when $x = 2$, then $y = 2 - 2 = 0$

So, points of intersection are (0, 2) and (2, 0). (1)

On drawing the graph, we get the shaded region as shown below:



(2)

Clearly, required area $= \int_0^2 [y_{(circle)} - y_{(line)}]\, dx$

$= \int_0^2 \left[\sqrt{4 - x^2} - (2 - x)\right] dx$ (1)

[from Eqs. (i) and (ii)]

$= \int_0^2 \sqrt{4 - x^2}\, dx - \int_0^2 (2 - x)\, dx$

$= \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 - \left[2x - \frac{x^2}{2}\right]_0^2$

$\left[\because \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$

$= \left[0 + 2\sin^{-1}\left(\frac{2}{2}\right) - 0 - 2\sin^{-1} 0\right] - \left(4 - \frac{4}{2} - 0\right)$ (1)

$= (2\sin^{-1} 1 - 0) - \left(4 - \frac{4}{2}\right)$

$= 2 \cdot \frac{\pi}{2} - 2 = (\pi - 2)$ sq units

Hence, required area of region is $(\pi - 2)$ sq units. (1)

36. Do same as Q. No. 35. **[Ans.** $\left(\frac{\pi}{4} - \frac{1}{2}\right)$ sq units]

37. First, we sketch the graph of

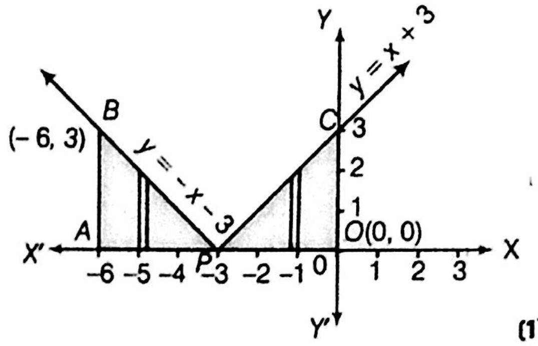
$$y = |x + 3|$$

$\therefore \quad y = |x + 3| = \begin{cases} x + 3, & \text{if } x + 3 \ge 0 \\ -(x + 3), & \text{if } x + 3 < 0 \end{cases}$

$\Rightarrow \quad y = |x + 3| = \begin{cases} x + 3, & \text{if } x \ge -3 \\ -x - 3, & \text{if } x < -3 \end{cases}$ (1½)

So, we have $y = x + 3$ for $x \ge -3$ and $y = -x - 3$ for $x < -3$.

A sketch of $y = |x + 3|$ is shown below:



(1½)

Here, $y = x + 3$ is the straight line which cuts X and Y-axes at (– 3, 0) and (0, 3), respectively.

Thus, $y = x + 3$ for $x \ge -3$ represents the part of line which lies on the right side of $x = -3$.

Similarly, $y = -x - 3$, $x < -3$ represents the part of line $y = -x - 3$, which lies on left side of $x = -3$. (1)

Clearly, required area

= Area of region $ABPA$ + Area of region $PCOP$

$= \int_{-6}^{-3} (-x - 3)\, dx + \int_{-3}^{0} (x + 3)\, dx$ (1)

$= \left[-\frac{x^2}{2} - 3x\right]_{-6}^{-3} + \left[\frac{x^2}{2} + 3x\right]_{-3}^{0}$

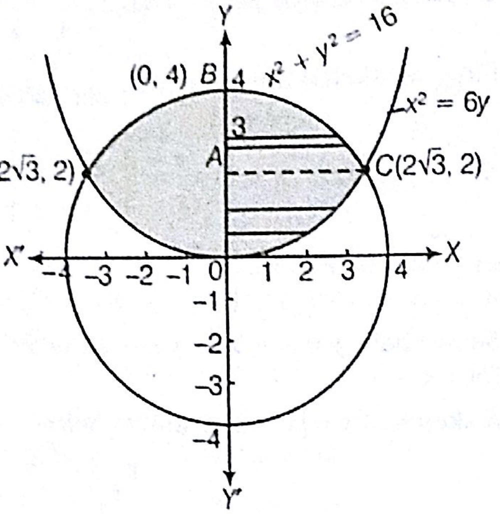
$= \left[\left(-\frac{9}{2} + 9\right) - (-18 + 18)\right] + \left[0 - \left(\frac{9}{2} - 9\right)\right]$

$= \left(-\frac{9}{2} - \frac{9}{2}\right) + (9 + 9) = 18 - 9 = 9$ sq units (1)

38. Given region is $\{(x, y): x^2 + y^2 \le 16, x^2 \le 6y\}$

Above region has a circle $x^2 + y^2 = 16$ whose centre is (0, 0) and radius 4 and a parabola whose vertex is (0, 0) and axis along Y-axis. (1)

First, let us sketch the region, as shown below:



(1)

For finding the points of intersection of two curves, we have

$$x^2 + y^2 = 16 \quad \ldots(i)$$

and
$$x^2 = 6y \quad \ldots(ii)$$

On putting $x^2 = 6y$ from Eq. (ii) in Eq. (i), we get

$$y^2 + 6y - 16 = 0 \Rightarrow y^2 + 8y - 2y - 16 = 0$$

$$\Rightarrow y(y+8) - 2(y+8) = 0 \Rightarrow (y-2)(y+8) = 0$$

$$\Rightarrow \quad y = 2 \text{ or } -8$$

When $y = 2$, then from Eq. (ii), we get

$$x = \pm\sqrt{12} = \pm 2\sqrt{3}$$

and when $y = -8$, then from Eq. (ii), we get $x^2 = -48$ which is not possible.

So, $y = -8$ is rejected. (1)

Thus, the two curves meet at points $C(2\sqrt{3}, 2)$ and $D(-2\sqrt{3}, 2)$.

Now, required area

= Area of shaded region *OCBDO*

= 2[Area of region *OACO*

+ Area of region *ABCA*]

$$= 2\left[\int_0^2 x_{\text{(parabola)}}\,dy + \int_2^4 x_{\text{(circle)}}\,dy\right] \quad (1)$$

$$= 2\left[\int_0^2 \sqrt{6y}\,dy + \int_2^4 \sqrt{16-y^2}\,dy\right]$$

$$= 2\left[\sqrt{6}\int_0^2 \sqrt{y}\,dy + \int_2^4 \sqrt{16-y^2}\,dy\right]$$

$$= 2\left\{\left[\sqrt{6}\cdot y^{3/2}\cdot\frac{2}{3}\right]_0^2 + \left[\frac{y}{2}\sqrt{16-y^2} + \frac{16}{2}\sin^{-1}\frac{y}{4}\right]_2^4\right\}$$

$$\left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= 2\left\{\left[\frac{2\sqrt{6}}{3}y^{3/2}\right]_0^2 + \left[\frac{y}{2}\sqrt{16-y^2} + 8\sin^{-1}\frac{y}{4}\right]_2^4\right\} \quad (1)$$

$$= 2\left\{\left[\frac{2\sqrt{6}}{3}\cdot(2)^{3/2}\right] + \left[0 + 8\sin^{-1}1 - \sqrt{12} - 8\sin^{-1}\frac{1}{2}\right]\right\}$$

$$= 2\left[\frac{2\times\sqrt{2}\times\sqrt{3}}{3}\times[(\sqrt{2})^2]^{3/2} + 8\sin^{-1}\left(\sin\frac{\pi}{2}\right) - 2\sqrt{3} - 8\sin^{-1}\left(\sin\frac{\pi}{6}\right)\right]$$

$$\left[\because 1 = \sin\frac{\pi}{2} \text{ and } \frac{1}{2} = \sin\frac{\pi}{6}\right]$$

$$= 2\left[\frac{2\sqrt{2}\times\sqrt{3}}{3}\times 2\sqrt{2} + 4\pi - 2\sqrt{3} - \frac{8\pi}{6}\right]$$

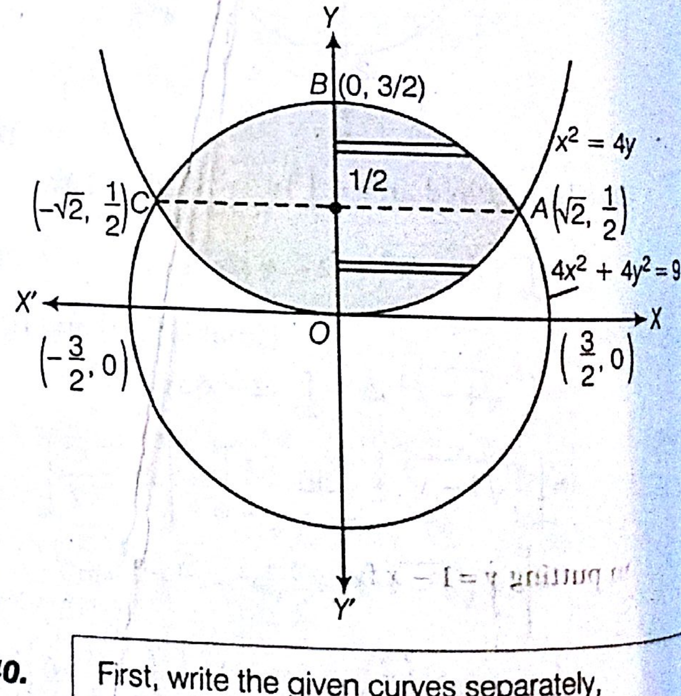
$$= 2\left[\frac{8\sqrt{3}}{3} + 4\pi - 2\sqrt{3} - \frac{4\pi}{3}\right]$$

$$= 2\left[\frac{2\sqrt{3}}{3} + \frac{8\pi}{3}\right] = \left(\frac{4\sqrt{3}}{3} + \frac{16\pi}{3}\right) \text{ sq units} \quad (1)$$

39. Do same as Q. No. 38.

$$\left[\textbf{Ans. } \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \text{sq units}\right]$$

Hint



40. First, write the given curves separately, i.e. $y = |x-1|$ and $y = \sqrt{5-x^2}$.

Then, sketch all the above defined functions and find the required area.

Given region is $\{(x, y) : |x-1| \le y \le \sqrt{5-x^2}\}$

Above region has two equations

$$y = |x-1| \quad \text{and} \quad y = \sqrt{5-x^2}$$

Now, $y=|x-1|=\begin{cases} x-1, & \text{if } x-1\ge 0 \\ -(x-1), & \text{if } x-1<0 \end{cases}$

$\therefore \quad y=\begin{cases} x-1, & \text{if } x\ge 1 \\ 1-x, & \text{if } x<1 \end{cases}$ **(1)**

Also, other curve is $y=\sqrt{5-x^2}$

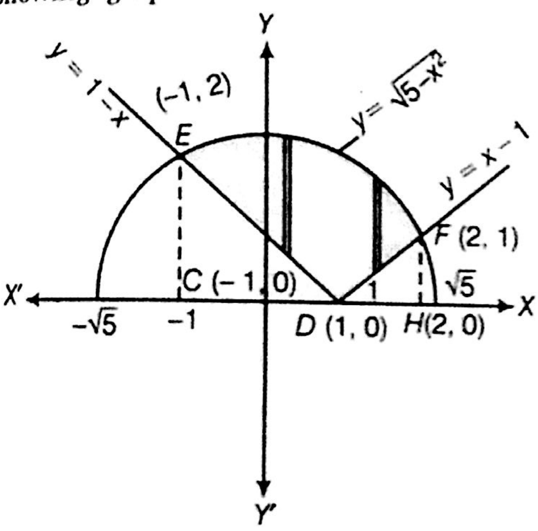
On squaring both sides, we get

$$y^2=5-x^2 \Rightarrow x^2+y^2=5$$

which represents equation of circle with centre (0, 0) and radius, $r=\sqrt{5}$.

But the actual equation of curve is $y=\sqrt{5-x^2}$ which represents a semi-circle whose centre is (0, 0) and radius $r=\sqrt{5}$.

On drawing the rough sketch, we get the following graph:



(1)

For finding the points of intersection of the curves, we have

$$y=1-x \quad \text{...(i)}$$

$$y=x-1 \quad \text{...(ii)}$$

and $$y=\sqrt{5-x^2} \quad \text{...(iii)}$$

On putting $y=1-x$ from Eq. (i) in Eq. (iii), we get

$$(1-x)=\sqrt{5-x^2}$$

$\Rightarrow \quad x^2+(1-x)^2=5$

$\Rightarrow \quad x^2+1+x^2-2x=5$

$\Rightarrow \quad 2x^2-2x-4=0$

$\Rightarrow \quad x^2-x-2=0$

$\Rightarrow \quad x^2-2x+x-2=0$

$\Rightarrow \quad x(x-2)+1(x-2)=0$

$\Rightarrow \quad (x+1)(x-2)=0$

$\therefore \quad x=-1$ or 2

Now, when $x=-1$, then

$$y=\sqrt{5-x^2}=\sqrt{5-1}=\sqrt{4} \Rightarrow y=2$$

and when $x=2$, then

$$y=\sqrt{5-x^2}=\sqrt{5-4}=1 \Rightarrow y=1$$

So, points of intersection of Eqs. (i) and (iii) are (− 1, 2) and (2, 1). **(1)**

Similarly, on solving Eq. (ii) and Eq. (iii), we get

$$x=-1 \text{ or } 2$$

From Eq. (iii), at $x=-1$, $y=2$ and at $x=2$, $y=1$.

Hence, the two curves intersect at (− 1, 2) and (2, 1). **(1)**

Now, required area

$$=\int_{-1}^{2} y_{(\text{circle})}\,dx-\int_{-1}^{1} y_{(\text{for line } DE)}\,dx-\int_{1}^{2} y_{(\text{for line } DF)}\,dx$$

$$=\int_{-1}^{2}\sqrt{5-x^2}\,dx-\int_{-1}^{1}(1-x)\,dx-\int_{1}^{2}(x-1)\,dx$$

$$=\left[\frac{x}{2}\sqrt{5-x^2}+\frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}\right]_{-1}^{2}-\left[x-\frac{x^2}{2}\right]_{-1}^{1}-\left[\frac{x^2}{2}-x\right]_{1}^{2} \quad \textbf{(1)}$$

$$\left[\because \int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$=\left[\left(1+\frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\right)-\left\{-1+\frac{5}{2}\sin^{-1}\left(-\frac{1}{\sqrt{5}}\right)\right\}\right]-\left[\left(1-\frac{1}{2}\right)-\left(-1-\frac{1}{2}\right)\right]-\left[\left(\frac{4}{2}-2\right)-\left(\frac{1}{2}-1\right)\right]$$

$$=1+\frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}+1-\frac{5}{2}\sin^{-1}\left(-\frac{1}{\sqrt{5}}\right)-\left(\frac{1}{2}+\frac{3}{2}\right)-\left(0+\frac{1}{2}\right)$$

$$=1+\frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}+1+\frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}}-2-\frac{1}{2}$$

$$[\because \sin^{-1}(-\theta)=-\sin^{-1}\theta]$$

$$=-\frac{1}{2}+\frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}}+\sin^{-1}\frac{1}{\sqrt{5}}\right)$$

Hence, required area

$$=\left[\frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}}+\sin^{-1}\frac{1}{\sqrt{5}}\right)-\frac{1}{2}\right] \text{ sq units} \quad \textbf{(1)}$$

Objective Questions

(For Complete Chapter)

1 Mark Questions

1. The area of the region bounded by the lines $y = mx, x = 1, x = 2$ and X-axis is 6 sq units, then m is equal to
(a) 3 (b) 1
(c) 2 (d) 4

2. Find the area of a curve $xy = 4$, bounded by the lines $x = 1$ and $x = 3$ and X-axis.
(a) log 12 (b) log 64 (c) log 81 (d) log 27

3. The area enclosed by $y = 3x - 5, y = 0, x = 3$ and $x = 5$ is
(a) 12 sq units (b) 13 sq units
(c) $13\frac{1}{2}$ sq units (d) 14 sq units

4. The area bounded by $y = \log x$, X-axis and ordinates $x = 1, x = 2$ is
(a) $\frac{1}{2}(\log 2)^2$ (b) $\log(2/e)$
(c) $\log(4/e)$ (d) $\log 4$

5. The area bounded by the curve $y = \frac{1}{2}x^2$, the X-axis and the ordinate $x = 2$ is
(a) $\frac{1}{3}$ sq unit (b) $\frac{2}{3}$ sq unit
(c) 1 sq unit (d) $\frac{4}{3}$ sq units

6. Area of the region satisfying $x \le 2, y \le |x|$ and $x \ge 0$ is
(a) 4 sq units (b) 1 sq unit
(c) 2 sq units (d) None of these

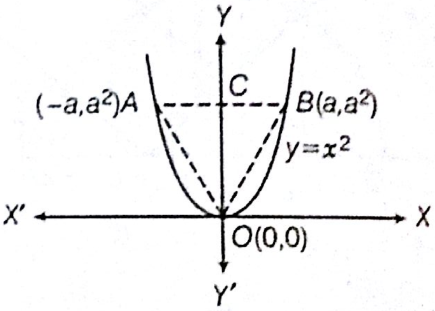
7. The area bounded by the parabola $y^2 = 8x$ and its latusrectum is
(a) 16/3 sq units
(b) 32/3 sq units
(c) 8/3 sq units
(d) 64/3 sq units

8. The area bounded by $y = |\sin x|$, X-axis and the lines $|x| = \pi$ is
(a) 2 sq units (b) 3 sq units
(c) 4 sq units (d) None of these

9. The area bounded by the curves $y^2 = 8x$ and $x^2 = 8y$ is
(a) 64 sq units (b) $\frac{64}{3}$ sq units
(c) $\frac{8}{3}$ sq units (d) None of these

10. The area enclosed between $y^2 = x$ and $y = x$ is
(a) $\frac{2}{3}$ sq unit (b) $\frac{1}{2}$ sq unit
(c) $\frac{1}{3}$ sq unit (d) $\frac{1}{6}$ sq unit

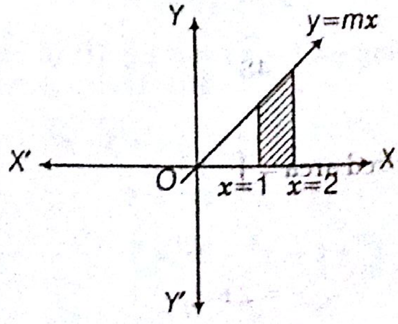
11. The given figure shows a ΔAOB and the parabola $y = x^2$. The ratio of the area of the ΔAOB to the area of the region AOB of the parabola $y = x^2$ is equal to



(a) $\frac{3}{5}$ (b) $\frac{3}{4}$ (c) $\frac{7}{8}$ (d) $\frac{5}{6}$

Solutions

1. (*d*) Given, equation of line is $y = mx$ and bounded by $x = 1$, $x = 2$ and X-axis.

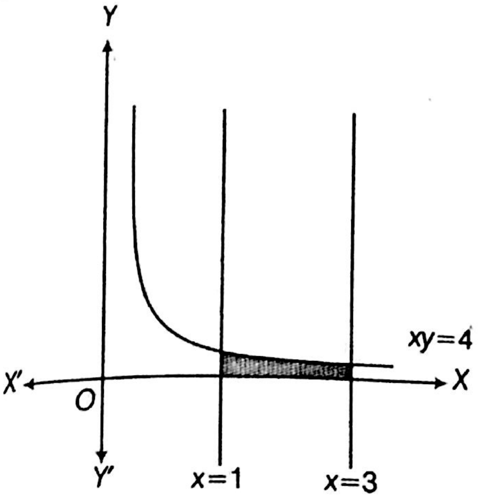


$\therefore$ Required area $= \int_1^2 mx\, dx$

$\Rightarrow 6 = m\left[\frac{x^2}{2}\right]_1^2 \Rightarrow 6 = m\left(\frac{4}{2} - \frac{1}{2}\right) \Rightarrow 6 = m \times \frac{3}{2}$

$\therefore m = 4$

2. (c) Given curve is $xy = 4$.

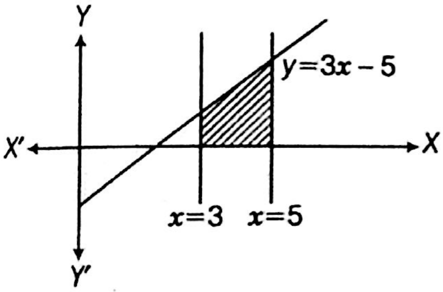


$\therefore$ Required area $= \int_1^3 \frac{4}{x} dx = 4 \cdot [\log x]_1^3$

$= 4(\log 3 - \log 1)$

$= 4 \log 3 = \log 81$

3. (d) The region is bounded by the curves $y = 3x - 5$, $y = 0$, $x = 3$ and $x = 5$.



$\therefore$ Required area $= \int_3^5 (3x - 5)\, dx$

$= \left[\frac{3x^2}{2} - 5x\right]_3^5$

$= \left(\frac{75}{2} - 25\right) - \left(\frac{27}{2} - 15\right)$

$= \frac{75}{2} - 25 - \frac{27}{2} + 15$

$= \frac{48}{2} - 10 = 14$ sq units

4. (c) Required area $= \int_1^2 \log x\, dx$

$= [x \log x - x]_1^2$

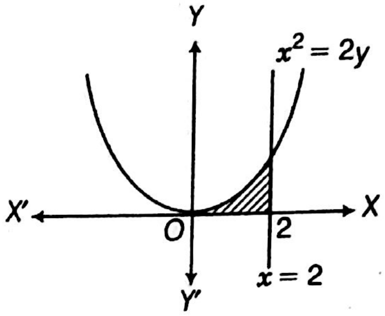
$= 2 \log 2 - 1$

$= \log 4 - \log e$

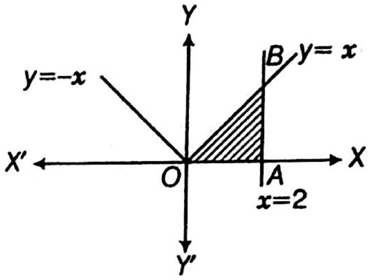
$= \log\left(\frac{4}{e}\right)$

5. (d) Required area $= \int_0^2 y\, dx$

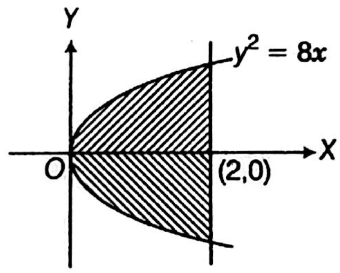
$= \int_0^2 \frac{x^2}{2} dx = \left[\frac{x^3}{6}\right]_0^2 = \frac{4}{3}$ sq units



6. (c) Required area $= \int_0^2 x\, dx = \left[\frac{x^2}{2}\right]_0^2 = 2$ sq units

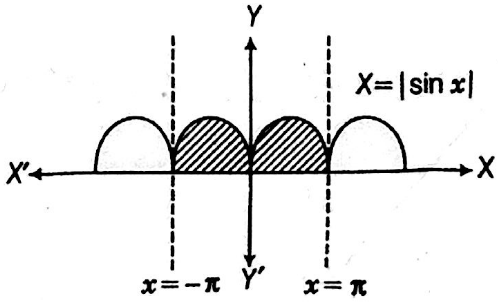


7. (b) Required area $= 2\int_0^2 \sqrt{8x}\, dx = 4\sqrt{2}\left[\frac{x^{3/2}}{3/2}\right]_0^2$



$= 4\sqrt{2}\left[\frac{2\sqrt{2}}{3/2}\right] = \frac{32}{3}$ sq units

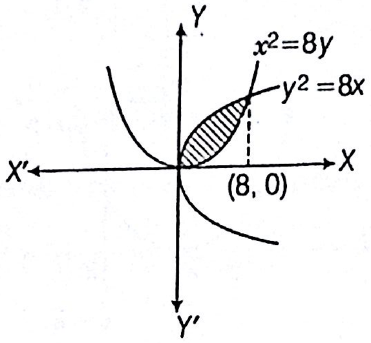
8. (c) Required area $= 2\int_0^{\pi} \sin x\, dx$



$= 2[-\cos x]_0^{\pi} = 2[1 + 1] = 4$ sq units

9. (*b*) Given, curves are $y^2 = 8x \Rightarrow y = \sqrt{8x}$

and $x^2 = 8y \Rightarrow y = \dfrac{x^2}{8}$



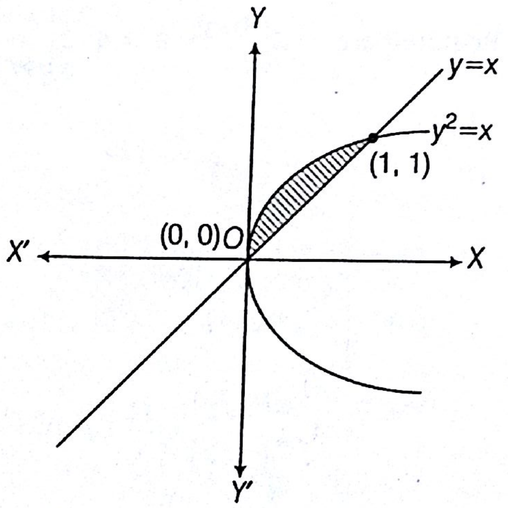
The points of intersection of two curves are (0, 0) and (8, 8).

∴ Required area

$$= \int_0^8 \left(\sqrt{8x} - \frac{x^2}{8}\right) dx = \left[\frac{\sqrt{8}\, x^{3/2}}{3/2} - \frac{x^3}{8 \cdot 3}\right]_0^8$$

$$= \frac{64}{3} \text{ sq units}$$

10. (*d*) The points of intersection of the curves $y = x$ and $y^2 = x$ are (0, 0) and (1, 1).



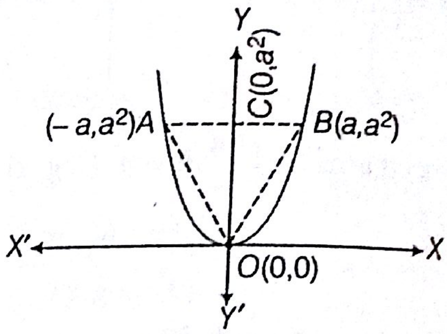
∴ Required area $= \int_0^1 (\sqrt{x} - x)\, dx$

$$= \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{2}$$

$$= \frac{1}{6} \text{ sq unit}$$

11. (*b*) Area of curve $AOB = 2\int_0^{a^2} x\, dy$



$$= 2\int_0^{a^2} \sqrt{y}\, dy$$

$$= 2\left[\frac{y^{3/2}}{3/2}\right]_0^{a^2}$$

$$= \frac{4}{3}[a^3]$$

Now, area of $\Delta AOB = \dfrac{1}{2} \times AB \times OC$

$$= \frac{1}{2} \times 2a \times a^2 = a^3$$

$$\therefore \frac{\text{Area of } \Delta AOB}{\text{Area of curve } AOB} = \frac{a^3}{\frac{4}{3}a^3}$$

$$= \frac{3}{4}$$

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