

Chapter 2 - Inverse trigonometric functions

Exercise 2.1

Question 1:

Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$

Solution 1:

Let $\sin^{-1}\left(-\frac{1}{2}\right) = y$, Then $\sin y = \left(-\frac{1}{2}\right) = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of \sin^{-1} is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

and $\sin\left(-\frac{\pi}{6}\right) = \frac{1}{2}$,

Therefore, the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is $-\frac{\pi}{6}$.

Question 2:

Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution 2:

Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$. Then $\cos y = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right)$

We know that the range of the principal value branch of \cos^{-1} is $[0, \pi]$

and $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

Therefore, the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$.

Question 3:

Find the principal value of $\operatorname{cosec}^{-1}(2)$

Solution 3:

Let $\operatorname{cosec}^{-1}(2) = y$.

Then, $\operatorname{cosec} y = 2 = \operatorname{cosec}\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of $\operatorname{cosec}^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Therefore, the principal value of $\operatorname{cosec}^{-1}(2)$ is $\frac{\pi}{6}$.

Question 4:

Find the principal value of $\tan^{-1}(-\sqrt{3})$

Solution 4:

Let $\tan^{-1}(-\sqrt{3}) = y$

Then, $\tan y = -\sqrt{3} = -\tan\frac{\pi}{3} = \tan\left(-\frac{\pi}{3}\right)$.

We know that the range of the principal value branch of \tan^{-1} is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

and $\tan\left(-\frac{\pi}{3}\right)$ is $-\sqrt{3}$.

Therefore, known that the principal value of $\tan^{-1}(-\sqrt{3})$ is $-\frac{\pi}{3}$.

Question 5:

Find the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$

Solution 5:

Let $\cos^{-1}\left(-\frac{1}{2}\right) = y$.

Then, $\cos y = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$.

We know that the range of the principal value branch of \cos^{-1} is $[0, \pi]$

and $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$.

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $\left(\frac{2\pi}{3}\right)$.

Question 6:

Find the principal value of $\tan^{-1}(-1)$

Solution 6:

Let $\tan^{-1}(-1) = y$.

Then, $\tan y = -1 = -\tan\left(\frac{\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right)$.

We know that the range of the principal value branch of \tan^{-1} is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

and $\tan\left(-\frac{\pi}{4}\right) = -1$.

Therefore, the principal value of $\tan^{-1}(-1)$ is $-\frac{\pi}{4}$.

Question 7:

Find the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Solution 7:

Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y$. Then, $\sec y = \frac{2}{\sqrt{3}} = \sec\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of \sec^{-1} is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

and $\sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$.

Therefore, the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is $\frac{\pi}{6}$.

Question 8:

Find the principal value of $\cot^{-1}(\sqrt{3})$

Solution 8:

Let $\cot^{-1}(\sqrt{3}) = y$. Then $\cot y = \sqrt{3} = \cot\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of \cot^{-1} is $(0, \pi)$

and $\cot\left(\frac{\pi}{6}\right) = \sqrt{3}$.

Therefore, the principal value of $\cot^{-1}(\sqrt{3})$ is $\frac{\pi}{6}$.

Question 9:

Find the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution 9:

Let $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = y$.

Then $\cos y = -\frac{1}{\sqrt{2}} = -\cos\left(\frac{\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right)$.

We know that the range of the principal value branch of \cos^{-1} is $[0, \pi]$

and $\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $\frac{3\pi}{4}$.

Question 10:

Find the principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$

Solution 10:

Let $\operatorname{cosec}^{-1}(-\sqrt{2}) = y$. Then, $\operatorname{cosec} y = -\sqrt{2} = -\operatorname{cosec}\left(\frac{\pi}{4}\right) = \operatorname{cosec}\left(-\frac{\pi}{4}\right)$.

We know that the range of the principal value branch of $\operatorname{cosec}^{-1}$ is

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \text{ and } \operatorname{cosec}\left(-\frac{\pi}{4}\right) = -\sqrt{2}.$$

Therefore, the principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$ is $-\frac{\pi}{4}$.

Question 11:

Find the value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

Solution 11:

Let $\tan^{-1}(1) = x$.

$$\text{Then, } \tan x = 1 = \tan\left(\frac{\pi}{4}\right).$$

$$\therefore \tan^{-1}(1) = \frac{\pi}{4}$$

Let $\cos^{-1}\left(-\frac{1}{2}\right) = y$.

$$\text{Then, } \cos y = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right).$$

$$\therefore \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

Let $\sin^{-1}\left(-\frac{1}{2}\right) = z$.

$$\text{Then, } \sin z = -\frac{1}{2} = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right).$$

$$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

$$\therefore \tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$$

$$= \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6}$$

$$= \frac{3\pi + 8\pi - 2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}$$

Question 12:

Find the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$

Solution 12:

Let $\cos^{-1}\left(\frac{1}{2}\right) = x$.

Then, $\cos x = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$.

$\therefore \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Let $\sin^{-1}\left(\frac{1}{2}\right) = y$.

Then, $\sin y = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + 2 \times \frac{\pi}{6} = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

Question 13:

If $\sin^{-1} x = y$, then

(A) $0 \leq y \leq \pi$

(B) $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

(C) $0 < y < \pi$

(D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Solution 13:

Exercise 2.2

Question 1:

Prove $3 \sin^{-1} x = \sin^{-1} (3x - 4x^3)$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Solution 1:

To Prove $3 \sin^{-1} x = \sin^{-1} (3x - 4x^3)$, where $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Let $x = \sin \theta$. Then, $\sin^{-1} x = \theta$.

We have,

R.H.S

$$\begin{aligned} \sin^{-1} (3x - 4x^3) &= \sin^{-1} (3 \sin \theta - 4 \sin^3 \theta) \\ &= \sin^{-1} (\sin 3\theta) \\ &= 3\theta \\ &= 3 \sin^{-1} x = L.H.S \end{aligned}$$

Question 2:

Prove $3 \cos^{-1} x = \cos^{-1} (4x^3 - 3x)$, $x \in \left[\frac{1}{2}, 1\right]$

Solution 2:

To Prove $3 \cos^{-1} x = \cos^{-1} (4x^3 - 3x)$, $x \in \left[\frac{1}{2}, 1\right]$

Let $x = \cos \theta$. Then, $\cos^{-1} x = \theta$

We have

R.H.S

$$\cos^{-1} (4x^3 - 3x)$$

$$\begin{aligned}
 &= \cos^{-1}(4\cos^3\theta - 3\cos\theta) \\
 &= \cos^{-1}(\cos 3\theta) \\
 &= 3\theta \\
 &= 3\cos^{-1}x = L.H.S
 \end{aligned}$$

Question 3:

Prove $\tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{7}{24}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

Solution 3:

To prove: $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

L.H.S.

$$\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24}$$

$$= \tan^{-1}\frac{\frac{2}{11} + \frac{7}{24}}{1 - \left(\frac{2}{11} \cdot \frac{7}{24}\right)} \quad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy} \right]$$

$$= \tan^{-1}\frac{11 \times 24}{11 \times 24 - 14}$$

$$= \tan^{-1}\frac{48+77}{264-14}$$

$$= \tan^{-1}\left(\frac{125}{250}\right) = \tan^{-1}\left(\frac{1}{2}\right) = R.H.S.$$

Question 4:

Prove $2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$

Solution 4:

To prove: $2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$

$$\text{L.H.S} = 2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7}$$

$$= \tan^{-1} \frac{2 \cdot \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2} + \tan^{-1} \frac{1}{7} \left[2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2} \right]$$

$$= \tan^{-1} \frac{1}{\left(\frac{3}{4}\right)} + \tan^{-1} \frac{1}{7}$$

$$= \tan^{-1} \frac{4}{3} + \tan^{-1} \frac{1}{7}$$

$$= \tan^{-1} \frac{\frac{4}{3} + \frac{1}{7}}{1 - \frac{4}{3} \cdot \frac{1}{7}} \left[\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right]$$

$$= \tan^{-1} \left(\frac{\frac{28+3}{21}}{\frac{21-4}{21}} \right)$$

$$= \tan^{-1} \frac{31}{17} = \text{R.H.S.}$$

Question 5:

Write the function in the simplest form: $\tan^{-1} \frac{\sqrt{1+x^2}-1}{x}, x \neq 0$

Solution 5:

$$\tan^{-1} \frac{\sqrt{1+x^2}-1}{x}$$

Put $x = \tan \theta \Rightarrow \theta = \tan^{-1} x$

$$\begin{aligned} \therefore \tan^{-1} \frac{\sqrt{1+x^2}-1}{x} &= \tan^{-1} \frac{\sqrt{1+\tan^2 \theta}-1}{\tan \theta} \\ &= \tan^{-1} \left(\frac{\sec \theta - 1}{\tan \theta} \right) = \tan^{-1} \left(\frac{1 - \cos \theta}{\sin \theta} \right) \\ &= \tan^{-1} \left(\frac{2 \sin^2 \left(\frac{\theta}{2} \right)}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \right) \\ &= \tan^{-1} \left(\tan \frac{\theta}{2} \right) = \frac{\theta}{2} = \frac{1}{2} \tan^{-1} x \end{aligned}$$

Question 6:

Write the function in the simplest form: $\tan^{-1} \left(\frac{1}{\sqrt{x^2-1}} \right), |x| > 1$

Solution 6:

$$\tan^{-1} \frac{1}{\sqrt{x^2-1}}, |x| > 1$$

$$\text{Put } x = \operatorname{cosec} \theta \Rightarrow \theta = \operatorname{cosec}^{-1} x$$

$$\begin{aligned} \therefore \tan^{-1} \frac{1}{\sqrt{x^2-1}} \\ &= \tan^{-1} \frac{1}{\sqrt{\operatorname{cosec}^2 \theta - 1}} \\ &= \tan^{-1} \left(\frac{1}{\cot \theta} \right) \\ &= \tan^{-1} (\tan \theta) \\ &= \theta \\ &= \operatorname{cosec}^{-1} x \\ &= \frac{\pi}{2} - \sec^{-1} x \quad \left[\text{As, } \operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2} \right] \end{aligned}$$

Question 7:

Write the function in the simplest form: $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), x < \pi$

Solution 7:

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), x < \pi$$

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$$

$$= \tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$$

$$= \tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$= \frac{x}{2}$$

Question 8:

Write the function in the simplest form: $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right), 0 < x < \pi$

Solution 8:

$$\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$$

$$= \tan^{-1}\left(\frac{1 - \left(\frac{\sin x}{\cos x}\right)}{1 + \left(\frac{\sin x}{\cos x}\right)}\right)$$

$$\begin{aligned}
 &= \tan^{-1} \left(\frac{1 - \tan x}{1 + \tan x} \right) \\
 &= \tan^{-1}(1) - \tan^{-1}(\tan x) \quad \left[\because \tan^{-1} \frac{-y}{1-xy} = \tan^{-1} x - \tan^{-1} y \right] \\
 &= \frac{\pi}{4} - x
 \end{aligned}$$

Question 9:

Write the function in the simplest form: $\tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}, |x| < a$

Solution 9:

$$\tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}$$

$$\text{Let, } x = a \sin \theta \Rightarrow \frac{x}{a} = \sin \theta \Rightarrow \sin^{-1} \left(\frac{x}{a} \right)$$

$$\therefore \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}$$

$$= \tan^{-1} \left(\frac{a \sin \theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} \right)$$

$$= \tan^{-1} \left(\frac{a \sin \theta}{a \sqrt{1 - \sin^2 \theta}} \right)$$

$$= \tan^{-1} \left(\frac{a \sin \theta}{a \cos \theta} \right)$$

$$= \tan^{-1}(\tan \theta) = \theta = \sin^{-1} \frac{x}{a}$$

Question 10:

Write the function in the simplest form: $\tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right), a > 0; \frac{-a}{\sqrt{3}} \leq x \leq \frac{a}{\sqrt{3}}$

Solution 10:

Consider, $\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$

Let

$$x = a \tan \theta \Rightarrow \frac{x}{a} = \tan \theta \Rightarrow \theta = \tan^{-1}\left(\frac{x}{a}\right)$$

$$\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$$

$$= \tan^{-1}\left(\frac{3a^2 \cdot a \tan \theta - a^3 \tan^3 \theta}{a^3 - 3a \cdot a^2 \tan^2 \theta}\right)$$

$$= \tan^{-1}\left(\frac{3a^3 \tan \theta - a^3 \tan^3 \theta}{a^3 - 3a^3 \tan^2 \theta}\right)$$

$$= \tan^{-1}(\tan 3\theta)$$

$$= 3\theta$$

$$= 3 \tan^{-1} \frac{x}{a}$$

Question 11:

Find the value of $\tan^{-1}\left[2 \cos\left(2 \sin^{-1} \frac{1}{2}\right)\right]$

Solution 11:

Let $\sin^{-1} \frac{1}{2} = x$.

$$\text{Then, } \sin x = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right).$$

$$\therefore \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$$

$$\therefore \tan^{-1}\left[2 \cos\left(2 \sin^{-1} \frac{1}{2}\right)\right]$$

$$= \tan^{-1}\left[2 \cos\left(2 \times \frac{\pi}{6}\right)\right]$$

$$\begin{aligned}
 &= \tan^{-1} \left[2 \cos \frac{\pi}{3} \right] \\
 &= \tan^{-1} \left[2 \times \frac{1}{2} \right] \\
 &= \tan^{-1} 1 = \frac{\pi}{4}
 \end{aligned}$$

Question 12:

Find the value of $\cot(\tan^{-1} a + \cot^{-1} a)$

Solution 12:

$$\begin{aligned}
 &\cot(\tan^{-1} a + \cot^{-1} a) \\
 &= \cot\left(\frac{\pi}{2}\right) \quad \left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \right] \\
 &= 0
 \end{aligned}$$

Question 13:

Find the value of $\tan \frac{1}{2} \left[\sin^{-1} \frac{2x}{1+x^2} + \cos^{-1} \frac{1-y^2}{1+y^2} \right]$, $|x| < 1$, $y > 0$ and $xy < 1$

Solution 13:

Let $x = \tan \theta$.

Then, $\theta = \tan^{-1} x$.

$$\begin{aligned}
 \therefore \sin^{-1} \frac{2x}{1+x^2} &= \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) \\
 &= \sin^{-1}(\sin 2\theta) \\
 &= 2\theta \\
 &= 2 \tan^{-1} x
 \end{aligned}$$

Let $y = \tan \theta$. Then, $\theta = \tan^{-1} y$.

$$\begin{aligned}
&\therefore \cos^{-1}\left(\frac{1-y^2}{1+y^2}\right) \\
&= \cos^{-1}\left(\frac{1-\tan^2 \theta}{1+\tan^2 \theta}\right) \\
&= \cos^{-1}(\cos 2\theta) \\
&= 2\theta = 2 \tan^{-1} y \\
&\therefore \tan \frac{1}{2} \left[\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \cos^{-1}\left(\frac{1-y^2}{1+y^2}\right) \right] \\
&= \tan \frac{1}{2} [2 \tan^{-1} x + 2 \tan^{-1} y] \quad \left[\text{As, } \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right] \\
&= \tan \left[\tan^{-1} \left(\frac{x+y}{1-xy} \right) \right] \\
&= \frac{x+y}{1-xy}
\end{aligned}$$

Question 14:

If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then find the value of x .

Solution 14:

$$\begin{aligned}
&\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1 \\
&\Rightarrow \sin\left(\sin^{-1}\frac{1}{5}\right)\cos(\cos^{-1}x) + \cos\left(\sin^{-1}\frac{1}{5}\right)\sin(\cos^{-1}x) = 1 \\
&[\because \sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B] \\
&\Rightarrow \frac{1}{5} \cdot x + \cos\left(\sin^{-1}\frac{1}{5}\right)\sin(\cos^{-1}x) = 1 \\
&\Rightarrow \frac{x}{5} + \cos\left(\sin^{-1}\frac{1}{5}\right)\sin(\cos^{-1}x) = 1 \quad \dots(1)
\end{aligned}$$

Now, let $\sin^{-1}\frac{1}{5} = y$

Then,

$$\sin^{-1} \frac{1}{5} = y$$

$$\sin y = \frac{1}{5}$$

$$\Rightarrow \cos y = \sqrt{1 - \left(\frac{1}{5}\right)^2} = \frac{2\sqrt{6}}{5}$$

$$\Rightarrow y = \cos^{-1} \left(\frac{2\sqrt{6}}{5} \right)$$

$$\therefore \sin^{-1} \frac{1}{5} = \cos^{-1} \left(\frac{2\sqrt{6}}{5} \right) \quad \dots(2)$$

Let $\cos^{-1} x = z$.

Then, $\cos z = x$

$$\Rightarrow \sin z = \sqrt{1 - x^2}$$

$$\Rightarrow z = \sin^{-1} \left(\sqrt{1 - x^2} \right)$$

$$\therefore \cos^{-1} x = \sin^{-1} \left(\sqrt{1 - x^2} \right) \quad \dots(3)$$

From (1), (2) and (3) we have:

$$\frac{x}{5} + \cos \left(\cos^{-1} \frac{2\sqrt{6}}{5} \right) \cdot \sin \left(\sin^{-1} \sqrt{1 - x^2} \right) = 1$$

$$\Rightarrow \frac{x}{5} + \frac{2\sqrt{6}}{5} \cdot \sqrt{1 - x^2} = 1$$

$$\Rightarrow x + 2\sqrt{6}\sqrt{1 - x^2} = 5$$

$$\Rightarrow 2\sqrt{6}\sqrt{1 - x^2} = 5 - x$$

On squaring both sides, we get:

$$\begin{aligned} (2\sqrt{6})^2(1-x^2) &= 25+x^2-10x \\ \Rightarrow (4 \times 6)(1-x^2) &= 25+x^2-10x \\ \Rightarrow 24-24x^2 &= 25+x^2-10x \\ \Rightarrow 25x^2-10x+1 &= 0 \\ \Rightarrow (5x-1)^2 &= 0 \\ \Rightarrow (5x-1) &= 0 \\ \Rightarrow x &= \frac{1}{5} \end{aligned}$$

Hence, the value of x is $\frac{1}{5}$.

Question 15:

If $\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}$, then find the value of x .

Solution 15:

$$\begin{aligned} \tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} &= \frac{\pi}{4} \\ \Rightarrow \tan^{-1} \left[\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x+2}\right)\left(\frac{x+1}{x+2}\right)} \right] &= \frac{\pi}{4} \quad \left[\text{As, } \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right] \\ \Rightarrow \tan^{-1} \left[\frac{(x-1)(x+2) + (x+1)(x-2)}{(x+2)(x-2) - (x-1)(x+1)} \right] &= \frac{\pi}{4} \\ \Rightarrow \tan^{-1} \left[\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1} \right] &= \frac{\pi}{4} \\ \Rightarrow \tan^{-1} \left[\frac{2x^2-4}{-3} \right] &= \frac{\pi}{4} \\ \Rightarrow \tan \left[\tan^{-1} \frac{4-2x^2}{3} \right] &= \tan \frac{\pi}{4} \\ \Rightarrow \frac{4-2x^2}{3} &= 1 \end{aligned}$$

$$\Rightarrow 4 - 2x^2 = 3$$

$$\Rightarrow 2x^2 = 4 - 3 = 1$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{2}}$$

Hence, the value of x is $\pm \frac{1}{\sqrt{2}}$.

Question 16:

Find the values of $\sin^{-1}\left(\sin \frac{2\pi}{3}\right)$

Solution 16:

Consider, $\sin^{-1}\left(\sin \frac{2\pi}{3}\right)$

We know that $\sin^{-1}(\sin x) = x$

If $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principal value branch of $\sin^{-1} x$.

Here, $\frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Now, $\sin^{-1}\left(\sin \frac{2\pi}{3}\right)$ can be written as:

$$\begin{aligned} & \sin^{-1}\left(\sin \frac{2\pi}{3}\right) \\ &= \sin^{-1}\left[\sin\left(\pi - \frac{2\pi}{3}\right)\right] \\ &= \sin^{-1}\left(\sin \frac{\pi}{3}\right), \text{ where } \frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \end{aligned}$$

$$\therefore \sin^{-1}\left(\sin \frac{2\pi}{3}\right) = \sin^{-1}\left[\sin \frac{\pi}{3}\right] = \frac{\pi}{3}$$

Question 17:

Find the values of $\tan^{-1}\left(\tan \frac{3\pi}{4}\right)$

Solution 17:

Consider, $\tan^{-1}\left(\tan \frac{3\pi}{4}\right)$

We know that $\tan^{-1}(\tan x) = x$

If $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1} x$.

Here, $\frac{3\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now, $\tan^{-1}\left(\tan \frac{3\pi}{4}\right)$ can be written as:

$$\tan^{-1}\left(\tan \frac{3\pi}{4}\right) = \tan^{-1}\left[-\tan\left(\frac{-3\pi}{4}\right)\right] = \tan^{-1}\left[-\tan\left(\pi - \frac{\pi}{4}\right)\right]$$

$$\tan^{-1}\left(-\tan \frac{\pi}{4}\right) = \tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right] \text{ where } -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\therefore \tan^{-1}\left(\tan \frac{3\pi}{4}\right) = \tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right] = \frac{-\pi}{4}$$

Question 18:

Find the values of $\tan\left(\sin^{-1} \frac{3}{5} + \cot^{-1} \frac{3}{2}\right)$

Solution 18:

Let $\sin^{-1} \frac{3}{5} = x$.

Then ,

$$\sin x = \frac{3}{5}$$

$$\Rightarrow \cos x = \sqrt{1 - \sin^2 x} = \frac{4}{5}$$

$$\Rightarrow \sec x = \frac{5}{4}$$

$$\therefore \tan x = \sqrt{\sec^2 x - 1} = \sqrt{\frac{25}{16} - 1} = \frac{3}{4}$$

$$\therefore x = \tan^{-1} \frac{3}{4}$$

$$\therefore \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{3}{4} \quad \dots(i)$$

$$\text{Now, } \cot^{-1} \frac{3}{2} = \tan^{-1} \frac{2}{3} \quad \dots(ii)$$

$$\left[\because \cot^{-1} x = \tan^{-1} \frac{1}{x} \right]$$

$$\text{Therefore, } \tan \left(\sin^{-1} \frac{3}{5} + \cot^{-1} \frac{3}{2} \right)$$

$$= \tan \left(\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{2}{3} \right)$$

[Using (i) and (ii)]

$$= \tan \left[\tan^{-1} \left(\frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \cdot \frac{2}{3}} \right) \right]$$

$$\left[\text{As, } \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right]$$

$$= \tan \left(\tan^{-1} \frac{9+8}{12-6} \right)$$

$$= \tan \left(\tan^{-1} \frac{17}{6} \right) = \frac{17}{6}$$

Question 19:

Find the values of $\cos^{-1} \left(\cos \frac{7\pi}{6} \right)$ is equal to

(A) $\frac{7\pi}{6}$

(B) $\frac{5\pi}{6}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{6}$

Solution 19:

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1} x$.

Here, $\frac{7\pi}{6} \notin [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ can be written as :

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{-7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{7\pi}{6}\right)\right] \quad \left[\because \cos(2\pi - x) = \cos x\right]$$

$$\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$$

The correct answer is B.

Question 20:

Find the values of $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to

(A) $\frac{1}{2}$

(B) $\frac{1}{3}$

(C) $\frac{1}{4}$

(D) 1

Solution 20:

Let $\sin^{-1}\left(-\frac{1}{2}\right) = x$.

Then, $\sin x = \frac{-1}{2} = -\sin\frac{\pi}{6} = \sin\left(\frac{-\pi}{6}\right)$.

We know that the range of the principal value branch of \sin^{-1} is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

$$\sin^{-1}\left(-\frac{1}{2}\right) = \frac{-\pi}{6}$$

$$\therefore \sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right) = \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{3\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

The correct answer is D.

Miscellaneous Exercise

Question 1:

Find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

Solution 1:

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1} x$.

Here, $\frac{13\pi}{6} \notin [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ can be written as :

$$\begin{aligned} & \cos^{-1}\left(\cos\frac{13\pi}{6}\right) \\ &= \cos^{-1}\left[\cos\left(2\pi + \frac{\pi}{6}\right)\right] \\ &= \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right] \quad \text{where } \frac{\pi}{6} \in [0, \pi]. \end{aligned}$$

$$\therefore \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right] = \frac{\pi}{6}$$

Question 2:

Find the value of $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

Solution 2:

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1} x$.

Here, $\frac{7\pi}{6} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now, $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ can be written as:

$$\begin{aligned}
 & \tan^{-1}\left(\tan \frac{7\pi}{6}\right) \\
 &= \tan^{-1}\left[\tan\left(2\pi - \frac{5\pi}{6}\right)\right] \quad [\because \tan(-x) = -\tan x] \\
 &= \tan^{-1}\left[-\tan\left(\frac{5\pi}{6}\right)\right] \quad [\because \tan(-x) = -\tan x] \\
 &= \tan^{-1}\left[\tan\left(\pi - \frac{5\pi}{6}\right)\right] \\
 &= \tan^{-1}\left[\tan\left(\frac{\pi}{6}\right)\right], \text{ where } \frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\
 \therefore \tan^{-1}\left(\tan \frac{7\pi}{6}\right) &= \tan^{-1}\left(\tan \frac{\pi}{6}\right) = \frac{\pi}{6}
 \end{aligned}$$

Question 3:

Prove $2 \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{24}{7}$

Solution 3:

Let $\sin^{-1} \frac{3}{5} = x$. Then $\sin x = \frac{3}{5}$.

$$\Rightarrow \cos x = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

$$\therefore \tan x = \frac{3}{4}$$

$$\therefore x = \tan^{-1} \frac{3}{4} \Rightarrow \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{3}{4}$$

Now, we have:

L.H.S

$$2 \sin^{-1} \frac{3}{5} = 2 \tan^{-1} \frac{3}{4}$$

$$= \tan^{-1} \left(\frac{2 \times \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2} \right) \quad \left[\because \tan^{-1} \frac{2x}{1-x^2} \right]$$

$$= \tan^{-1} \left(\frac{\frac{3}{2}}{\frac{16-9}{16}} \right) = \tan^{-1} \left(\frac{3}{2} \times \frac{16}{7} \right)$$

$$= \tan^{-1} \frac{24}{7} = R.H.S.$$

Question 4:

Prove $\sin^{-1} \frac{8}{17} + \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{77}{36}$

Solution 4:

$$\sin^{-1} \frac{8}{17} = x.$$

$$\text{Then, } \sin x = \frac{8}{17} \Rightarrow \cos x = \sqrt{1 - \left(\frac{8}{17}\right)^2} = \sqrt{\frac{225}{289}} = \frac{15}{17}.$$

$$\therefore \tan x = \frac{8}{15} \Rightarrow x = \tan^{-1} \frac{8}{15}$$

$$\therefore \sin^{-1} \frac{8}{17} = \tan^{-1} \frac{8}{15} \quad \dots(1)$$

$$\text{Now, let } \sin^{-1} \frac{3}{5} = y$$

$$\text{Then, } \sin y = \frac{3}{5} \Rightarrow \cos y = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

$$\therefore \tan y = \frac{3}{4} \Rightarrow y = \tan^{-1} \frac{3}{4}$$

$$\therefore \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{3}{4} \quad \dots(2)$$

Now, we have:

L.H.S.

$$\sin^{-1} \frac{8}{17} + \sin^{-1} \frac{3}{5}$$

Using (1) and (2), we get

$$\begin{aligned}
 &= \tan^{-1} \frac{8}{15} + \tan^{-1} \frac{3}{4} \\
 &= \tan^{-1} \frac{\frac{8}{15} + \frac{3}{4}}{1 - \frac{8}{15} \times \frac{3}{4}} \quad \left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right] \\
 &= \tan^{-1} \left(\frac{32+45}{60-24} \right) \\
 &= \tan^{-1} \frac{77}{36} = R.H.S.
 \end{aligned}$$

Question 5:

Prove $\cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} = \cos^{-1} \frac{33}{65}$

Solution 5:

Let $\cos^{-1} \frac{4}{5} = x$

Then, $\cos x = \frac{4}{5} \Rightarrow \sin x = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \frac{3}{5}$

$\therefore \tan x = \frac{3}{4} \Rightarrow x = \tan^{-1} \frac{3}{4}$

$\therefore \cos^{-1} \frac{4}{5} = \tan^{-1} \frac{3}{4} \quad \dots(1)$

Now, let $\cos^{-1} \frac{12}{13} = y$.

Then, $\cos y = \frac{12}{13} \Rightarrow \sin y = \frac{5}{13}$.

$\therefore \tan y = \frac{5}{12} \Rightarrow y = \tan^{-1} \frac{5}{12}$

$\therefore \cos^{-1} \frac{12}{13} = \tan^{-1} \frac{5}{12} \quad \dots(2)$

Let $\cos^{-1} \frac{33}{65} = z$

$$\text{Then, } \cos z = \frac{33}{65} \Rightarrow \sin z = \frac{56}{65}$$

$$\therefore \tan z = \frac{56}{33} \Rightarrow z = \tan^{-1} \frac{56}{33}$$

$$\therefore \cos^{-1} \frac{33}{65} = \tan^{-1} \frac{56}{33} \quad \dots(3)$$

Now,

L.H.S.

$$\cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13}$$

Using (1) and (2), we get

$$= \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{5}{12}$$

$$= \tan^{-1} \left(\frac{\frac{3}{4} + \frac{5}{12}}{1 - \left(\frac{3}{4} \times \frac{5}{12} \right)} \right) \quad \left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right) \right]$$

$$= \tan^{-1} \left(\frac{36+20}{48-15} \right)$$

$$= \tan^{-1} \frac{56}{33}$$

$$= \cos^{-1} \frac{56}{65}$$

$$= \text{R.H.S.}$$

Question 6:

$$\text{Prove } \cos^{-1} \frac{12}{13} + \sin^{-1} \frac{3}{5} = \sin^{-1} \frac{56}{65}$$

Solution 6:

$$\text{Let } \sin^{-1} \frac{3}{5} = x. \text{ Then, } \sin x = \frac{3}{5} \Rightarrow \cos x = \sqrt{1 - \left(\frac{3}{5} \right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

$$\therefore \tan x = \frac{3}{4} \Rightarrow x = \tan^{-1} \frac{3}{4}$$

$$\therefore \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{3}{4} \quad \dots(1)$$

Now, let $\cos^{-1} \frac{12}{13} = y$

Then, $\cos y = \frac{12}{13} \Rightarrow \sin y = \frac{5}{13}$.

$$\therefore \tan y = \frac{5}{12} \Rightarrow y = \tan^{-1} \frac{5}{12}$$

$$\therefore \cos^{-1} \frac{12}{13} = \tan^{-1} \frac{5}{12} \quad \dots(2)$$

Let $\sin^{-1} \frac{56}{65} = z$.

Then, $\sin z = \frac{56}{65} \Rightarrow \cos z = \frac{33}{65}$.

$$\therefore \tan z = \frac{56}{33} \Rightarrow z = \tan^{-1} \frac{56}{33}$$

$$\therefore \sin^{-1} \frac{56}{65} = \tan^{-1} \frac{56}{33} \quad \dots(3)$$

Now, we have

L.H.S

$$\begin{aligned} & \cos^{-1} \frac{12}{13} + \sin^{-1} \frac{3}{5} \\ &= \tan^{-1} \frac{5}{12} + \tan^{-1} \frac{3}{4} \end{aligned}$$

Using (1) and (2), we get

$$\begin{aligned}
 &= \tan^{-1} \frac{\frac{5}{12} + \frac{3}{4}}{1 - \left(\frac{5}{12} \cdot \frac{3}{4}\right)} \quad \left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right] \\
 &= \tan^{-1} \left(\frac{20+36}{48-15} \right) \\
 &= \tan^{-1} \frac{56}{33} \\
 &= \sin^{-1} \frac{56}{65} \quad [\text{Using (3)}] \\
 &= R.H.S
 \end{aligned}$$

Question 7:

Prove $\tan^{-1} \frac{63}{16} = \sin^{-1} \frac{5}{13} + \cos^{-1} \frac{3}{5}$

Solution 7:

Let $\sin^{-1} \frac{5}{13} = x$

Then, $\sin x = \frac{5}{13} \Rightarrow \cos x = \frac{12}{13}$.

$\therefore \tan x = \frac{5}{12} \Rightarrow x = \tan^{-1} \frac{5}{12}$

$\therefore \sin^{-1} \frac{5}{13} = \tan^{-1} \frac{5}{12} \quad \dots(1)$

Let $\cos^{-1} \frac{3}{5} = y$.

Then, $\cos y = \frac{3}{5} \Rightarrow \sin y = \frac{4}{5}$.

Thus, $\tan y = \frac{4}{3} \Rightarrow y = \tan^{-1} \frac{4}{3}$

$\therefore \cos^{-1} \frac{3}{5} = \tan^{-1} \frac{4}{3} \quad \dots(2)$

Using (1) and (2), we have

R.H.S.

$$\begin{aligned}
 & \sin^{-1} \frac{5}{12} + \cos^{-1} \frac{3}{5} \\
 &= \tan^{-1} \frac{5}{12} + \tan^{-1} \frac{4}{3} \\
 &= \tan^{-1} \left(\frac{\frac{5}{12} + \frac{4}{3}}{1 - \frac{5}{12} \times \frac{4}{3}} \right) \quad \left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right] \\
 &= \tan^{-1} \left(\frac{15+48}{36-20} \right) \\
 &= \tan^{-1} \frac{63}{16} \\
 &= \text{L.H.S.}
 \end{aligned}$$

Question 8:

Prove $\tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8} = \frac{\pi}{4}$

Solution 8:

L.H.S

$$\begin{aligned}
 & \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8} \\
 &= \tan^{-1} \left(\frac{\frac{1}{5} + \frac{1}{7}}{1 - \frac{1}{5} \times \frac{1}{7}} \right) + \tan^{-1} \left(\frac{\frac{1}{3} + \frac{1}{8}}{1 - \frac{1}{3} \times \frac{1}{8}} \right) \quad \left[\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right] \\
 &= \tan^{-1} \left(\frac{7+5}{35-1} \right) + \tan^{-1} \left(\frac{8+3}{24-1} \right) \\
 &= \tan^{-1} \frac{12}{34} + \tan^{-1} \frac{11}{23} \\
 &= \tan^{-1} \frac{6}{17} + \tan^{-1} \frac{11}{23} \\
 &= \tan^{-1} \left(\frac{\frac{6}{17} + \frac{11}{23}}{1 - \frac{6}{17} \times \frac{11}{23}} \right)
 \end{aligned}$$

$$= \tan^{-1} \left(\frac{325}{325} \right) = \tan^{-1} 1$$

$$= \frac{\pi}{4} = \text{R.H.S.}$$

Question 9:

Prove $\tan^{-1} \sqrt{x} = \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right), x \in [0, 1]$

Solution 9:

Let $x = \tan^2 \theta$.

Then $\sqrt{x} = \tan \theta \Rightarrow \theta = \tan^{-1} \sqrt{x}$.

$$\therefore \frac{1-x}{1+x} = \frac{1-\tan^2 \theta}{1+\tan^2 \theta} = \cos 2\theta$$

Now, we have:

R.H.S

$$\begin{aligned} & \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right) \\ &= \frac{1}{2} \cos^{-1} (\cos 2\theta) \\ &= \frac{1}{2} \times 2\theta = \theta = \tan^{-1} \sqrt{x} = \text{L.H.S.} \end{aligned}$$

Question 10:

Prove $\cot^{-1} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4} \right)$

Solution 10:

$$\begin{aligned}
 &\text{Consider } \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \\
 &= \frac{(\sqrt{1+\sin x} + \sqrt{1-\sin x})^2}{(\sqrt{1+\sin x} - \sqrt{1-\sin x})^2} \quad (\text{by rationalizing}) \\
 &= \frac{(1+\sin x) + (1-\sin x) + 2\sqrt{(1+\sin x)(1-\sin x)}}{1+\sin x - 1 + \sin x} \\
 &= \frac{2(1 + \sqrt{1-\sin^2 x})}{2\sin x} \\
 &= \frac{1 + \cos x}{\sin x} \\
 &= \frac{2\cos^2 \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}} \\
 &= \cot \frac{x}{2} \\
 &\therefore L.H.S. = \cot^{-1} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right) \\
 &= \cot^{-1} \left(\cot \frac{x}{2} \right) = \frac{x}{2} = R.H.S.
 \end{aligned}$$

Question 11:

Prove $\tan^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right) = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x, -\frac{1}{\sqrt{2}} \leq x \leq 1$

[Hint : put $x = \cos 2\theta$]

Solution 11:

Let, $x = \cos 2\theta$ then, $\theta = \frac{1}{2} \cos^{-1} x$.

Thus, we have:

$$\begin{aligned}
 L.H.S. &= \tan^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right) \\
 &= \tan^{-1} \left(\frac{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}} \right) \\
 &= \tan^{-1} \left(\frac{\sqrt{2\cos^2 \theta} - \sqrt{2\sin^2 \theta}}{\sqrt{2\cos^2 \theta} + \sqrt{2\sin^2 \theta}} \right) \\
 &= \tan^{-1} \left(\frac{\sqrt{2}\cos \theta - \sqrt{2}\sin \theta}{\sqrt{2}\cos \theta + \sqrt{2}\sin \theta} \right) \\
 &= \tan^{-1} \left(\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} \right) \\
 &= \tan^{-1} \left(\frac{1 - \tan \theta}{1 + \tan \theta} \right) \\
 &= \tan^{-1} 1 - \tan^{-1} (\tan \theta) \quad \left[\because \tan^{-1} \left(\frac{x-y}{1+xy} \right) = \tan^{-1} x - \tan^{-1} y \right] \\
 &= \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x = \text{R.H.S.}
 \end{aligned}$$

Question 12:

Prove $\frac{9\pi}{8} - \frac{9}{4} \sin^{-1} \frac{1}{3} = \frac{9}{4} \sin^{-1} \frac{2\sqrt{2}}{3}$

Solution 12:

$$\begin{aligned}
 L.H.S. &= \frac{9\pi}{8} - \frac{9}{4} \sin^{-1} \frac{1}{3} \\
 &= \frac{9}{4} \left(\frac{\pi}{2} - \sin^{-1} \frac{1}{3} \right) \\
 &= \frac{9}{4} \left(\cos^{-1} \frac{1}{3} \right) \quad \dots\dots(1) \quad \left[\because \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x \right]
 \end{aligned}$$

Now, let $\cos^{-1} \frac{1}{3} = x$

$$\text{Then, } \cos x = \frac{1}{3} \Rightarrow \sin x = \sqrt{1 - \left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}.$$

$$\therefore x = \sin^{-1} \frac{2\sqrt{2}}{3} \Rightarrow \cos^{-1} \frac{1}{3} \Rightarrow \sin^{-1} \frac{2\sqrt{2}}{3}$$

$$\therefore \text{L.H.S.} = \frac{9}{4} \sin^{-1} \frac{2\sqrt{2}}{3} = \text{R.H.S.}$$

Question 13:

$$\text{Solve } 2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$$

Solution 13:

$$2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$$

$$\Rightarrow \tan^{-1}\left(\frac{2 \cos x}{1 - \cos^2 x}\right) = \tan^{-1}(2 \operatorname{cosec} x) \quad \left[\because \quad \zeta = \tan^{-1} \frac{2x}{1-x^2} \right]$$

$$\Rightarrow \frac{2 \cos x}{1 - \cos^2 x} = 2 \operatorname{cosec} x$$

$$\Rightarrow \frac{2 \cos x}{\sin^2 x} = \frac{2}{\sin x}$$

$$\Rightarrow \cos x = \sin x$$

$$\Rightarrow \tan x = 1$$

$$\therefore x = \frac{\pi}{4}$$

Question 14:

$$\text{Solve } \tan^{-1} \frac{1-x}{1+x} = \frac{1}{2} \tan^{-1} x, (x > 0)$$

Solution 14:

$$\tan^{-1} \frac{1-x}{1+x} = \frac{1}{2} \tan^{-1} x$$

$$\Rightarrow \tan^{-1} 1 - \tan^{-1} x = \frac{1}{2} \tan^{-1} x \quad \left[\because -\tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy} \right]$$

$$\Rightarrow \frac{\pi}{4} = \frac{3}{2} \tan^{-1} x$$

$$\Rightarrow \tan^{-1} x = \frac{\pi}{6}$$

$$\Rightarrow x = \tan \frac{\pi}{6}$$

$$\therefore x = \frac{1}{\sqrt{3}}$$

Question 15:

Solve $\sin(\tan^{-1} x)$, $|x| < 1$ is equal to

- (A) $\frac{x}{\sqrt{1-x^2}}$ (B) $\frac{1}{\sqrt{1-x^2}}$
 (C) $\frac{1}{\sqrt{1+x^2}}$ (D) $\frac{x}{\sqrt{1+x^2}}$

Solution 15:

$$\tan y = x \Rightarrow \sin y = \frac{x}{\sqrt{1+x^2}}$$

Let $\tan^{-1} x = y$. Then,

$$y = \sin^{-1} \left(\frac{x}{\sqrt{1+x^2}} \right) \Rightarrow \tan^{-1} x = \sin^{-1} \left(\frac{x}{\sqrt{1+x^2}} \right)$$

$$\Rightarrow \sin(\tan^{-1} x) = \sin \left(\sin^{-1} \left(\frac{x}{\sqrt{1+x^2}} \right) \right) = \frac{x}{\sqrt{1+x^2}}$$

The correct answer is **D**.

Question 16:

Solve $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$, then x is equal to

- (A) $0, \frac{1}{2}$ (B) $1, \frac{1}{2}$
 (C) 0 (D) $\frac{1}{2}$

Solution 16:

$$\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow -2\sin^{-1}x = \frac{\pi}{2} - \sin^{-1}(1-x)$$

$$\Rightarrow -2\sin^{-1}x = \cos^{-1}(1-x) \quad \dots(1)$$

$$\text{Let } \sin^{-1}x = \theta \Rightarrow \sin\theta = x \Rightarrow \cos\theta = \sqrt{1-x^2}.$$

$$\therefore \theta = \cos^{-1}(\sqrt{1-x^2})$$

$$\therefore \sin^{-1}x = \cos^{-1}(\sqrt{1-x^2})$$

Therefore, from equation (1), we have

$$-2\cos^{-1}(\sqrt{1-x^2}) = \cos^{-1}(1-x)$$

Let, $x = \sin y$. Then, we have:

$$-2\cos^{-1}(\sqrt{1-\sin^2 y}) = \cos^{-1}(1-\sin y)$$

$$\Rightarrow -2\cos^{-1}(\cos y) = \cos^{-1}(1-\sin y)$$

$$\Rightarrow -2y = \cos^{-1}(1-\sin y)$$

$$\Rightarrow 1-\sin y = \cos(-2y) = \cos 2y$$

$$\Rightarrow 1-\sin y = 1-2\sin^2 y$$

$$\Rightarrow 2\sin^2 y - \sin y = 0$$

$$\Rightarrow \sin y(2\sin y - 1) = 0$$

$$\Rightarrow \sin y = 0 \text{ or } \frac{1}{2}$$

$$\therefore x = 0 \text{ OR } x = \frac{1}{2}$$

When $x = \frac{1}{2}$, it can be observed that:

$$\begin{aligned} \text{L.H.S.} &= \sin^{-1}\left(1 - \frac{1}{2}\right) - \sin^{-1}\frac{1}{2} \\ &= \sin^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\frac{1}{2} \\ &= -\sin^{-1}\frac{1}{2} \\ &= \frac{\pi}{6} \neq \frac{\pi}{2} \neq \text{R.H.S.} \end{aligned}$$

$\therefore x = \frac{1}{2}$ is not a solution of the given equation.

Thus, $x = 0$.

Hence, the correct answer is C.

Question 17:

Solve $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\frac{x-y}{x+y}$ is equal to

- (A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$
 (C) $\frac{\pi}{4}$ (D) $\frac{-3\pi}{4}$

Solution 17:

$$\begin{aligned} &\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right) \\ &= \tan^{-1}\left[\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1 + \left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right] \quad \left[\because -\tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy} \right] \\ &= \tan^{-1}\left[\frac{\frac{x(x+y) - y(x-y)}{y(x+y)}}{\frac{y(x+y) + x(x-y)}{y(x+y)}}\right] \\ &= \tan^{-1}\left(\frac{x^2 + xy - xy + y^2}{xy + y^2 + x^2 - xy}\right) \end{aligned}$$

$$= \tan^{-1} \left(\frac{x^2 + y^2}{x^2 + y^2} \right) = \tan^{-1} 1 = \frac{\pi}{4}$$

Hence, the correct answer is C.