Chapter-06 - Application of Derivatives
Exercise 6.1

Question 1:
Find the rate of change to the area of a circle with respect to its radius r when
(a) $r = 3\ cm$ **(b)** $r = 4cm$

Solution 1:
The area of a circle (A) with radius (r) is given by,

$A = \pi r^2$

Now, the area of the circle is changing of the area with respect to its radius is given by,

$$\frac{dA}{dr} = \frac{d}{dr}\left(\pi r^2\right) = 2\pi r$$

1. When $r = 3\ cm$,

$$\frac{dA}{dr} = 2\pi(3) = 6\pi$$

Hence, the area of the circle is changing at the rate of 6π cm when its radius is $3\ cm$.

2. When $r = 4cm$,

$$\frac{dA}{dr} = 2\pi(4) = 8\pi$$

Hence, the area of the circle is changing at the rate of 8π cm when its radius is $4cm$.

Question 2:
The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of an edge is 12 cm?

Solution 2:
Let x be the length of a side, v be the volume, and s be the surface area of the cube.
Then, $V = x^3$ and $S = 6x^2$ when x is a function of time t.

It is given that $\frac{dv}{dt} = 8cm^3/s$

Then, by using the chain rule, we have:

$$\therefore 8 = \frac{dv}{dt} = \left(x^3\right) = \frac{d}{x}(x).\frac{dx}{dt} = 3x^2.\frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{8}{3x^2} \qquad \ldots (1)$$

$$\text{Now, } \Rightarrow \frac{ds}{dt} = \frac{d}{dt}(6x^2) = \frac{d}{dx}(6x^2)\cdot\frac{dx}{dt} \qquad [\text{By chain rule}]$$

$$= 12x.\frac{dx}{dt} = 12x.\left(\frac{8}{3x^2}\right) = \frac{32}{x}$$

Thus, when $x = 12cm$, $\frac{dS}{dt} = \frac{32}{12}cm^2/s = \frac{8}{3}cm^2/s.$

Hence, if the length of the edge of the cube is $12cm,$ then the surface area is increasing at the rate of $\frac{8}{3}$ cm^2/s.

Question 3:
The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.

Solution 3:
The area of a circle (A) with radius (r) is given by,

$A = \pi r^2$

Now, the rate of change of area (A) with respect to time (t) is given by,

$$\frac{dA}{dt} = \frac{d}{d}\left(\pi r^2\right).\frac{dr}{dx} = 2\pi r\frac{dr}{dt} \qquad \text{[By chain rule]}$$

It is given that,

$$\frac{dr}{dt} = 3cm/s$$

$$\therefore \frac{dA}{dt} = 2\pi r(3) = 6\pi r$$

Thus, when $r = 10cm,$

$$\frac{dA}{dt} = 6\pi(10) = 60\pi\ cm^2/s$$

Hence, the rate at which the area of the circle is increasing when the radius is 10 cm is $60\pi\ cm^2/s$.

Question 4:
An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?

Solution 4:
Let x be the length of a side and v be the volume of the cube. Then,

$V = x^3$

$$\therefore \frac{dV}{dt} = 3x^2.\frac{dx}{dt} \qquad \text{(by chain rule)}$$

It is given that,

$$\frac{dx}{dt} = 3\ cm/s$$

$$\therefore \frac{dV}{dt} = 3x^2(3) = 9x^2$$

Then, when $x = 10cm,$

$$\frac{dV}{dt}=9(10)^2=900cm^3/s$$

Hence, the volume of the cube is increasing at the rate of 900 cm^3/s when the edge is 10 cm long.

Question 5:

A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

Solution 5:

The area of a circle (A) with radius (r) is given by

$A=\pi r^2$

Therefore, the rate of change of area (A) with respect to time (t) is given by,

$$\therefore \frac{dA}{dt}=\frac{d}{dt}\left(\pi r^2\right)=\frac{d}{dr}\left(\pi r^2\right)\frac{dr}{dt}=2\pi r\frac{dr}{dt} \qquad (\text{by chain rule})$$

It is given that $\frac{dr}{dt}=5cm/s$

Thus, when $r=8cm,$

$$\frac{dA}{dt}=2\pi(8)(5)=80\pi$$

Hence, when the radius of the circular wave is $8cm,$ the enclosed area is increasing at the rate of $80\pi\ cm^2/s.$

Question 6:

The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?

Solution 6:

The circumference of a circle (C) with radius (r) is given by

$C=2nr.$

Therefore, the rate of change of circumference (C) with respect to time (t) is given by,

$$\frac{dC}{dt}=\frac{dC}{dr}.\frac{dr}{dt} \qquad (\text{by chain rule})$$

$$=\frac{d}{dr}(2\pi r)\frac{dr}{dt}$$

$$=2\pi.\frac{dr}{dt}$$

It is given that $\frac{dr}{dt}=0.7\ cm/s$

Hence, the rate of increase of the circumference is

$2\pi(0.7)=1.4\pi$ cm/s

Question 7:
The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x=8cm$ and $y=6cm,$ find the rates of change of (a) the perimeter, and (b) the area of the rectangle.

Solution 7:
Since the length (x) is decreasing at the rate of $5cm/\text{minute}$ and the width (y) is increasing at the rate of $4cm/\text{minute}$, we have:

$$\frac{dt}{dy}=-5cm/\min \text{ and}$$

$$\frac{dy}{dt}=4cm/\min$$

(a) The perimeter (P) of a rectangle is given by,

$$P=2(x+y)$$

$$\therefore \frac{dP}{dt}=2\left(\frac{dx}{dt}+\frac{dy}{dt}\right)=2(-5+4)=-2 \text{ cm/min}$$

Hence, the perimeter is decreasing at the rate of 2 cm/min,

(b) The area (A) of a rectangle is given by,

$$A=x\times y$$

$$\frac{dA}{dt}=\frac{dx}{dt}.y+x.\frac{dy}{dt}=-5y+4x$$

When $x=8\ cm$ and $y=6\ cm,$ $\frac{dA}{dt}=(-5\times6+4\times8)\text{cm}^2/\text{min}=2\text{ cm}^2/\text{min}$

Hence, the area of the rectangle is increasing at the rate of $2\text{cm}^2/\text{min}$.

Question 8:
A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

Solution 8:
The volume of a sphere (V) with radius (r) is given by,

$$V = \frac{4}{3}\pi r^3$$

$\therefore$ Rate of change of volume (V) with respect to time (t) is given by,

$$\frac{dV}{dt} = \frac{dV}{dr}.\frac{dr}{dt} \qquad \text{(by chain rule)}$$

$$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right).\frac{dr}{dt}$$

$$= 4\pi r^2.\frac{dr}{dt}$$

It is given that

$$\frac{dv}{dt} = 900\ cm^3 / S$$

$$\therefore 900 = 4\pi r^2.\frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{900}{4\pi r^2} = \frac{225}{\pi r^2}$$

Therefore, when radius = 15 cm,

$$\frac{dr}{dt} = \frac{225}{\pi(15)^2} = \frac{1}{\pi}$$

Hence, the rate at which the radius of the balloon increases when the radius is 15 cm is $\frac{1}{\pi}$ cm/s

Question 9:
A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.

Solution 9:
The volume of a sphere (v) with radius (r) is given by $V = \frac{4}{3}\pi r^2$

Rate of change of volume (v) with respect to its radius (r) is given by,

$$\frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi\left(3r^2\right) = 4\pi r^2$$

Therefore, when radius = 10 cm,

$$\frac{dV}{dr} = 4\pi(10)^2 = 400\pi$$

Hence, the volume of the balloon is increasing at the rate of $400\pi \ cm^3/s$.

Question 10:

A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?

Solution 10:

Let y m be the height of the wall at which the ladder touches. Also, let the foot of the ladder be x m away from the wall.

Then, by Pythagoras Theorem, we have:

$x^2 + y^2 = 25$ (Length of the ladder = 5 m)

$\Rightarrow y = \sqrt{25 - x^2}$

Then, the rate of change of height (y) with respect to time (t) is given by,

$$\frac{dy}{dt} = \frac{-x}{\sqrt{25 - x^2}} \cdot \frac{dx}{dt}$$

It is given that $\frac{dx}{dt} = 2 \ cm/s$

$$\therefore \frac{dy}{dt} = \frac{-2x}{\sqrt{25 - x^2}}$$

Now, when $x = 4m$, we have:

$$\frac{dy}{dt} = \frac{-2x4}{\sqrt{25\text{-}4^2}} = \frac{8}{3}$$

Hence, the height of the ladder on the wall is decreasing at the rate of $\frac{8}{3}$ cm/s.

Question 11:

A particle moving along the curve $6y = x^3 + 2$, Find the points on the curve at which the y coordinate is changing 8 times as fast as the x-coordinate.

Solution 11:

The equation of the curve is given as:

$6y = x^3 + 2$

The rate of change of the position of the particle with respect to time (t), is given by,

$$6\frac{dy}{dt} = 3x^2\frac{dx}{dt} + 0$$

$$\Rightarrow 2\frac{dy}{dt} = x^2\frac{dx}{dt}$$

When the y-coordinate of the particle changes 8 times as fast as the x-coordinate i.e., $\left(\frac{dy}{dt}=8\frac{dx}{dt}\right)$, we have:

$$2\left(8\frac{dx}{dt}\right)=x^2\frac{dx}{dt}$$
$$\Rightarrow 16\frac{dx}{dt}=x^2\frac{dx}{dt}$$
$$\Rightarrow \left(x^2-16\right)\frac{dx}{dt}=0$$
$$\Rightarrow x^2=16$$
$$\Rightarrow x=\pm 4$$

When, $x=4, y=\frac{4^3+2}{6}=\frac{66}{6}=11$

When,

$$x=-4, y=\frac{(-4)^3+2}{6}=\frac{62}{6}=\frac{31}{3}$$

Hence, the points required on the curve are $(4,11)$ and $\left(-4,\frac{-31}{3}\right)$.

Question 12:

The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubbles of the bubble increasing when the radius is 1 cm?

Solution 12:

The air bubble is in the shape of a sphere.

Now, the volume of an air bubble (V) with radius (r) is given by, $V=\frac{4}{3}\pi r^2$

The rate of change of volume (V), with respect to time (t) is given by,

$$\frac{dV}{dt}=\frac{4}{3}\pi\frac{d}{dr}\left(r^3\right).\frac{dr}{dt} \qquad \text{(By chain rule)}$$
$$=\frac{4}{3}\pi\left(3r^2\right)\frac{dr}{dt}.$$
$$=4\pi r^2\frac{dr}{dt}$$

It is given that $\frac{dr}{dt}=\frac{1}{2}cm/s$

Therefore, when $r=1\,cm$,

$$\frac{dV}{dt}=4\pi(1)^2\left(\frac{1}{2}\right)=2\pi\ cm^3/s$$

Hence, the rate at which the volume of the bubble increases in $2\pi\ cm^3/s$.

Question 13:

A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x.

Solution 13:

The volume of a sphere (V) with radius (r) is given by,

$$V=\frac{4}{3}\pi r^2$$

Diameter = $\frac{3}{2}(2x+1)$

$$r=\frac{3}{4}(2x+1)$$

$$V=\frac{4}{3}\pi r^3=\frac{4}{3}\pi\left(\frac{3}{4}(2x+1)\right)^3=\frac{9}{16}\pi(2x+1)^3$$

Hence, the rate of change of volume with respect to x is as

$$\frac{dV}{dt}=\frac{9}{16}\pi\frac{d}{dt}(2x+1)^3=\frac{9}{16}\pi \text{x} 3(2x+1)^2 \text{ x } 2=\frac{27}{8}\pi(2x+1)^3.$$

Question 14:

Sand is pouring from a pipe at the rate of $12\,cm^3/s$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

Solution 14:

The volume of a cone (V) with radius (r) and height (h) is given by.

$$V=\frac{1}{3}\pi r^2 h$$

It is given that,

$$h=\frac{1}{6}r \Rightarrow r=6h$$

$$\therefore V=\frac{1}{3}\pi(6h)^2 h=12\pi h^3$$

The rate of change of volume with respect to time (t) is given by,

$$\frac{dV}{dt}=12\pi\frac{d}{dh}\left(h^3\right).\frac{dh}{dt} \qquad \text{(By chain rule)}$$

$$=12\pi\left(3h^2\right)\frac{dh}{dt}$$

$$=36\pi h^2\frac{dh}{dt}$$

It is also given that $\frac{dV}{dt}=12\,cm^2/s$

Therefore, when $h = 4cm,$ we have:

$$12 = 36\pi (4)^2 \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{12}{36\pi (16)} = \frac{1}{48\pi}$$

Hence, when the height of the sand cone is 4 cm, its height is increasing at the rate of $\frac{1}{48\pi}$ cm/s.

Question 15:

The total cost $C(x)$ in Rupees associated with the production of x units of an item is given by

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$$

Find the marginal cost when 17 units are produced.

Solution 15:

Marginal cost is the rate of change of total cost with respect to output.

$$\therefore \text{ Marginal cost } (MC) = \frac{dC}{dx} = 0.007(3x^2) - 0.003(2x) + 15$$

$$= 0.021x^2 - 0.006x + 15$$

When $x = 17,$ $MC = 0.021(17^2) - 0.006(17) + 15$

$$= 0.021(289) - 0.006(17) + 15$$

$$= 6.069 - 0.102 + 15$$

$$= 20.967$$

Hence, when 17 units are produced, the marginal cost is Rs. 20.967.

Question 16:

The total revenue in Rupees received from the sale of x units of a product is given by

$$R(x) = 13x^2 + 26x + 15$$

Find the marginal revenue when $x = 7$.

Solution 16:

Marginal revenue is the rate of change of total revenue with respect to the number of units sold.

$$\therefore \text{ Marginal Revenue } (MR) = \frac{dR}{dx} = 13(2x) + 26 = 26x + 26$$

When $x = 7,$

$$MR = 26(7) + 26 = 182 + 26 = 208$$

Hence, the required marginal revenue is Rs. 208.

Question 17:

The rate of change of the area of a circle with respect to its radius r at $r = 6\ cm$ is

(A) 10π **(B)** 12π **(C)** 8π **(D)** 11π

Solution 17:
The area of a circle (A) with radius (r) is given by,

$A = \pi r^2$

Therefore, the rate of change of the area with respect to its radius r is

$\frac{dA}{dt} = \frac{d}{dr}(\pi r^2) = 2\pi r$

$\therefore$ When $r = 6\ cm$,

$\frac{dA}{dr} = 2\pi \text{ x } 6 = 12\pi \text{ cm}^2 / s$

Hence, the required rate of change of the area of a circle is $12\pi\ cm^2 / s$

The correct answer is **B.**

Question 18:
The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. The marginal revenue, when $x = 15$ is

(A) 116 **(B)** 96 **(C)** 90 **(D)** 126

Solution 18:
Marginal revenue is the rate of change of total revenue with respect to the number of units sold.

$\therefore$ Marginal Revenue $(MR) = \frac{dR}{dx} = 3(2x) + 36 = 6x + 36$

$\therefore$ When $x = 15$,

$MR = 6(15) + 36 = 90 + 36 = 126$

Hence, the required marginal revenue is Rs. 126
The correct answer is **D.**

Exercise 6.2

Question 1:
Show, that the function given by $f(x) = 3x + 17$ is strictly increasing on **R**.

Solution 1:
Let x_1 and x_2, be any two numbers in **R**.

$x_1 < x_2 \Rightarrow 3x_1 < 3x_2 \Rightarrow 3x_1 + 17 < 3x_2 + 17 = f(x_2) < f(x_2)$

Hence, f is strictly increasing on **R**.

Alternate Method:
$f'(x) = 3 > 0$, in every interval, on **R**.
Thus, the function is strictly increasing on **R**.

Question 2:
Show that the function given by $f(x)=e^{2x}$ is strictly increasing on **R**.

Solution 2:
Let x_1 and x_2 be any two numbers in **R**.
Then, we have:
$x_1 < x_2 \Rightarrow 2x_1 < 2x_2 \Rightarrow e^{2x_1} < e^{2x_2} \Rightarrow f(x_1) < f(x_2)$
Hence, f is strictly increasing on **R**.

Question 3:
Show that the function given by $f(x)=\sin x$ is

(A) Strictly increasing in $\left(0,\frac{\pi}{2}\right)$

(B) Strictly decreasing $\left(\frac{\pi}{2},\pi\right)$

(C) Neither increasing nor decreasing in (O,n)

Solution 3:
The given function is $f(x)=\sin x$

$\therefore f'(x)=\cos x$

(A) Since for each $x\in\left(0,\frac{\pi}{2}\right)$, $\cos x>0$, we have $f'(x)>0$.

Hence, f is strictly increasing in $\left(0,\frac{\pi}{2}\right)$.

(B) Since for each $x\in\left(\frac{\pi}{2},\pi\right)$, $\cos x<0$, we have $f'(x)<0$.

Hence, f is strictly increasing in $\left(\frac{\pi}{2},\pi\right)$.

(C) From the results obtained in **(A)** and **(B)** it is clear that f is neither increasing nor decreasing in $(0,n)$.

Question 4:
Find the intervals in which the function f given by $f(x)=2x^2-3x$ is

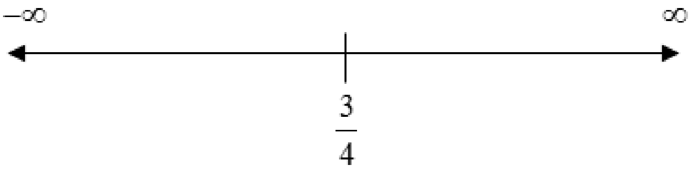
(A) Strictly increasing **(B)** strictly decreasing

Solution 4:
The given function is $f(x)=2x^2-3x$.

$f'(x) = 4x - 3$

$\therefore f'(x) = 0 \Rightarrow x = \frac{3}{4}$

Now, the point $\frac{3}{4}$ divides the real line into two disjoint intervals i.e., $\left(-\infty, \frac{3}{4}\right)$ and $\left(\frac{3}{4}, \infty\right)$.



In interval $\left(-\infty, \frac{3}{4}\right)$, $f'(x) = 4x - 3 < 0.$

Hence, the given function (f) is strictly decreasing in interval $\left(-\infty, \frac{3}{4}\right)$.

In interval $\left(\frac{3}{4}, \infty\right)$, $f'(x) = 4x - 3 > 0.$

Hence, the given function (f) is strictly increasing in interval $\left(\frac{3}{4}, \infty\right)$.

Question 5:

Find the intervals in which the function f given is

$f(x) = 2x^3 - 3x^2 - 36x + 7$

(A) strictly increasing **(B)** strictly decreasing

Solution 5:

The given function is

$f(x) = 2x^3 - 3x^2 - 36x + 7$

$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6) = 6(x+2)(x-3)$

$\therefore f'(x) = 0 \Rightarrow x = -2, 3$

The points $x = -2$ and $x = 3$ divide the real line into three disjoints intervals i.e., $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$.

In intervals $(-\infty, -2)$ and $(3, \infty)$, $f'(x)$ is positive while in interval $(-2, 3)$, $f'(x)$ is negative.

Hence, the given function (f) is strictly increasing in intervals $(-\infty, -2) \text{ U } (3, \infty)$, while function (f) is strictly decreasing in interval $(-2, 3)$.

Question 6:
Find the intervals in which the following functions are strictly increasing or decreasing :

(a) $x^2 + 2x - 5$
(b) $10 - 6x - 2x^2$
(c) $-2x^3 - 9x^2 - 12x + 1$
(d) $6 - 9x - x^2$
(e) $(x+1)^3 (x-3)^3$

Solution 6:
We have,

$f(x) = x^2 + 2x - 5$

$\therefore f'(x) = 2x + 2$

Now,

$f'(x) = 0 \Rightarrow x = -1$

Point $x = -1$ divides the real line into two disjoint intervals i.e., $(-\infty, -1)$ and $(-1, \infty)$.

In interval $(-\infty, -1)$,

$\therefore f'(x) = 2x + 2 < 0$

$\therefore f$ is strictly decreasing in interval $(-\infty, -1)$.

Thus, f is strictly decreasing for $x < -1$.

In interval $(-1, \infty)$,

$\therefore f'(x) = 2x + 2 > 0$

$\therefore f$ is strictly decreasing in interval $(-1, \infty)$.

Thus f is strictly increasing for $x > -1$.

(b) We have,

$f(x) = 10 - 6x - 2x^2$

$\therefore f'(x) = -6 - 4x$

Now,

$f'(x) = 0 \Rightarrow x = -\frac{3}{2}$

The point $x = -\frac{3}{2}$ divides the real line into two disjoint intervals

i.e., $\left(-\infty, -\frac{3}{2}\right)$ and $\left(-\frac{3}{2}, \infty\right)$.

In interval $\left(-\infty, -\frac{3}{2}\right)$ i.e., when $x < -\frac{3}{2}$, $f'(x) = -6 - 4x < 0$.

$\therefore f$ is strictly increasing for $x < -\frac{3}{2}$.

In interval $\left(-\frac{3}{2}, \infty\right)$ i.e., when $x > \frac{3}{2}$, $f'(x) = -6 - 4x < 0$.

$\therefore f$ is strictly increasing for $x < -\frac{3}{2}$.

(c) we have,

$f(x)=-2x^3-9x^2-12x+1$

$\therefore f'(x)=-6x^2-18x-12=-6(x^2+3x+2)=-6(x+1)(x+2)$

Now,

$f'(x)=0 \Rightarrow x=-1$ and $x=-2$

Points $x=-1$ and $x=-2$ divide the real line into three disjoint intervals.

i.e., $(-\infty,-2),(-2,-1)$ and $(-1,\infty)$

in intervals $(-\infty,-2)$ and $(-1,\infty)$ i.e., when $x<-2$ and $x>-1$,

$f'(x)=-6(x+1)(x+2)<0$

$\therefore f$ is strictly increasing for $x<-2<x>-1$.

Now, in interval (–2, –1) i.e., when –2 < x < –1,

$f'(x)=-6(x+1)(x+2)>0.$

$\therefore f$ is strictly increasing for –2 < x < –1.

(d) We have,

$f(x)=6-9x-x^2$

$\therefore f'(x)=-9-2x$

Now, $f'(x)=0$ gives $x=-\frac{9}{2}$

The point $x=-\frac{9}{2}$ divides the real line two disjoint intervals i.e.,

$\left(-\infty,-\frac{9}{2}\right)$ and $\left(-\frac{9}{2},\infty\right)$.

In interval $\left(-\infty,-\frac{9}{2}\right)$ i.e., for $x<\frac{9}{2},.$

$\therefore f$ is strictly increasing for $x<\frac{9}{2}$

In interval $\left(\frac{9}{2},\infty\right)$ i.e., for $x<-\frac{9}{2}$, $f'(x)=-9-2x>0.$

$\therefore f$ is strictly decreasing for $x<-\frac{9}{2}$

In the interval $\left(-\frac{9}{2},\infty\right)$ i.e., for $x>-\frac{9}{2}$, $f'(x)=-9-2x<0$

$\therefore f$ is strictly decreasing for x $>-\frac{9}{2}$

(e) We have,

$f(x)=(x+1)^3(x-3)^3$

$f'(x)=3x(x+1)^2(x-3)^3+3(x-3)^2(x+1)^3$

$=3(x+1)^2(x-3)^2[x-3+x+1]$

$=3(x+1)^2(x-3)^2(2x-2)$

$=6(x+1)^2(x-3)^2(x-1)$

Now,

$f'(x)=0 \Rightarrow x=-1,3,1$

The points $x=-1$, $x=1$, and $x=3$ divided the real line into four disjoint intervals.
i.e., $(-\infty,-1)$, $(-1,1)$, $(1,3)$ and $(3,\infty)$

In intervals $(-\infty,-1)$ and $(-1,1)$, $f'(x)=6(x+1)^2(x-3)^2(x-1)<0$

$\therefore f$ is strictly decreasing in intervals $(-\infty,-1)$ and $(-1,1)$

In intervals $(1,3)$ and $(3,\infty)$, $f'(x)=6(x+1)^2(x-3)^2(x-1)>0$

$\therefore f$ is strictly increasing in intervals $(1,3)$ and $(3,\infty)$.

Question 7:

Show that $y=\log(1+x)-\dfrac{2x}{2+x}, x>-1$, is an increasing function of x throughout its domain.

Solution 7:

We have,

$y=\log(1+x)-\dfrac{2x}{2+x}$

$\therefore \dfrac{dy}{dx}=\dfrac{1}{1+x}-\dfrac{(2+x)(2)-2x(1)}{(2+x)^2}=\dfrac{1}{1+x}-\dfrac{4}{(2+x)^2}=\dfrac{x^2}{(2+x)^2}$

Now, $\dfrac{dy}{dx}=0$

$\Rightarrow \dfrac{x^2}{(2+x)^2}=0$

$\Rightarrow x^2=0 \qquad [(2+x)\neq 0 \text{ as } x>-1]$

$\Rightarrow x=0$

Since $x>-1$, point $x=0$ divides the domain $(-1,\infty)$ in two disjoint intervals i.e., $-1<x<0$ and $x>0$.

When $-1<x<0$ we have:

$x<0 \Rightarrow x^2>0$

$x>-1 \Rightarrow (2+x)>0 \Rightarrow (2+x)^2>0$

$\therefore y'=\dfrac{x^2}{(2+x)^2}>0$

Also, when $x>0$:

$x<0 \Rightarrow x^2>0,(2+x)^2>0$

$\therefore y' = \dfrac{x^2}{(2+x)^2} > 0$

Hence, function f is increasing throughout this domain.

Question 8:

Find the values of x for which $y = [x(x-2)]^2$ is an increasing function.

Solution 8:

We have,

$y = [x(x-2)]^2 = [x^2 - 2x]^2$

$\therefore \dfrac{dy}{dx} = 2(x^2 - 2x)(2x-2) = 4x(x-2)(x-1)$

$\therefore \dfrac{dy}{dx} = 0 \Rightarrow x = 0, x = 2, x = 1$

The points $x = 0$, $x = 1$ and $x = 2$ divide the real line into four disjoint intervals i.e., $(-\infty, 0)$, $(0,1)$, $(1,2)$ and $(2,\infty)$.

In intervals $(-\infty, 0)$ and $(1,2)$, $\dfrac{dy}{dx} < 0$

$\therefore y$ is strictly decreasing in intervals $(-\infty, 0)$ and $(1,2)$

However, in intervals $(0,1)$ and $(2,\infty)$, $\dfrac{dy}{dx} > 0$

$\therefore y$ is strictly decreasing in intervals $(0,1)$ and $(2,\infty)$

$\therefore y$ is strictly decreasing in intervals $0 < x < 1$ and $x > 2$.

Question 9:

Prove that $y = \dfrac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function of θ in $\left[0, \dfrac{\pi}{2}\right]$.

Solution 9:

We have,

$y = \dfrac{4\sin\theta}{(2+\cos\theta)} - \theta$

$\therefore \dfrac{dy}{d\theta} = \dfrac{(2+\cos\theta)(4\cos\theta) - 4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2} - 1$

$= \dfrac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$

$= \dfrac{8\cos\theta + 4}{(2+\cos\theta)^2} - 1$

Now,

$$\frac{dy}{d\theta}=0$$

$$\Rightarrow \frac{8\cos\theta+4}{\left(2+\cos\theta\right)^2}=1$$

$$\Rightarrow 8\cos\theta+4=4+\cos^2\theta+4\cos\theta$$

$$\Rightarrow \cos^2\theta-4\cos\theta=0$$

$$\Rightarrow \cos\theta\left(\cos\theta-4\right)=0$$

$$\Rightarrow \cos\theta=0 \ or\ \cos\theta=4$$

Since $\cos\theta\neq 4, \cos\theta=0.$

$$\cos\theta=0\Rightarrow\theta=\frac{\pi}{2}$$

Now,

$$\frac{dy}{d\theta}=0\frac{8\cos\theta+-(4+\cos^2\theta+4\cos\theta)}{(2+\cos\theta)h2}=\frac{4\cos\theta-\cos^{2\theta}}{(2+\cos\theta)^2}=\frac{\cos(4-\cos\theta)}{(2+\cos\theta)^2}$$

In interval $\left[0,\frac{\pi}{2}\right]$, we have $\cos\theta>0$, Also $4>\cos\theta\Rightarrow 4-\cos\theta>0$.

$$\therefore \cos\theta\left(4-\cos\theta\right)>0 \text{ and also } \left(2+\cos\theta\right)^2>0$$

$$\Rightarrow \frac{\cos\theta\left(4-\cos\theta\right)}{\left(2+\cos\theta\right)^2}>0$$

$$\Rightarrow \frac{dy}{dx}>0$$

Therefore, y is strictly increasing in interval $\left(0,\frac{\pi}{2}\right)$

Also, the given function is continuous at $x=0$ and $x=\frac{\pi}{2}$.

Hence, y is increasing in interval $\left[0,\frac{\pi}{2}\right]$.

Question 10:

Prove that the logarithmic function is strictly increasing on $\left(0,\infty\right)$.

Solution 10:

The given function is $f\left(x\right)\log x$.

$$\therefore f'\left(x\right)=\frac{1}{x}$$

It is clear that for $x>0$, $f'\left(x\right)=\frac{1}{x}>0$

Hence, $f\left(x\right)\log x$ is strictly increasing in interval $\left(0,\infty\right)$.

Question 11:

Prove that the function f is given by $f(x)=x^2-x+1$ is neither strictly increasing nor strictly decreasing on $(-1,1)$.

Solution 11:

The given function is $f(x)=x^2-x+1$

$\therefore f'(x)=2x-1$

Now, $f'(x)=0 \Rightarrow x=\frac{1}{2}$.

The point $\frac{1}{2}$ divides the interval $(-1,1)$ into two disjoint intervals i.e., $\left(-1,\frac{1}{2}\right)$ and $\left(\frac{1}{2},1\right)$.

Now, in interval $\left(-1,\frac{1}{2}\right)$, $f'(x)=2x-1<0$.

Therefore, f is strictly decreasing in interval $\left(-1,\frac{1}{2}\right)$

However, in interval $\left(\frac{1}{2},1\right)$, $f'(x)=2x-1>0$.

Therefore, f is strictly increasing in interval $\left(\frac{1}{2},1\right)$.

Hence, f is neither strictly increasing nor decreasing in interval $(-1,1)$.

Question 12:

Which of the following functions are strictly decreasing on $\left(0,\frac{\pi}{2}\right)$?

(A) cos x (B) cos $2x$ (C) cos $3x$ (D) tan x

Solution 12:

(A) let $f_1(x)=\cos x$.

$\therefore f_1'(x)=-\sin x$

In interval $\left(0,\frac{\pi}{2}\right)$, $f_1'(x)=-\sin x<0$.

$\therefore f_1(x)=\cos x$ is strictly decreasing in interval $\left(0,\frac{\pi}{2}\right)$.

(B) let $f_2(x)=\cos 2x$

$\therefore f_2'(x)=-2\sin 2x$

Now, $0<x<\frac{\pi}{2} \Rightarrow 0<2x<\pi \Rightarrow \sin 2x>0 \Rightarrow -2\sin 2x<0$

$\therefore f_2'(x) = -2\sin 2x < 0 \ on \left(0, \frac{\pi}{2}\right)$

$\therefore f_2(x) = \cos 2x$ is strictly decreasing in interval $\left(0, \frac{\pi}{2}\right)$.

(C) let $f_3(x) = \cos 3x$

$\therefore f_3'(x) = -3\sin 3x$

Now, $f'_3(x) = 0.$

$\Rightarrow \sin 3x = 0 \Rightarrow 3x = \pi, as \ x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow x = \frac{\pi}{3}$

The point $x = \frac{\pi}{3}$ divides the interval $\left(0, \frac{\pi}{2}\right)$ into two disjoint intervals

i.e., $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$.

Now, in interval $\left(0, \frac{\pi}{3}\right)$, $f_3(x) = -3\sin 3x < 0 \left[as 0 < x < \frac{\pi}{3} \Rightarrow 0 < 3x < \pi\right]$

$\therefore f_3$ is strictly decreasing in interval $\left(0, \frac{\pi}{3}\right)$.

However, in interval $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, $f_3(x) = -3\sin 3x > 0 \left[as \frac{\pi}{3} < x < \frac{\pi}{2} \Rightarrow \pi < 3x < \frac{3\pi}{2}\right]$

$\therefore f_3$ is strictly increasing in interval $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$.

Hence, f_3 is neither increasing nor decreasing in interval $\left(0, \frac{\pi}{2}\right)$.

(D) let $f_4(x) = \tan x$

$\therefore f_4'(x) = \sec^2 x$

In interval $\left(0, \frac{\pi}{2}\right)$, $f_4'(x) = \sec^2 x > 0.$

$\therefore f_4$ is strictly increasing in interval $\left(0, \frac{\pi}{2}\right)$.

Therefore, function $\cos x$ and $\cos 2x$ are strictly decreasing in $\left(0, \frac{\pi}{2}\right)$

Hence, the correct answer are **A** and **B.**

Question 13:

On which of the following intervals is the function f is given by $f(x) = x^{100} + \sin x - 1$ strictly decreasing?

A. $(0,1)$

B. $\left(\frac{\pi}{2},\pi\right)$

C. $\left(0,\frac{\pi}{2}\right)$

D. None of these

Solution 13:

We have,

$f(x)=x^{100}+\sin x-1$

$\therefore f'(x)=100x^{90}+\cos x$

In interval, $(0,1)$, $\cos x>0$ and $100x^{90}>0$

$\therefore f'(x)>0.$

Thus, function f is strictly increasing in interval $(0,1)$.

In interval $\left(\frac{\pi}{2},\pi\right)$, $\cos x<0$ *and* $100x^{90}>0$.Also $100x^{90}>\cos x$

$\therefore f'(x)>0 in\left(\frac{\pi}{2},\pi\right)$

Thus, function f is strictly increasing in interval $\left(\frac{\pi}{2},\pi\right)$.

In interval $\left(0,\frac{\pi}{2}\right)$, $\cos x>0$ *and* $100x^{90}>0$.

$\therefore 100x^{90}+\cos x>0$

$\Rightarrow f'(x)>0$ *on* $\left(0,\frac{\pi}{2}\right)$

$\therefore f$ is strictly increasing in interval $\left(0,\frac{\pi}{2}\right)$.

Hence, function f is strictly decreasing in none of the intervals.

The correct answer is **D.**

Question 14:

Find the least value of a such that the function f given $f(x)=x^2+ax+1$ is strictly increasing on $(1,2)$.

Solution 14:

We have,

$f(x)=x^2+ax+1$

$\therefore f'(x)=2x+a$

Now, function f will be increasing in $(1,2)$ if $f'(x)>0$ in $(1,2)$.

$\Rightarrow 2x+a>0$

$\Rightarrow 2x>-a$

$\Rightarrow x>\dfrac{-a}{2}$

Therefore, we have to find the least value of a such that

$x>\dfrac{-a}{2}$, when $x\in(1,2)$.

$\Rightarrow x>\dfrac{-a}{2}$ (when $1<x<2$)

Thus, the least value of a for f to be increasing on $(1,2)$ is given by,

$\dfrac{-a}{2}=1$

$\dfrac{-a}{2}=1\Rightarrow a=-2$

Hence, the required value of a is –2.

Question 15:

Let $\mathbf{I}$ be any interval disjoint from $(-1,1)$, prove that the function f given by $f(x)=x+\dfrac{1}{x}$ is strictly increasing on $\mathbf{I}$.

Solution 15:

We have,

$f(x)=x+\dfrac{1}{x}$

$\therefore f'(x)=1-\dfrac{1}{x^2}$

Now,

$f(x)=0\Rightarrow x=\pm 1$

The points $x=1$ and $x=-1$ divided the real line in three disjoint intervals i.e., $(-\infty,-1)$, $(-1,1)$ and $(1,\infty)$.

In interval $(-1,1)$, it is observed that:

$-1<x<1$

$\Rightarrow x^2<1$

$\Rightarrow 1<\dfrac{1}{x^2}, x\neq 0$

$\Rightarrow 1-\dfrac{1}{x^2}<0, x\neq 0$

$\therefore f'(x)=1-\dfrac{1}{x^2}<0\, on(-1,1)\sim\{0\}$.

$\therefore f$ is strictly decreasing on $(-1,1) \sim \{0\}$.

In interval $(-\infty,-1)$ and $(1,\infty)$, it is observed that:

$x < -1 \; or \; 1 < x$

$\Rightarrow x^2 > 1$

$\Rightarrow 1 > \frac{1}{x^2}$

$\Rightarrow 1 - \frac{1}{x^2} > 0$

$\therefore f'(x) = 1 - \frac{1}{x^2} > 0$ on $(-\infty,-1)$ and $(1,\infty)$.

$\therefore f$ is strictly increasing on $(-\infty,-1)$ and $(1,\infty)$.

Hence, function f is strictly increasing in interval in interval $\mathbf{I}$ disjoint from $(-1,1)$.

Hence, the given result is proved.

Question 16:

Prove that the function f given by $f(x) = \log \sin x$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$

Solution 16:

We have,

$f(x) = \log \sin x$

$\therefore f'(x) = \frac{1}{\sin x} \cos x = \cot x$

In interval $\left(0, \frac{\pi}{2}\right)$, $f'(x) = \cot x > 0$

$\therefore f$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$.

In interval $\left(\frac{\pi}{2}, \pi\right)$, $f'(x) = \cot x < 0$

$\therefore f$ is strictly increasing in $\left(\frac{\pi}{2}, \pi\right)$.

Question 17:

Prove that the function f is given by $f(x)=\log\cos x$ is strictly decreasing on $\left(0,\frac{\pi}{2}\right)$ and strictly increasing on $\left(\frac{\pi}{2},\pi\right)$

Solution 17:
We have,
$f(x)=\log\cos x$
$\therefore f'(x)=\frac{1}{\cos x}(-\sin x)=-\tan x$
In interval $\left(0,\frac{\pi}{2}\right)$, $\tan x>0\Rightarrow -\tan x<0.$
$\therefore f'(x)=<0\ on\left(0,\frac{\pi}{2}\right)$
$\therefore f$ is strictly decreasing on $\left(0,\frac{\pi}{2}\right)$.
In interval $\left(\frac{\pi}{2},\pi\right)$, $\tan x<0\Rightarrow -\tan x>0.$
$\therefore f'(x)>0\ on\left(\frac{\pi}{2},\pi\right)$
$\therefore f$ is strictly increasing on $\left(\frac{\pi}{2},\pi\right)$.

Question 18:
Prove that the function given by $f(x)=x^3-3x^2+3x=100$ is increasing in **R**.

Solution 18:
We have,
$f(x)=x^3-3x^2+3x=100$
$f'(x)=3x^2-6x+3$
$=3(x^2-2x+1)$
$=3(x-1)^2$
For any $x\in\mathbf{R}$
$(x-1)^2>0$
Thus $f'(x)$ is always positive in **R**.
Hence, the given function (f) is increasing in **R**.

Question 19:
The interval in which $y=x^2e^{-x}$ is increasing is

A. $(-\infty,\infty)$
B. $(-2,0)$
C. $(2,\infty)$
D. $(0,2)$

Solution 19:
We have,
$y=x^2e^{-x}$
$\therefore \frac{dy}{dx}=2xe^{-x}-x^2e^{-x}=xe^{-x}(2-x)$
Now , $\frac{dy}{dx}=0.$
$\Rightarrow x=0 \; and \; x=2$
The points $x=0$ and $x=2$ divided the real line into the three disjoint intervals i.e., $(-\infty,0)$, $(0,2)$ and $(2,\infty)$.
In intervals $(-\infty,0)$ and $(2,\infty)$, $f'(x)<0 \; as \; e^{-x}$ is always positive.
$\therefore f$ is decreasing on $(-\infty,0)$ and $(2,\infty)$.
In interval $(0,2)$, $f'(x)>0$.
$\therefore f$ is strictly increasing on $(0,2)$.
Hence, f is strictly increasing in interval $(0,2)$.
The correct answer is **D.**

Exercise 6.3

Question 1:
Find the slope of the tangent to the curve $y=3x^4-4x$ at $x=4$.

Solution 1:
The given curve is $y=3x^4-4x$.
Then, the slope of the tangent to the given curve at $x=4$ is given by,
$$\left.\frac{dx}{dy}\right]_{x=4}=12x^3-4\Big]_{x=4}=12(4)^3-4=12(64)-4=764.$$

Question 2:
Find the slope of the tangents to the curve $y=\frac{x-1}{x-2},\; x\neq 2$ at $x=10$.

Solution 2:

The given curve is $y=\dfrac{x-1}{x-2}$.

$$\therefore \frac{dx}{dy}=\frac{(x-2)(1)-(x-1)(1)}{(x-2)^2}$$

$$=\frac{x-2-x+1}{(x-2)^2}=\frac{-1}{(x-2)^2}$$

Thus, the slope of the tangent at $x=10$ is given by,

$$\left.\frac{dx}{dy}\right]_{x=10}=\left.\frac{-1}{(x-2)^2}\right]_{x=10}=\frac{-1}{(10-2)^2}=\frac{-1}{64}.$$

Hence, the slope of the tangent at $x=10$ is $\dfrac{-1}{64}$.

Question 3:

Find the slope of the tangent to curve $y=x^3-x+1$ at the point whose *x*-coordinate is 2.

Solution 3:

The given curve is $y=x^3-x+1$

$$\therefore \frac{dx}{dy}=3x^2-1$$

The slope of the tangent to a curve at (x_0,y_0) is $\left.\dfrac{dx}{dy}\right]_{(x_0,y_0)}$.

It is given that $x_0=2$.

Hence, the slope of the tangent at the point where the *x*-coordinate is 2 is given by,

$$\left.\frac{dx}{dy}\right]_{x=2}=3x^2-1\Big]_{x=2}=3(2)^2-1=12-1=11$$

Question 4:

Find the slope of the tangent to the curve $y=x^3-3x+2$ at the point whose *x*-coordinate is 3.

Solution 4:

The given curve is $y=x^3-3x+2$

$$\therefore \frac{dx}{dy}=3x^2-3$$

The slope of the tangent to a curve at $(x_0\ ,\ y_0)$ is $\left.\dfrac{dx}{dy}\right]_{(x_0\ ,\ y_0)}$

Hence, the slope of the tangent at the tangent at the point where the *x*-coordinate is 3 is given by,

$$\left.\frac{dx}{dy}\right]_{x=3}=3x^2-3\Big]_{x=3}=3(3)^2-3=27-3=24$$

Question 5:

Find the slope of the normal to the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.

Solution 5:

It is given that $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

$$\therefore \frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta) = -3a\cos^2\theta\sin\theta$$

$$\frac{dy}{d\theta} = 3a\sin^2\theta(\cos\theta)$$

$$\therefore \frac{dx}{dy} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} == \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

Therefore, the slope of the tangent at $\theta = \frac{\pi}{4}$ is given by,

$$\left.\frac{dx}{dy}\right]_{\theta=\frac{\pi}{4}} = -\tan\theta\Big]_{\theta=\frac{\pi}{4}} = -\tan\frac{\pi}{4} = -1$$

Hence, the slope of the normal at $\theta = \frac{\pi}{4}$ is given by,

$$\frac{1}{\text{slope of the tangent at } \theta = \frac{\pi}{4}} = \frac{-1}{-1} = 1$$

Question 6:

Find the slope of the normal to the curve $x = 1 - a\sin\theta$ and $y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$.

Solution 6:

It is given, that $x = 1 - a\sin\theta$ and $y = bco^2\theta$.

$$\therefore \frac{dx}{d\theta} = -a\cos\theta \text{ and } \frac{dy}{d\theta} = 2b\cos\theta(-\sin\theta) = -2b\sin\theta\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-2b\sin\theta\cos\theta}{-a\cos\theta} = \frac{2b}{a}\sin\theta$$

Therefore, the slope of the tangent at $\theta = \frac{\pi}{2}$ is given by,

$$\left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{2}} = \left.\frac{2b}{a}\sin\theta\right]_{\theta=\frac{\pi}{2}} = \frac{2b}{a}\sin\frac{\pi}{2} = \frac{2b}{a}$$

Hence, the slope of the normal at $\theta = \frac{\pi}{2}$ is given by,

$$\frac{1}{\text{slope of the tangent at } \theta = \frac{\pi}{4}} = \frac{-1}{\left(\frac{2b}{a}\right)} = -\frac{a}{2b}$$

Question 7:

Find the points at which tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x - axis.

Solution 7:

The equation of the given curve is $y = x^3 - 3x^2 - 9x + 7$.

$$\therefore \frac{dy}{dx} = 3x^2 - 6x - 9$$

Now, the tangent is parallel to the x - axis if the slope of the tangent is zero.

$\therefore 3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x-3)(x+1) = 0$

$\Rightarrow x = 3 \text{ or } x = -1$

When $x = 3$, $y = (3)^3 - 9(3) + 7 = 27 - 27 - 27 + 7 = -20$.

When, $x = 1$, $y = (-1)^3 - 3(-1)^2 - 9(-1) + 7 = -1 - 3 + 9 + 7 = 12.$

Hence, the points at which the tangent is parallel to the x-axis are $(3, -20)$ and $(-1, 12)$.

Question 8:

Find a point on the curve $y = (x-2)^2$ at which the tangent is parallel to the chord joining the points $(2, 0)$ and $(4, 4)$.

Solution 8:

If a tangent is parallel to the chord joining the points $(2, 0)$ and $(4, 4)$, then the slope of the tangent = the slope of the chord.

The slope of the chord is $\frac{4-0}{4-2} = \frac{4}{2} = 2.$

Now, the slope of the tangent to the given curve at a point (x, y) is given by,

$$\frac{dy}{dx} = 2(x-2)$$

Since the slope of the tangent = slope of the chord, we have:

$2(x-2)=2$

$\Rightarrow x-2=1 \Rightarrow x=3$

When $x=3,\ y=(3-2)^2=1.$

Hence, the required point is $(3,1)$.

Question 9:

Find the point on the curve $y=x^3-11x+5$ at which the tangent is $y=x-11.$

Solution 9:

The equation of the given curve is $y=x^3-11x+5$.

The equation of the tangent to the given curve is given as $y=x-11$ (which is of the form $y=mx+c$).

$\therefore$ Slope of the tangent = 1

Now, the slope of the tangent to the given curve at the point (x,y) is given by,

$$\frac{dy}{dx}=3x^2-11$$

Then, we have:

$3x^2-11=1$

$\Rightarrow 3x^2=12$

$\Rightarrow x^2=4$

$\Rightarrow x=\pm 2$

When $x=2, y=(2)^3-11(2)+5=8-22+5=-9.$

When $x=-2, y=(-2)^3-11(-2)+5=-8+22+5=19.$

Hence, the required points are $(2,-9)$ and $(-2,19)$.

Question 10:

Find the equation of all lines having slope –1 that are tangents to the curve $y=\frac{1}{x-1}, x\neq 1$

Solution 10:

The equation of the given curve is $y=\frac{1}{x-1}, x\neq 1$

The slope of the tangents to the given curve at any point (x,y) is given by,

$$\frac{dy}{dx}=\frac{-1}{(x-1)^2}$$

If the slope of the tangents is –1, then we have:

$\frac{-1}{(x-1)^2}=-1$

$\Rightarrow (x-1)^2=1$

$\Rightarrow x-1=\pm 1$

$\Rightarrow x=2,\ 0$

When $x=0$, $y=-1$ and when $x=2$, $y=1$.

Thus, there are two tangents to the given curve having slope –1. These are passing through the points $(0,-1)$ and (2, 1).

$\therefore$ The equation of the tangent through (0, –1) is given by,

$y-(-1)=-1(x-0)$

$\Rightarrow y+1=-x$

$\Rightarrow y+x+1=0$

$\therefore$ The equation of the tangent through $(2,1)$, is given by,

$y-1=-1(x-2)$

$\Rightarrow y-1=x+2$

$\Rightarrow y+x-3=0$

Hence, the equations of the required lines are $y+x+1=0$ and $y+x-3=0$.

Question 11:

Find the equation of all lines having slope 2 which are tangents to the curve $y=\frac{1}{x-3}, x\neq 3$.

Solution 11:

The equation of the given curve is $y=\frac{1}{x-3}, x\neq 3$

The slope of the tangent to the given curve at any point (x,y) is given by,

$\frac{dy}{dx}=\frac{-1}{(x-3)^2}$

If the slope of the tangent is 2, then we have:

$\frac{-1}{(x-3)^2}=2$

$\Rightarrow 2(x-3)^2=-1$

$\Rightarrow (x-3)^2=\frac{-1}{2}$

This is not possible since the **L.H.S.** is positive while the **R.H.S.** is negative.

Hence, there is no tangent to the given curve having slope 2.

Question 12:

Find the equations of all lines having slope 0 which are tangent to the curve $y=\frac{1}{x^2-2x+3}$.

Solution 12:

The equation of the given curve is $y=\frac{1}{x^2-2x+3}$.

The slope of the tangent to the given curve at any point (x,y) is given by,

$$\frac{dy}{dx}=\frac{-(2x-2)}{(x^2-2x+3)^2}=\frac{-2(x-1)}{(x^2-2x+3)^2}$$

If the slope of the tangent is 0, then we have:

$$\frac{-2(x-1)}{(x^2-2x+3)^2}=0$$

$$\Rightarrow -2(x-1)=0$$

$$\Rightarrow x=1$$

When $x=1,\ y=\frac{1}{1-2+3}=\frac{1}{2}$.

$\therefore$ The equation of the tangent through $\left(1,\frac{1}{2}\right)$ is given by,

$$y-\frac{1}{2}=0(x-1)$$

$$\Rightarrow y-\frac{1}{2}=0$$

$$\Rightarrow y=\frac{1}{2}$$

Hence, the equation of the required line is $y=\frac{1}{2}$.

Question 13:

Find points on the curve $\frac{x^2}{9}+\frac{y^2}{16}=1$ at which the tangents are

i) Parallel to x-axis ii) Parallel to y-axis

Solution 13:

The equation of the given curve is $\frac{x^2}{9}+\frac{y^2}{16}=1$

On differentiating both sides with respect to x, we have:

$$\frac{2x}{9}+\frac{2y}{16}.\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-16x}{9y}$$

(i) The tangent is parallel to the x-axis if the slope of the tangent is i.e., $\frac{-16x}{9y}=0$, which is possible if $x=0$.

Then, $\frac{x^2}{9}+\frac{y^2}{16}=1$ for $x=0$

$\Rightarrow y^2=16 \Rightarrow y=\pm 4$

Hence, the points at which the tangents are parallel to the x-axis are $(0,4)$ and $(0,-4)$.

(ii) The tangent is parallel to the y-axis if the slope of the normal is 0, which gives

$$\frac{-1}{\left(\frac{-16x}{9y}\right)}=\frac{9y}{16x}=0 \Rightarrow y=0.$$

Then, $\frac{x^2}{9}+\frac{y^2}{16}=1$ for $y=0$.

$\Rightarrow x=\pm 3$

Hence, the points at which the tangents are parallel to the y-axis are $(3,0)$ and $(-3,0)$.

Question 14:

Find the equations of the tangents and normal to the given curves at the indicated points:

I. $y=x^4-6x^3+13x^2-10x+5$ $at\,(0,5)$

II. $y=x^4-6x^3+13x^2-10x+5$ $at\ (1,3)$

III. $y=x^3$ $at\ (1,1)$

IV. $y=x^2$ $at\ (0,0)$

V. $x=\cos t,\ y=\sin t\ at\ t=\frac{\pi}{4}$

Solution 14:

(i). The equation of the curve is $y=x^4-6x^3+13x^2-10x+5$.

on differentiating with respect to x, we get:

$$\frac{dy}{dx}=4x^3-18x^2+26x-10$$

$$\left.\frac{dy}{dx}\right]_{(0,5)}=-10$$

Thus, the slope of the tangent at $(0,5)$ is -10. The equation of the tangent id given as:

$$y-5=-10(x-0)$$

$$\Rightarrow y-5=-10x$$

$$\Rightarrow 10x+y=5$$

The slope of the normal at $(0,5)$ is $\dfrac{-1}{\text{Slope of the tangent at }(0,5)}=\dfrac{1}{10}$.

Therefore, the equation of the normal at $(0,5)$ is given as:

$$y-5=\frac{1}{10}(x-0)$$
$$\Rightarrow 10y-50=x$$
$$\Rightarrow x-10y+50=0$$

(ii) The equation of the curve is $y=x^4-6x^3+13x^2-10x+5$.

On the differentiating with respect to x, we get:

$$\frac{dy}{dx}=4x^3-18x^2+26x-10$$
$$\left.\frac{dy}{dx}\right]_{(1,3)}=4-18+26-10=2$$

Thus, the slope of the tangent at $(1,3)$ is 2. The equation of the tangent is given as:

$$y-3=2(x-1)$$
$$\Rightarrow y-3=2x-2$$
$$\Rightarrow y=2x+1$$

The slope of the normal at $(1,3)$ is $\dfrac{-1}{\text{Slope of the tangent at }(1,3)}=\dfrac{-1}{2}$.

Therefore, the equation of the normal at $(1,3)$ is given as:

$$y-3=\frac{1}{2}(x-1)$$
$$\Rightarrow 2y-6=x+1$$
$$\Rightarrow x+2y-7=0$$

(iii) The equation of the curve is $y=x^3$.

On differentiating with respect to x, we get:

$$\frac{dy}{dx}=3x^2$$
$$\left.\frac{dy}{dx}\right]_{(1,1)}=3(1)^2=3$$

Thus, the slope of the tangent at $(1,1)$ is 3 and the equation of the tangent is given as:

$$y-1=3(x-1)$$
$$\Rightarrow y=3x-2$$

The slope of the normal at $(1,1)$ is $\dfrac{-1}{\text{Slope of the tangent at }(1,1)}=\dfrac{-1}{3}$.

Therefore, the equation of the normal at $(1,1)$ is given as:

$y-1=\frac{-1}{3}(x-1)$

$\Rightarrow 3y-3=x+1$

$\Rightarrow x+3y-4=0$

(iv) The equation of the curve is $y=x^2$.

On differentiating with respect to x, we get:

$\frac{dy}{dx}=2x$

$\left.\frac{dy}{dx}\right]_{(0,0)}=0$

Thus, the slope of the tangent at $(0,0)$ is 0 and the equation of the tangent is given as:

$y-0=0(x-0)$

$\Rightarrow y=0$

The slope of the normal at $(0,0)$ is $\frac{-1}{\text{Slope of the tangent at }(0,0)}=-\frac{1}{0},$.which is not defined.

Therefore, the equation of the normal at $(x_0 y_0)=(0,0)$ is given by

$x=x_0=0.$

(v) The equation of the curve is $x=\cos t, y=\sin t.$

$x=\cos t, y=\sin t.$

$\therefore \frac{dx}{dt}=-\sin t, \frac{dy}{dt}=\cos t$

$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\cos t}{-\sin t}=-\cot t$

$\left.\frac{dy}{dx}\right]_{t=\frac{\pi}{4}}=-\cot t=-1$

The slope of the tangent at $t=\frac{\pi}{4}$ is –1.

When $t=\frac{\pi}{4}$, $x=\frac{1}{\sqrt{2}}$ and $y=\frac{1}{\sqrt{2}}$.

Thus, the equation of the tangent to the given curve at $t=\frac{\pi}{4}$ i.e., $\left[\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right]$ is

$y-\frac{1}{\sqrt{2}}=-1\left(x-\frac{1}{\sqrt{2}}\right).$

$\Rightarrow x+y-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0$

$\Rightarrow x+y-\sqrt{2}=0$

The slope of the normal at $t=\frac{\pi}{4}$ is $\frac{-1}{\text{Slope of the tangent at } t=\frac{\pi}{4}}=1.$

Therefore, the equation of the normal to the given curve at $t=\frac{\pi}{4}$ i.e., at $\left[\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right]$ is

$$y-\frac{1}{\sqrt{2}}=1\left(x-\frac{1}{\sqrt{2}}\right).$$

$$\Rightarrow x=y$$

Question 15:

Find the equation of the tangent line to the curve $y=x^2-2x+7$ which is

(a) parallel to the line $2x-y+9=0$

(b) Perpendicular to the line $5y-15x=13.$

Solution 15:

The equation of the given curve is $y=x^2-2x+7$

On differentiating with respect to x, we get:

$$\frac{dy}{dx}=2x-2.$$

(a) The equation of the line is $2x-y+9=0$

$2x-y+9=0$, $y=2x+9$

This is of the form $y=mx+c.$

Slope of the line = 2

If a tangent is parallel to the line $2x-y+9=0$, then the slope of the tangent is equal to the slope of the line.

Therefore, we have:

$$2=2x-2$$

$$\Rightarrow 2x=4$$

$$\Rightarrow x=2$$

Now, $x=2$

$$\Rightarrow y=4-4+7=7$$

Thus, the equation of the equation of the tangent passing through $(2,7)$ is given by,

$$y-7=2(x-2)$$

$$\Rightarrow y-2x-3=0$$

Hence, the equation of the tangent line to the given curve (which is parallel to line $2x-y+9=0$) is $y-2x-3=0$.

(b) The equation of the line is $5y-15x=13.$

$5y-15x=13,\ y=3x+\frac{13}{5}$

This is form of $y=mx+c.$

Slope of the line = 3
If a tangent id perpendicular to the line $5y-15x=13$, then the slope of the tangent is

$$\frac{1}{\text{slope of the line}}=\frac{-1}{3}.$$

$$\Rightarrow 2x-2=\frac{-1}{3}$$

$$\Rightarrow 2x=\frac{-1}{3}+2$$

$$\Rightarrow 2x=\frac{5}{3}$$

$$\Rightarrow x=\frac{5}{6}$$

Now, $x=\frac{5}{6}$

$$\Rightarrow y=\frac{25}{36}+\frac{10}{6}+7=\frac{25-60+252}{36}=\frac{217}{36}$$

Thus, the equation of the tangent passing through $\left(\frac{5}{6},\frac{217}{36}\right)$ is given by,

$$y-\frac{217}{36}=\frac{1}{3}\left(x-\frac{5}{6}\right)$$

$$\Rightarrow \frac{36y-217}{36}=\frac{-1}{18}(6x-5)$$

$$\Rightarrow 36y-217=-2(6x-5)$$

$$\Rightarrow 36y-217=-12x+10$$

$$\Rightarrow 36y+12x-227=0$$

Hence, the equation of the tangent line to the given curve (which is perpendicular to line $5y-15x=13$) is $36y+12x-227=0$.

Question 16:

Show that the tangents to the curve $y=7x^3+11$ at the points where $x=2$ and $x=-2$ are parallel.

Solution 16:

The equation of the given curve is $y=7x^3+11$.

$$\therefore \frac{dy}{dx}=21x^2$$

The slope of the tangent to a curve at $(x_0 y_0)$ is $\left.\frac{dy}{dx}\right]_{(x_0,y_0)}$.

Therefore, the slope of the tangent at the point where $x=2$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=-2} = 21(2)^2 = 84$$

It is observed that the slope of the tangents at the points where $x = 2$ and $x = -2$ are equal. Hence, the two tangents are parallel.

Question 17:

Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.

Solution 17:

The equation of the given curve is $y = x^3$.

$$\therefore \frac{dy}{dx} = 3x^2$$

The slope of the tangent at the point (x, y) is given by,

$$\left.\frac{dy}{dx}\right]_{(x,y)} = 3x^2$$

When the slope of the tangent is equal to the y-coordinate of the point, then $y = 3x^2$.

Also, we have $y = x^3$.

$$3x^2 = x^3$$

$$x^2(x-3) = 0$$

$$x = 0, x = 3$$

When $x = 0$, then $y = 0$ and when $x = 3$ then $y = 3(3)^2 = 27$.

Hence, the required points are $(0,0)$ and $(3,27)$.

Question 18:

For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangents passes through the origin.

Solution 18:

The equation of the given curve is $y = 4x^3 - 2x^5$.

$$\therefore \frac{dy}{dx} = 12x^2 - 10x^4$$

Therefore, the slope of the tangent at a point (x, y) is $12x^2 - 10x^4$.

The equation of the tangent at (x, y) is given by,

$$Y - y = \left(12x^2 - 10x^4\right)(X - x) \qquad \text{..... (1)}$$

When the tangent passes through the origin $(0,0)$, then $X = Y = 0$.

Therefore, equation (1) reduces to:

$-y=\left(12x^2-10x^4\right)(-x)$

$y=12x^3-10x^5$

Also, we have $y=4x^3-2x^5$.

$\therefore 12x^3-10x^5=4x^3-2x^5$

$\Rightarrow 8x^5-8x^3=0$

$\Rightarrow x^5-x^3=0$

$\Rightarrow x^3\left(x^2-1\right)=0$

$\Rightarrow x=0,\pm1$

When $x=0, y=4(0)^3-2(0)^5=0.$

When $x=1, y=4(1)^3-2(1)^5=2.$

When $x=-1, y=4(-1)^3-2(-1)^5=-2$.

Hence, the required points are $(0,0)$, $(1,2)$ and $(-1,-2)$.

Question 19:

Find the points on the curve $x^2+y^2-2x-3=0$ at which the tangents are parallel to the x-axis.

Solution 19:

The equation of the given curve is $x^2+y^2-2x-3=0$.

On differentiating with respect to x, we have:

$2x+2y\dfrac{dy}{dx}-2=0$

$\Rightarrow y\dfrac{dy}{dx}=1-x$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{1-x}{y}$

Now, the tangents are parallel to the x-axis if the slope of the tangent is 0.

$\therefore \dfrac{1-x}{y}=0 \Rightarrow 1-x=0 \Rightarrow x=1$

But, $x^2+y^2-2x-3=0$ for $x=1$.

$\Rightarrow y^2=4$, $y=\pm2$

Hence, the points at which the tangents are parallel to the x-axis are $(1,2)$ and $(1,-2)$.

Question 20:

Find the equation of the normal at the point $\left(am^2, am^3\right)$ for the curve $ay^2=x^3$.

Solution 20:

The equation of the given curve is $ay^2 = x^3$.

On differentiating with respect to x, we have:

$$2ay\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{2ay}$$

The slope of a tangent to the curve at (x_0, y_0) is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$.

$\Rightarrow$ The slope of the tangent to the given curve at (am^2, am^3) is

$$\left.\frac{dy}{dx}\right]_{(am^2, am^3)} = \frac{3(am^2)^2}{2a(am^3)} = \frac{3a^2m^4}{2a^2m^3} = \frac{3m}{2}.$$

Slope of normal at (am^2, am^3)

$$= \frac{1}{\text{slope of the tangent at }(am^2, am^3)} = \frac{-2}{3m}$$

Hence, the equation of the normal at (am^2, am^3) is given by,

$$y - am^3 = \frac{-2}{3m}(x - am^2)$$

$$\Rightarrow 3my - 3am^4 = 2x + 2am^2$$

$$\Rightarrow 2x + 3my - am^2(2 + 3m^2) = 0$$

Question 21:

Find the equation of the normal to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.

Solution 21:

The equation of the given curve is $y = x^3 + 2x + 6$.

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = 3x^2 + 2$$

Slope of the normal to the given curve at any point (x, y)

$$= \frac{-1}{\text{Slope of the tangent at the point}(x, y)}$$

$$= \frac{-1}{3x^2 + 2}$$

The equation of the given line is $x + 14y + 4 = 0$.

$x + 14y + 4 = 0$, $y = -\frac{1}{14}x - \frac{4}{14}$ (which is of the form $y = mx + c$)

Slope of the given line = $\frac{-1}{14}$

If the normal is parallel to the line, then we must have the slope of the normal being equal to the slope of the line.

$\therefore \frac{-1}{3x^2+2} = \frac{-1}{14}$

$\Rightarrow 3x^2+2=14$

$\Rightarrow 3x^2=12$

$\Rightarrow x^2=4$

$\Rightarrow x=\pm 2$

When $x=2, y=8+4+6=18.$

When $x=-2, y=-8-4+6=-6.$

Therefore, there are two normal to the given curve with slope $\frac{-1}{14}$ and passing through the points $(2,18)$ and $(-2,-6)$.

Thus, the equation of the normal through $(2,18)$ is given by,

$y-18=\frac{-1}{14}(x-2)$

$\Rightarrow 14y-252=x+2$

$\Rightarrow x+14y-254=0$

And, the equation of the normal through $(-2,-6)$ is given by,

$y-(-6)=\frac{-1}{14}[x-(-2)]$

$\Rightarrow y+6=\frac{-1}{14}(x+2)$

$\Rightarrow 14y+84=-x-2$

$\Rightarrow x+14y+86=0$

Hence, the equations of the normal to the given curve (which are parallel to the given line) are $x+14y-254=0$ and $x+14y+86=0.$

Question 22:

Find the equations of the tangent and normal to the parabola $y^2=4ax$ at the point $(at^2, 2at)$.

Solution 22:

The equation of the given parabola is $y^2=4ax$.

On differentiating $y^2=4ax$ with respect to x, we have:

$$2y\frac{dy}{dx}=4a$$
$$\Rightarrow \frac{dy}{dx}=\frac{2a}{y}$$

The slope of the tangent at $\left(at^2,2at\right)$ is $\left.\frac{dy}{dx}\right]_{\left(at^2,2at\right)}=\frac{2a}{2at}=\frac{1}{t}$.

Then, the equation of the tangent at $\left(at^2,2at\right)$ is given by,

$$y-2at=\frac{1}{t}\left(x-at^2\right)$$
$$\Rightarrow ty-2at^2=x-at^2$$
$$\Rightarrow ty=x+at^2$$

Now, the slope of the normal at $\left(at^2,2at\right)$ is given by,

$$\frac{-1}{\text{Slope of the tangent }\left(at^2,2at\right)}=-t$$

Thus, the equation of the normal at $\left(at^2,2at\right)$ is given as:

$$y-2at=t\left(x-at^2\right)$$
$$\Rightarrow y-2at=tx+at^3$$
$$\Rightarrow y=-tx+2at+at^3$$

Question 23:

Prove that the curves $x=y^2$ and $xy=k$ cut at angles if $8k^2=1$.

[*Hint*: Two curves intersect at right angle if the tangents to the curves at the point of intersection are perpendicular to each other.]

Solution 23:

The equation of the given curves are given as $x=y^2$ and $xy=k$

Putting $x=y^2$ in $xy=k$, we get:

$$y^3=k\Rightarrow y=k^{\frac{1}{3}}$$
$$\therefore x=k^{\frac{2}{3}}$$

Thus, the point of intersection of the given curves is $\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)$

Differentiating $x=y^2$ with respect to x, we have:

$$1=2y\frac{dy}{dx}\Rightarrow \frac{dy}{dx}=\frac{1}{2y}$$

Therefore, the slope of the tangent to the curve $x=y^2$ at $\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)$ is $\left.\frac{dy}{dx}\right]_{\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)}=\frac{1}{2k^{\frac{1}{3}}}$

On differentiating $xy=k$ with respect to x, we have:

$$x\frac{dy}{dx}+y=0\Rightarrow\frac{dy}{dx}=\frac{-y}{x}$$

Slope of the tangent to the curve $xy=k$ at $\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)$ is given by,

$$\left.\frac{dy}{dx}\right]_{\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)}=\left.\frac{-y}{x}\right]_{\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)}=-\frac{k^{\frac{1}{3}}}{k^{\frac{2}{3}}}=\frac{-1}{k^{\frac{1}{3}}}$$

We know that two curves intersect at right angles if the tangents to the curves at the point of intersection i.e., at $\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)$ are perpendicular to each other.

This implies that we should have the product of the tangents as –1.

Thus, the given two curves cut at right angles if the product of the slopes of their respective tangents at $\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)$ is –1.

i.e., $\left(\frac{1}{2k^{\frac{1}{3}}}\right)\left(\frac{-1}{k^{\frac{1}{3}}}\right)=-1$

$$\Rightarrow 2k^{\frac{2}{3}}=1$$

$$\Rightarrow\left(2k^{\frac{2}{3}}\right)^3=(1)^3$$

$$\Rightarrow 8k^2=1$$

Hence, the given two curves cut at right angles if $8k^2=1$.

Question 24:

Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_0y_0).

Solution 24:

Differentiating $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with respect to x, we have:

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$
$$\Rightarrow \frac{2y}{b^2}\frac{dy}{dx}=\frac{2x}{a^2}$$
$$\Rightarrow \frac{dy}{dx}=\frac{b^2x}{a^2y}$$

Therefore, the slope of the tangent at (x_0, y_0) is $\left.\frac{dy}{dx}\right]_{(x_0,y_0)}=\frac{b^2x_0}{a^2y_0}$

Then, the equation of the tangent at (x_0, y_0) is given by,

$$y-y_0=\frac{b^2x_0}{a^2y_0}(x-x_0)$$
$$\Rightarrow a^2yy_0-a^2{y_0}^2=b^2xx_0-b^2{x_0}^2$$
$$\Rightarrow b^2xx_0-a^2yy_0-b^2{x_0}^2+a^2{y_0}^2=0$$
$$\Rightarrow \frac{xx_0}{a^2}-\frac{yy_0}{b^2}-\left(\frac{{x_0}^2}{a^2}-\frac{{y_0}^2}{b^2}\right)=0 \qquad \left[\text{on dividing both sides by } a^2b^2\right]$$
$$\Rightarrow \frac{xx_0}{a^2}-\frac{yy_0}{b^2}-1=0 \qquad \left[(x_0,y_0)\text{ lies on the hyperbola } \frac{x^2}{a^2}-\frac{y^2}{b^2}=1\right]$$
$$\Rightarrow \frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1$$

Now, the slope of the normal at (x_0, y_0) is given by,

$$\frac{-1}{\text{Slope of the tangent }(x_0,y_0)}=\frac{-a^2y_0}{b^2x_0}$$

Hence, the equation of the normal at (x_0, y_0) is given by,

$$y-y_0=\frac{-a^2y_0}{b^2x_0}(x-x_0)$$
$$\Rightarrow \frac{y-y_0}{a^2y_0}=\frac{-(x-x_0)}{b^2x_0}$$
$$\Rightarrow \frac{y-y_0}{a^2y_0}+\frac{-(x-x_0)}{b^2x_0}=0$$

Question 25:

Find the equation of the tangent to the curve $y=\sqrt{3x-2}$ which is parallel to the line $4x-2y+5=0$.

Solution 25:

The equation of the given curve is $y=\sqrt{3x-2}$.

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx}=\frac{3}{2\sqrt{3x-2}}$$

The equation of the given line is $4x-2y+5=0$.

$4x-2y+5=0$, $y=2x+\frac{5}{2}$ (which is of the form $y=mx+c$)

Slope of the line = 2

Now, the tangent to the given curve is parallel to the line $4x-2y-5=0$ if the slope of the tangent is equal to the slope of the line.

$$\frac{3}{2\sqrt{3x-2}}=2$$

$$\Rightarrow \sqrt{3x-2}=\frac{3}{4}$$

$$\Rightarrow 3x-2=\frac{9}{16}$$

$$\Rightarrow 3x=\frac{9}{16}+2=\frac{41}{16}$$

$$\Rightarrow x=\frac{41}{48}$$

When $x=\frac{41}{48}, y=\sqrt{3\left(\frac{41}{48}\right)-2}=\sqrt{\frac{41}{16}-2}=\sqrt{\frac{41-32}{16}}=\sqrt{\frac{9}{16}}=\frac{3}{4}$.

Equation of the tangent passing through the point $\left(\frac{41}{48}, \frac{3}{4}\right)$ is given by,

$$y-\frac{3}{4}=2\left(x-\frac{41}{48}\right)$$

$$\Rightarrow \frac{4y-3}{4}=2\left(\frac{48x-41}{48}\right)$$

$$\Rightarrow 4y-3=\left(\frac{48x-41}{6}\right)$$

$$\Rightarrow 24y-18=48x-41$$

$$\Rightarrow 48x-24y=23$$

Hence, the equation of the required tangent is $48x-24y=23$.

Question 26:

The slope of the normal to the curve $y=2x^2+3\sin x$ at $x=0$ is

(A) 3, **(B)** $\frac{1}{3}$, **(C)** -3, **(D)** $-\frac{1}{3}$

Solution 26:

The equation of the given curve is $y = 2x^2 + 3\sin x$.
Slope of the tangent to the given curve at $x = 0$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=0} = 4x + 3\cos x\Big]_{x=0} = 0 + 3\cos 0 = 3$$

Hence, the slope of the normal to the given curve at $x = 0$ is

$$\frac{-1}{\text{Slope of the tangent at } x = 0} = \frac{-1}{3}.$$

The correct answer is **D.**

Question 27:

The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point

(A) $(1,2)$, **(B)** $(2,1)$, **(C)** $(1,-2)$, **(D)** $(-1,2)$

Solution 27:

The equation of the given curve is $y^2 = 4x$.
Differentiating with respect to x, we have:

$$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

Therefore, the slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{2}{y}$$

The given line is $y = x + 1$(which is of the form $y = mx + c$)
Slope of the line = 1.
The line $y = x + 1$ is a tangent to the given curve if the slope of the line is equal to the slope of the tangent. Also, the line must intersect the curve.
Thus, we must have:

$$\frac{2}{y} = 1$$

$$\Rightarrow y = 2$$

Now, $y = x + 1 \Rightarrow x = y - 1 \Rightarrow x = 2 - 1 = 1$

Hence, the line $y = x + 1$is a tangent to the given curve at the point $(1, 2)$.

The correct answer is **A.**

Exercise 6.4

Question 1:

Using differentials, find the approximate value of each of the following up to 3 places of decimal.

(i) $\sqrt{25.3}$, **(ii)** $\sqrt{49.5}$, **(iii)** $\sqrt{0.6}$, **(iv)** $(0.009)^{\frac{1}{3}}$, **(v)** $(0.999)^{\frac{1}{10}}$, **(vi)** $(15)^{\frac{1}{4}}$, **(vii)** $(26)^{\frac{1}{3}}$,
(viii) $(255)^{\frac{1}{4}}$, **(ix)** $(82)^{\frac{1}{4}}$, **(x)** $(401)^{\frac{1}{2}}$, **(xi)** $(0.0037)^{\frac{1}{2}}$, **(xii)** $(26.57)^{\frac{1}{3}}$, **(xiii)** $(81.5)^{\frac{1}{4}}$,
(xiv) $(3.968)^{\frac{3}{2}}$, **(xv)** $(32.15)^{\frac{1}{5}}$

Solution 1:

(i) $\sqrt{25.3}$

Consider $y=\sqrt{x}$. Let $x=25$ and $\Delta x=0.3.$

Then,

$$\Delta y=\sqrt{x+\Delta x}-\sqrt{x}=\sqrt{25.3}-\sqrt{25}=\sqrt{25.3}-5$$

$$\Rightarrow \sqrt{25.3}=\Delta y+5$$

Now, dy is approximately equal to Δy and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{2\sqrt{x}}(0.3) \qquad \left[as\ y=\sqrt{x}\right]$$

$$=\frac{1}{2\sqrt{25}}(0.3)=0.03$$

Hence, the approximate value of $\sqrt{25.3}$ is $0.03+5=5.03.$

(ii) $\sqrt{49.5}$

Consider $y=\sqrt{x}$. Let $x=49$ and $\Delta x=0.5.$

Then,

$$\Delta y=\sqrt{x+\Delta x}-\sqrt{x}=\sqrt{49.5}-\sqrt{49}=\sqrt{49.5}-7$$

$$\Rightarrow \sqrt{49.5}=7+\Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{2\sqrt{x}}(0.5) \qquad \left[as\ y=\sqrt{x}\right]$$

$$=\frac{1}{2\sqrt{49}}(0.5)=\frac{1}{14}(0.5)=0.035$$

Hence, the approximate value of $\sqrt{49.5}$ is $7+0.035=7.035.$

(iii) $\sqrt{0.6}$

Consider $y=\sqrt{x}$. Let $x=1$ and $\Delta x=-0.4.$

Then,

$$\Delta y=\sqrt{x+\Delta x}-\sqrt{x}=\sqrt{0.6}-1$$

$$\Rightarrow \sqrt{0.6}=1+\Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{2\sqrt{x}}(\Delta x) \qquad \left[as\ y=\sqrt{x}\right]$$

$=\frac{1}{2}(-0.4)=-0.2$

Hence, the approximate value of $\sqrt{0.6}$ is $1+(-0.2)=1-0.2=0.8.$

(iv) $(0.009)^{\frac{1}{3}}$

Consider $y=x^{\frac{1}{3}}$. Let $x=0.008$ and $\Delta x=0.001.$

Then,

$\Delta y=(x+\Delta x)^{\frac{1}{3}}-(x)^{\frac{1}{3}}=(0.009)^{\frac{1}{3}}-(0.008)^{\frac{1}{3}}=(0.009)^{\frac{1}{3}}-0.2$

$\Rightarrow(0.009)^{\frac{1}{3}}=0.2+\Delta y$

Now, dy is approximately equal to Δy and is given by,

$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{3(x)^{\frac{2}{3}}}(\Delta x)$ $\left[as\ y=x^{\frac{1}{3}}\right]$

$=\frac{1}{3\text{x}0.04}(0.001)=\frac{0.001}{0.12}=0.008$

Hence, the approximate value of $(0.009)^{\frac{1}{3}}$ is $0.2+0.008=0.208.$

(v) $(0.999)^{\frac{1}{10}}$

Consider $y=(x)^{\frac{1}{10}}$. Let $x=1$ and $\Delta x=-0.001.$

Then,

$\Delta y=(x+\Delta x)^{\frac{1}{10}}-(x)^{\frac{1}{10}}=(0.999)^{\frac{1}{10}}-1$

$\Rightarrow(0.999)^{\frac{1}{10}}=1+\Delta y$

Now, dy is approximately equal to Δy and is given by,

$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{10(x)^{\frac{9}{10}}}(\Delta x)$ $\left[as\ y=(x)^{\frac{1}{10}}\right]$

$=\frac{1}{10}(-0.001)=-0.0001$

Hence, the approximate value of $(0.999)^{\frac{1}{10}}$ is $1+(-0.0001)=0.9999.$

(vi) $(15)^{\frac{1}{4}}$

Consider $y=x^{\frac{1}{4}}$. Let $x=16$ and $\Delta x=-1.$

Then,

$\Delta y=(x+\Delta x)^{\frac{1}{4}}-x^{\frac{1}{4}}=(15)^{\frac{1}{4}}-(16)^{\frac{1}{4}}=(15)^{\frac{1}{4}}-2$

$\Rightarrow(15)^{\frac{1}{4}}=2+\Delta y$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[as\ y = x^{\frac{1}{4}}\right]$$

$$= \frac{1}{4(16)^{\frac{3}{4}}}(-1) = \frac{-1}{4x8} = \frac{-1}{32} = -0.03125$$

Hence, the approximate value of $(15)^{\frac{1}{4}}$ is $2 + (-0.03125) = 1.96875$.

(vii) $(26)^{\frac{1}{3}}$

Consider $y = (x)^{\frac{1}{3}}$. Let $x = 27$ and $\Delta x = -1$.

$$\Delta y = (x + \Delta x)^{\frac{1}{3}} - (x)^{\frac{1}{3}} = (26)^{\frac{1}{3}} - (27)^{\frac{1}{3}} = (26)^{\frac{1}{3}} - 3$$

$$\Rightarrow (26)^{\frac{1}{3}} = 3 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3(x)^{\frac{2}{3}}}(\Delta x) \qquad \left[as\ y = (x)^{\frac{1}{3}}\right]$$

$$= \frac{1}{3(27)^{\frac{2}{3}}}(-1) = \frac{-1}{27} = -0.0\overline{370}$$

Hence, the approximate value of $(26)^{\frac{1}{3}}$ is $3 + (-0.0370) = 2.9629$.

(viii) $(255)^{\frac{1}{4}}$

Consider $y = (x)^{\frac{1}{4}}$. Let $x = 256$ and $\Delta x = -1$.

Then,

$$\Delta y = (x + \Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}} = (255)^{\frac{1}{4}} - (256)^{\frac{1}{4}} = (255)^{\frac{1}{4}} - 4$$

$$\Rightarrow (255)^{\frac{1}{4}} = 4 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[as\ y = x^{\frac{1}{4}}\right]$$

$$= \frac{1}{4(256)^{\frac{3}{4}}}(-1) = \frac{-1}{4x4^3} = -0.0039$$

Hence, the approximate value of $(255)^{\frac{1}{4}}$ is $4 + (-0.0039) = 3.9961$.

(iX) $(82)^{\frac{1}{4}}$

Consider $y = x^{\frac{1}{4}}$. Let $x = 81$ and $\Delta x = 1$.

$\Delta y = (x+\Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}} = (82)^{\frac{1}{4}} - (81)^{\frac{1}{4}} = (82)^{\frac{1}{4}} - 3$

$\Rightarrow (82)^{\frac{1}{4}} = \Delta y + 3$

Now, dy is approximately equal to Δy and is given by,

$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x)$ $\left[as\ y = (x)^{\frac{1}{4}}\right]$

$= \frac{1}{4(81)^{\frac{3}{4}}}(1) = \frac{1}{4(3)^3} = \frac{1}{108} = 0.009$

Hence, the approximate value of $(82)^{\frac{1}{4}}$ is $3 + 0.009 = 3.009$.

(x) $(401)^{\frac{1}{2}}$

Consider $y = x^{\frac{1}{2}}$. Let $x = 400$ and $\Delta x = 1$.

$\Delta y = \sqrt{x+\Delta x} - \sqrt{x} = \sqrt{401} - \sqrt{400} = \sqrt{401} - 20$

$\Rightarrow \sqrt{401} = 20 + \Delta y$

Now, dy is approximately equal to Δy and is given by,

$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(\Delta x)$ $\left[as\ y = x^{\frac{1}{2}}\right]$

$= \frac{1}{2x20}(1) = \frac{1}{40} = 0.025$

Hence, the approximate value of $\sqrt{401}$ is $20 + 0.025 = 20.025$.

(xi) $(0.0037)^{\frac{1}{2}}$

Consider $y = x^{\frac{1}{2}}$. Let $x = 0.0036$ and $\Delta x = 0.0001$.

Then,

$\Delta y = (x+\Delta x)^{\frac{1}{2}} - (x)^{\frac{1}{2}} = (0.0037)^{\frac{1}{2}} - (0.0036)^{\frac{1}{2}} = (0.0037)^{\frac{1}{2}} - 0.06$

$\Rightarrow (0.0037)^{\frac{1}{2}} = 0.06 + \Delta y$

Now, dy is approximately equal to Δy and is given by,

$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(\Delta x)$ $\left[as\ y = x^{\frac{1}{2}}\right]$

$= \frac{1}{2x0.06}(0.0001)$

$= \frac{0.0001}{0.12} = 0.00083$

Thus, the approximately value of $(0.0037)^{\frac{1}{2}}$ is $0.06+0.00083=0.6083$.

(xii) $(26.57)^{\frac{1}{3}}$

Consider $y=x^{\frac{1}{3}}$. Let $x=27$ and $\Delta x=-0.43$.
Then,

$$\Delta y=(x+\Delta x)^{\frac{1}{3}}-x^{\frac{1}{3}}=(26.57)^{\frac{1}{3}}-(27)^{\frac{1}{3}}=(26.57)^{\frac{1}{3}}-3$$

$$\Rightarrow (26.57)^{\frac{1}{3}}=3+\Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{3(x)^{\frac{2}{3}}}(\Delta x) \qquad \left[as\ y=x^{\frac{1}{3}}\right]$$

$$=\frac{1}{3(9)}(-0.43)$$

$$=\frac{-0.43}{27}=-0.015$$

Hence, the approximate value of $(26.57)^{\frac{1}{3}}$ is $3+(-0.015)=2.984$.

(xiii) $(81.5)^{\frac{1}{4}}$

Consider $y=x^{\frac{1}{4}}$. Let $x=81$ and $\Delta x=0.5$.
Then,

$$\Delta y=(x+\Delta x)^{\frac{1}{4}}-(x)^{\frac{1}{4}}=(81.5)^{\frac{1}{4}}-(81)^{\frac{1}{4}}=(81.5)^{\frac{1}{4}}-3$$

$$\Rightarrow (81.5)^{\frac{1}{4}}=3+\Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[as\ y=x^{\frac{1}{4}}\right]$$

$$=\frac{1}{4(3)^3}(0.5)=\frac{0.5}{108}=0.0046$$

Hence, the approximate value of $(81.5)^{\frac{1}{4}}$ is $3+0.0046=3.0046$.

(xiv) $(3.968)^{\frac{3}{2}}$

Consider, $y=x^{\frac{3}{2}}$. Let $x=4$ and $\Delta x=-0.032$.
Then,

$$\Delta y=(x+\Delta x)^{\frac{3}{2}}-x^{\frac{3}{2}}=(3.968)^{\frac{3}{2}}-(4)^{\frac{3}{2}}=(3.968)^{\frac{3}{2}}-8$$

$\Rightarrow (3.968)^{\frac{3}{2}} = 8 + \Delta y$

Now, dy is approximately equal to Δy and is given by,

$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{3}{2}(x)^{\frac{1}{2}}(\Delta x)$ $\left[as\ y = x^{\frac{3}{2}}\right]$

$= \frac{3}{2}(2)(-0.032)$

$= -0.096$

Hence, the approximate value of $(3.968)^{\frac{3}{2}}$ is $8 + (-0.096) = 7.904$.

(XV) $(32.15)^{\frac{1}{15}}$

Consider $y = x^{\frac{1}{5}}$. Let $x = 32$ and $\Delta x = -0.15$.

Then,

$\Delta y = (x+\Delta x)^{\frac{1}{5}} - x^{\frac{1}{5}} = (32.15)^{\frac{1}{5}} - (32)^{\frac{1}{5}} = (32.15)^{\frac{1}{5}} - 2$

$\Rightarrow (32.15)^{\frac{1}{15}} = 2 + \Delta y$

Now, dy is approximately equal to Δy and is given by,

$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{5(x)^{\frac{4}{5}}}.(\Delta x)$ $\left[as\ y = x^{\frac{1}{5}}\right]$

$= \frac{1}{5\text{x}(2)^4}(0.15)$

$= \frac{0.15}{80} = 0.00187$

Hence, the approximate value of $(32.15)^{\frac{1}{15}}$ is $2 + 0.00187 = 2.00187$.

Question 2:

Find the approximate value of (2.01), where $f(x) = 4x^2 + 5x + 2$.

Solution 2:

Let $x = 2$ and $\Delta x = 0.01$. Then, we have:

$f(2.01) = f(x+\Delta x) = 4(x+\Delta x)^2 + 5(x+\Delta x) + 2$

Now, $\Delta y = f(x+\Delta x) - f(x)$

$f(x+\Delta x) = f(x) + \Delta y$

$\approx f(x) + f'(x).\Delta x$ $(as\ dx = \Delta x)$

$\Rightarrow f(2.01) \approx (4x^2 + 5x + 2) + (8x+5)\Delta x$

$=\left[4(2)^2+5(2)+2\right]+\left[8(2)+5\right](0.01)$ $\quad [as\ \ x=2,\ \Delta x=0.01]$

$=(16+10+2)+(16+5)(0.01)$

$=28+(21)(0.01)$

$=28+0.21$

$=28.21$

Hence, the approximate value of $f(2.01)$ is 28.21.

Question 3:

Find the approximate value of $f(5.001)$, where $f(x)=x^3-7x^2+15$.

Solution 3:

Let $x=5$ and $\Delta x=0.001$. Then, we have:

$f(5.001)=f(x+\Delta x)=(x+\Delta x)^3-7(x+\Delta x)^2+15$

Now, $\Delta y=f(x+\Delta x)-f(x)$

$\therefore f(x+\Delta x)=f(x)+\Delta y$

$\approx f(x)+f'(x).\Delta x$ $\quad (as\ dx=\Delta x)$

$\Rightarrow f(5.001)\approx(x^3-7x^2+15)+(3x^2-14x)\Delta x$

$=\left[(5)^3-7(5)^2+15\right]+\left[3(5)^2-14(5)\right](0.001)$ $\quad [x=5, \Delta x=0.001]$

$=(125-175+15)+(75-70)(0.001)$

$=-35+(5)(0.001)$

$=-35+0.005$

$=-34.995$

Hence, the approximate value of $f(5.001)$ is -34.995.

Question 4:

Find the approximate change in the volume V of a cube side x meters caused by increasing side by 1%.

Solution 4:

The volume of a cube (V) of side x is given by $V=x^3$.

$\therefore dV=\left(\frac{dV}{dx}\right)\Delta x$

$=(3x^2)\Delta x$

$=(3x^2)(0.01x)$ $\quad [as\ 1\%\ of\ \ x\ is\ 0.01x]$

$=0.03x^3$

Hence, the approximate change in the volume of the cube is $0.03x^3$ m^3.

Question 5:
Find the approximate change in the surface area of a cube of side x meters caused by decreasing the side by 1%.

Solution 5:
The surface area of a cube (S) of side x is given by $S = 6x^2$.

$$\therefore \frac{ds}{dx} = \left(\frac{ds}{dx}\right)\Delta x$$

$$= (12x)\Delta x$$

$$= (12x)(0.01x) \qquad [as\ 1\%\ of\ \ x\ is\ 0.01x]$$

$$= 0.12x^2$$

Hence, the approximate change in the surface area of the cube is $0.12x^2$ m^2.

Question 6:
If the radius of a sphere is measured as 7 m with an error of 0.02 m, then find the approximate error in calculating its volume.

Solution 6:
Let r be the radius of the sphere and Δr be the error in measuring the radius.
Then,
$r = 7m$ and $\Delta r = 0.02m$
Now, the volume V of the sphere is given by,

$$V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dr} = 4\pi r^2$$

$$\therefore dV = \left(\frac{dV}{dr}\right)\Delta r$$

$$= \left(4\pi r^2\right)\Delta r$$

$$= 4\pi(7)^2(0.02)m^3 = 3.92\pi m^3$$

Hence, the approximate error in calculating the volume is 3.92π m^3.

Question 7:
If the radius of a sphere is measured as 9 m with an error of 0.03 m, then find the approximate error in calculating in surface area.

Solution 7:

Let r be the radius of the sphere and Δr be the error in measuring the radius.

Then,

$r = 9m$ and $\Delta r = 0.03m$

Now, the surface area of the sphere (S) is given by,

$\therefore \frac{dS}{dr} = 8\pi r$

$\therefore dS = \left(\frac{dS}{dr}\right)\Delta r$

$= (8\pi r)\Delta r$

$= 8\pi(9)(0.03)m^2$

$= 2.16\pi m^2$

Hence, the approximate error in calculating the surface area is 2.16π m^2.

Question 8:

If $f(x) = 3x^2 + 15x + 5$, then the approximate value of (3.02) is

(A) 47.66, **(B)** 57.66, **(C)** 67.66, **(D)** 77.66

Solution 8:

Let $x = 3$ and $\Delta x = 0.02$ Then, we have:

$f(3.02) = f(x + \Delta x) = 3(x + \Delta x)^2 + 15(x + \Delta x) + 5$

Now, $\Delta y = f(x + \Delta x) - f(x)$

$\Rightarrow f(x + \Delta x) = f(x) + \Delta y$

$\approx f(x) + f'(x)\Delta x$ $\qquad (As\ dx = \Delta x)$

$\Rightarrow f(3.02) \approx (3x^2 + 15x + 5) + (6x + 15)\Delta x$

$= [3(3^2) + 15(3) + 5] + [6(3) + 15](0.02)$ $\qquad [As\ x = 3,\ \Delta x = 0.02]$

$= (27 + 45 + 5) + (18 + 15)(0.02)$

$= 77 + (33)(0.02)$

$= 77 + 0.66$

$= 77.66$

Hence, the approximate value of (3.02) is 77.66.

The correct answer is **D.**

Question 9:

The approximate change in the volume of a cube of side x meters caused by increasing the side by 3% is

A. $0.06x^3m^3$ **B.** $0.6x^3m^3$ **C.** $0.09x^3m^3$ **D.** $0.9x^3m^3$

Solution 9:

The volume of a cube (V) of side x is given by $V = x^3$.

$\therefore dV = \left(\frac{dV}{dx}\right)\Delta x$

$= (3x^2)\Delta x$

$= (3x^2)(0.03x)$ $[As\ 3\%\ of\ x\ is\ 0.03x]$

$= 0.09x^3 m^3$

Hence, the approximate change in the volume of the cube is $0.09x^3$ m^3.

The correct answer is **C**.

Exercise 6.5

Question 1:

Find the maximum and minimum values, if any, of the following given by

(i) $f(x) = (2x-1)^2 + 3$ **(ii)** $f(x) = 9x^2 + 12x + 2$

(iii) $f(x) = -(x-1)^2 + 10$ **(iv)** $g(x) = x^3 + 1$

Solution 1:

(i) The given function is $f(x) = (2x-1)^2 + 3$

It can be observed that $(2x-1)^2 \geq 0$ for every $x \in \mathbf{R}$.

Therefore, $f(x) = (2x-1)^2 + 3 \geq 3$ for every $x \in \mathbf{R}$.

The minimum value of f is attained when $2x - 1 = 0$.

$2x - 1 = 0$, $x = \frac{1}{2}$

Minimum value of $f\left(\frac{1}{2}\right) = \left(2.\frac{1}{2} - 1\right)^2 + 3 = 3.$

Hence, function f does not have a maximum value.

(ii) The given function is $f(x) = 9x^2 + 12x + 2 = (3x^2 + 2)^2 - 2.$

It can be observed that $(3x^2 + 2)^2 \geq 0$ for every $x \in \mathbf{R}$.

Therefore, $f(x) = (3x^2 + 2)^2 - 2 \geq -2$ for every $x \in \mathbf{R}$.

The minimum value of f is attained when $3x + 2 = 0$.

$3x + 2 = 0 = 0$, $x = \frac{-2}{3}$

Minimum value of $f\left(-\frac{2}{3}\right)=\left(3\left(\frac{-2}{3}\right)+2\right)^2-2=-2$

Hence, function f does not have a maximum value.

(iii) The given function is $f(x)=-(x-1)^2+10$.

It can be observed that $(x-1)^2\geq 0$ for every $x\in\mathbf{R}$.

Therefore, $f(x)=-(x-1)^2+10\leq 10$ for every $x\in\mathbf{R}$.

The maximum value of f is attained when $(x-1)=0$.

$(x-1)=0$, $x=0$

Maximum value of $f=f(1)=-(1-1)^2+10=10$

Hence, function f does not have a maximum value.

(iv) The given function is $g(x)=x^3+1$.

Hence, function g neither has a maximum value nor a minimum value.

Question 2:

Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x)=|x+2|-1$ **(ii)** $g(x)=-|x+1|+3$ **(iii)** $h(x)=\sin(2x)+5$

(iv) $f(x)=|\sin 4x+3|$ **(v)** $h(x)=x+4, x\in(-1,1)$

Solution 2:

(i) $f(x)=|x+2|-1$

We know that $|x+2|\geq 0$ for every $x\in\mathbf{R}$.

Therefore, $f(x)=|x+2|-1\geq -1$ for every $x\in\mathbf{R}$.

The minimum value of f is attained when $|x+2|=0$.

$|x+2|=0$

$\Rightarrow x=-2$

Minimum value of $f=f(-2)==|-2+2|-1=-1$

Hence, function f does not have a maximum value.

(ii) $g(x)=-|x+1|+3$

We know that $-|x+1|\leq 0$ for every $x\in\mathbf{R}$.

Therefore, $g(x)=-|x+1|+3\leq 3$ for every $x\in\mathbf{R}$.

The maximum value of g is attained when $|x+1|=0$.

$|x+1|=0$

$\Rightarrow x=-1$

Maximum value of $g=g(-1)=-|-1+1|+3=3$

Hence, function g does not have a maximum value.

(iii) $h(x)=\sin 2x+5$

We know that $-1\le \sin 2x\le 1$.

$-1+5\le \sin 2x+5\le 1+5$

$4\le \sin 2x+5\le 6$

Hence, the maximum and minimum values of h are 6 and 4 respectively.

(iv) $f(x)=|\sin 4x+3|$

We know that $-1\le \sin 4x\le 1$.

$2\le \sin 4x+3\le 4$

$2\le |\sin 4x+3|\le 4$

Hence, the maximum and minimum values of f are 4 and 2 respectively.

(v) $h(x)=x+4, x\in(-1,1)$

Here, if a point x_0 is closest to -1, then we find $\frac{x_0}{2}+1<x_0+1$ for all $x_0\in(-1,1)$.

Also if x_1 is closet to -1, then we find $x_1+1<\frac{x_1+1}{2}+1$ for all $x_0\in(-1,1)$.

Hence, function $h(x)$ has neither maximum nor minimum value in $(-1,1)$.

Question 3:

Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

(i) $f(x)=x^2$ **(ii)** $g(x)=x^3-3x$ **(iii)** $h(x)=\sin x+\cos .0<x<\frac{\pi}{2}$

(iv) $f(x)=\sin x-\cos x, 0<x<2\pi$ **(v)** $f(x)=x^3-6x^2+9x+15$

(vi) $g(x)=\frac{x}{2}+\frac{2}{x}, x>0$ **(vii)** $g(x)=\frac{1}{x^2+2}$ **(vii)** $f(x)=x\sqrt{1-x}, x>0$

Solution 3:

(i) $f(x)=x^2$

$\therefore f'(x)=2x$

Now,

$f'(x)=0\Rightarrow x=0$

Thus, $x=0$ is the only critical point which could possibly be the point of local maxima or local minima of f.

We have f''(0)=2, which is positive.

Therefore, by second derivative test, $x=0$ is a point of local minima and local minimum value of f

at $x=0$ is $f(0)=0$.

(ii) $g(x)=x^3-3x$

$\therefore g'(x)=3x^2-3$

Now,

$g'(x)=0 \Rightarrow 3x^2=3 \Rightarrow x=\pm 1$

g''(x) = 6x

g''(1) = 6 > 0

g''(–1) = –6 < 0

By second derivative test, $x=1$ is a point of local minima and local minimum value of g

At $x=1$ is $g(1)=1^3-3=1-3=-2$. However,

$x=-1$ is a point of local maxima and local maximum value of g at

$x=-1$ is $g(1)=(-1)^3-3(-1)=-1+3=-2$.

(iii) $h(x)=\sin x+\cos .0<x<\dfrac{\pi}{2}$

$\therefore h'(x)=\cos x+\sin x$

$h'(x)=0 \Rightarrow \sin x=\cos x \Rightarrow \tan x=1 \Rightarrow x=\dfrac{\pi}{4}\in\left(0,\dfrac{\pi}{2}\right)$

h''(x) = –sin x – cos x

$h''\left(\dfrac{\pi}{4}\right)=-\left(\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}\right)=-\dfrac{1}{\sqrt{2}}=-\sqrt{2}<0$

Therefore, by second derivative test, $x=\dfrac{\pi}{4}$ is a point of local maxima and the local Maximum value of h at $x=\dfrac{\pi}{4}$ is $h\left(\dfrac{\pi}{4}\right)=\sin\dfrac{\pi}{4}+\cos\dfrac{\pi}{4}=\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}=\sqrt{2}.$

(iv) $f(x)=\sin x-\cos x, 0<x<2\pi$

$\therefore f'(x)=\cos x+\sin x$

$f'(x)=0 \Rightarrow \cos x=-\sin x \Rightarrow \tan x=-1 \Rightarrow x=\dfrac{3\pi}{4},\dfrac{7\pi}{4}\in(0,2\pi)$

$f''(x)=-\sin x+\cos x$

$f''\left(\dfrac{3\pi}{4}\right)=-\sin\dfrac{3\pi}{4}+\cos\dfrac{3\pi}{4}=-\dfrac{1}{\sqrt{2}}-\dfrac{1}{\sqrt{2}}=-\sqrt{2}>0$

$f''\left(\dfrac{7\pi}{4}\right)=-\sin\dfrac{7\pi}{4}+\cos\dfrac{7\pi}{4}=\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}=\sqrt{2}>0$

Therefore, by second derivative test, $x=\dfrac{3\pi}{4}$ is appoint of local maxima and the local maximum value of f at $x=\dfrac{3\pi}{4}$ is

$f\left(\dfrac{3\pi}{4}\right)=\sin\dfrac{3\pi}{4}\cos\dfrac{3\pi}{4}=\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}=\sqrt{2}.$ However, $x=\dfrac{7\pi}{4}$ is a point of local minima and the local minimum value of f at $x=\dfrac{7\pi}{4}$ is $f\left(\dfrac{7\pi}{4}\right)=\sin\dfrac{7\pi}{4}-\cos\dfrac{7\pi}{4}=-\dfrac{1}{\sqrt{2}}-\dfrac{1}{\sqrt{2}}=-\sqrt{2}.$

(v) $f(x)=x^3-6x+9x+15$

$\therefore f^{'}(x) = 3x^2 - 12x + 9$

$f(x) = 0 \Rightarrow 3(x^2 - 4x + 3) = 0$

$\Rightarrow 3(x-1)(x-3) = 0$

$\Rightarrow x = 1, 3$

Now, $f^{''}$

$(x) = 6x - 12 = 6(x-2)$

$f^{''}(1) = 6(1-2) = -6 < 0$

$f^{''}(3) = 6(3-2) = 6 > 0$

Therefore, by second derivative test, $x = 1$ is a point of local maxima and the local maximum value of f at $x = 1$ is $f(1) = 1 - 6 + 9 + 15 = 19.$ However, $x = 3$ is a point of local minima and the local minimum value of f at $x = 3$ is $f(3) = 27 - 54 + 27 + 15 = 15.$

(vi) $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$

$\therefore g^{'}(x) = \frac{1}{2} - \frac{2}{x^2}$

Now,

$g^{'}(x) = 0$ gives $\frac{2}{x^2} = \frac{1}{2} \Rightarrow x^3 = 4 \Rightarrow x = \pm 2$

Since $x > 0$, we take $x = 2.$

Now,

$g"(x) = \frac{4}{x^3}$

$g"(2) = \frac{4}{2^3} = \frac{1}{2} > 0$

Therefore, by second derivative test, $x = 2$ is a point of local minima and the local minimum value of g at $x = 2$ is $g(2) = \frac{2}{2} + \frac{2}{2} = 1 + 1 = 2.$

(vii) $g(x) = \frac{1}{x^2 + 2}$

$\therefore g^{'}(x) = \frac{-(2x)}{(x^3 + 2)^2}$

$g^{'}(x) = 0 \Rightarrow \frac{-2x}{(x^3 + 2)^2} = 0 \Rightarrow x = 0$

Now , for values close to $x = 0$ and to the left of 0, $g^1(x) > 0.$ Also, for values close to $x = 0$ and to the right of $g^1(x) < 0.$

Therefore, by first derivative test $x = 0$ is a point of local maxima and the local maximum value of $g(0)$ is $\frac{1}{0+2} = \frac{1}{2}.$

(viii) $f(x)=x\sqrt{1-x}, x>0$

$$\therefore f'(x)=x\sqrt{1-x}+x\cdot\frac{1}{2\sqrt{1-x}}(-1)=\sqrt{1-x}-\frac{x}{2\sqrt{1-x}}$$

$$=\frac{2(1-x)-x}{2\sqrt{1-x}}=\frac{2-3x}{2\sqrt{1-x}}$$

$$f'(x)=0\Rightarrow\frac{2-3x}{2\sqrt{1-x}}=0\Rightarrow 2-3x=0\Rightarrow x=\frac{2}{3}$$

$$f''(x)=\frac{1}{2}\left[\frac{\sqrt{1-x}(-3)-(2-3x)\left(\frac{-1}{2\sqrt{1-x}}\right)}{1-x}\right]$$

$$=\frac{\sqrt{1-x}(-3)+2(2-3x)\left(\frac{1}{2\sqrt{1-x}}\right)}{2(1-x)}$$

$$=\frac{-6(1-x)+2(2-3x)}{4(1-x)^{\frac{3}{2}}}$$

$$=\frac{3x-4}{4(1-x)^{\frac{3}{2}}}$$

$$f'\left(\frac{2}{3}\right)=\frac{3\left(\frac{2}{3}\right)-4}{4\left(1-\frac{2}{3}\right)^{\frac{3}{2}}}=\frac{2-4}{4\left(\frac{1}{3}\right)^{\frac{3}{2}}}=\frac{-1}{2\left(\frac{1}{3}\right)^{\frac{3}{2}}}<0$$

Therefore, by second derivative test, $x=\frac{2}{3}$ is a point of local maxima and the local maximum value of f at $x=\frac{2}{3}$ is

$$f\left(\frac{2}{3}\right)=\frac{2}{3}\sqrt{1-\frac{2}{3}}=\frac{2}{3}\sqrt{\frac{1}{3}}=\frac{2}{3\sqrt{3}}=\frac{2\sqrt{3}}{9}.$$

Question 4:

Prove that the following functions do not have maxima or minima:

(i) $f(x)=e^x$ **(ii)** $g(x)=\log x$ **(iii)** $h(x)=x^3+x^2+x+1$

Solution 4:

(i) $f(x)=e^x$

$\therefore f'(x)=e^x$

Now, if $f'(x)=0$, then $e^x=0$. But the exponential function can never assume 0 for any value of x.

Therefore, there does not exist $c\in\mathbf{R}$ such that $f'(c)=0$.

Hence, function f does not have maxima or minima.

(ii) We have,

$g(x)=\log x$

$\therefore g'(x)=\frac{1}{x}$

Since $\log x$ is defined for a positive number x, $g'(x)>0$ for any x.

Therefore, there does not exist $c\in\mathbf{R}$ such that $g'(c)=0$.

Hence, function g does not have maxima or minima.

(iii) We have,

$h(x)=x^3+x^2+x+1$

$\therefore h'(x)=3x^2+2x+1$

Now,

Therefore, there does not exist $c\in\mathbf{R}$ such that $h'(c)=0$.

Hence, function h does not have maxima or minima.

Question 5:

Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(i) $f(x)=x^3, x\in[-2,2]$ **(ii)** $f(x)=\sin x+\cos x, x\in[0,\pi]$

(iii) $f(x)=4x-\frac{1}{2}x^2, x\in\left[-2,\frac{9}{2}\right]$ **(iv)** $f(x)=(x-1)^2+3, x\in[-3,1]$

Solution 5:

(i) The given function is $f(x)=x^3$.

$\therefore f'(x)=3x^2$

Now,

$f'(x)=0\Rightarrow x=0$

Then, we evaluate the value of f at critical point $x=0$ and at end points of the interval $[-2,2]$.

$f(0)=0$

$f(-2)=(-2)^3=-8$

$f(2)=(2)^3=8$

Hence, we can conclude that the absolute maximum value of f on $[-2,2]$ is 8 occurring at $x=-2$. Also, the absolute minimum value of f on $[-2,2]$ is -8 occurring at $x=-2$.

(ii) The given function is $f(x)=\sin x+\cos x$.

$\therefore f'(x)=\cos x-\sin x$

Now,

$f'(x)=0 \Rightarrow \sin x=\cos x \Rightarrow \tan x=1 \Rightarrow x=\frac{\pi}{4}$

Then, we evaluate the value of f at critical point $x=\frac{\pi}{4}$ and at the end points of the interval $[0,\pi]$.

$f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$

$f(0)=\sin 0+\cos 0=0+1=1$

$f(\pi)=\sin\pi+\cos\pi=0-1=1$

Hence, we can conclude that the absolute maximum value of f on $[0,\pi]$ is $\sqrt{2}$ occurring at $x=\frac{\pi}{4}$ and the absolute minimum value of f on $[0,\pi]$ is -1 occurring $x=\pi$.

(iii) The given function is $f(x)=4x-\frac{1}{2}x^2$

$\therefore f'(x)=4x-\frac{1}{2}(2x)=4-x$

Now,

$f'(x)=0 \Rightarrow x=4$

Then, we evaluate the value of f at critical point $x=4$, and at the end points of the interval $\left[-2,\frac{9}{2}\right]$.

$f(4)=16-\frac{1}{2}(16)=16-8=8$

$f(-2)=-8-\frac{1}{2}(4)=-8-2=-10$

$f\left(\frac{9}{2}\right)=4\left(\frac{9}{2}\right)-\frac{1}{2}\left(\frac{9}{2}\right)^2=18-\frac{81}{8}=18-10.125=7.875$

Hence, we can conclude that the absolute maximum value of f on $\left[-2,\frac{9}{2}\right]$ is 8 occurring at $x=4$ and the absolute minimum value of f on $\left[-2,\frac{9}{2}\right]$ is -10 occurring at $x=-2$.

(iv) The given function is $f(x)=(x-1)^2+3$.

$\therefore f'(x)=2(x-1)$

Now,

$f'(x)=0 \Rightarrow 2(x-1)=0,\ x=1$

Then, we evaluate the value of f at critical point $x=1$ and at the end points of the interval $[-3,1]$.

$f(1)=(1-1)^2+3=0+3=3$

$f(-3)=(-3-1)^2+3=16+3=19$

Hence, we can conclude that the absolute maximum value of f on $[-3,1]$ is 19 occurring at $x=-3$ and the minimum value of f on $[-3,1]$ is occurring at $x=1$.

Question 6:

Find the maximum profit that a company can make, if the profit function is given by $p(x)=41-24x-18x^2$

Solution 6:

The profit function is given as $p(x)=41-24x-18x^2$.

$\therefore p'(x)=-24-36x$

$p''(x)=-36$

Now,

$p'(x)=0 \Rightarrow x=\frac{-24}{36}=-\frac{2}{3}$

Also,

$p''\left(\frac{-2}{3}\right)=-36<0$

By second derivatives test, $x=-\frac{2}{3}$ is the point of local maximum of p.

$\therefore$ Maximum profit = p(-2/3)

$=41-24\left(-\frac{2}{3}\right)-18\left(-\frac{2}{3}\right)^2$

$=41+16-8$

$=49$

Hence, the maximum profit that the company can make is 49 units.

Question 7:

Find the intervals in which the function f given by $f(x)=x^3+\frac{1}{x^3}, x\neq 0$ is

(i) Increasing **(ii)** Decreasing

Solution 7:

$f(x)=x^3+\frac{1}{x^3}$

$\therefore f'(x) = 3x^2 - \frac{3}{x^4} = \frac{3x^6 - 3}{x^4}$

Then, $f'(x) = 0 \Rightarrow 3x^6 - 3 = 0 \Rightarrow x^6 = 1 \Rightarrow x = \pm 1$

Now, the points $x = 1$ and $x = -1$ divided the real line into three disjoint intervals i.e., $(-\infty, -1), (-1, 1)$ and $(1, \infty)$.

In intervals $(-\infty, -1)$ and $(1, \infty)$ i.e., when $x < -1$ and $x > 1$, $f'(x) > 0$.

Thus, when $x < -1$ and $x > 1$, f is increasing.

In intervals $(-1, 1)$ i.e., when $-1 < x < 1$, $f'(x) < 0$.

Thus, when $-1 < x < 1$, f is decreasing.

Question 8:

At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain, its maximum value?

Solution 8:

Let $f(x) = \sin 2x$.

$\therefore f'(x) = 2\cos 2x$

Now,

$f'(x) = 0 \Rightarrow \cos 2x = 0$

$\Rightarrow 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Then, we evaluate the values of f at critical points $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ and at the end points of the interval $[0, 2\pi]$.

$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{2} = 1,\ f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{2} = -1$

$f\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{2} = 1,\ f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{2} = -1$

$f(0) = \sin 0 = 0,\ f(2\pi) = \sin 2\pi = 0$

Hence, we can conclude that the absolute maximum value of f $[0, 2\pi]$ is occurring at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Question 9:

What is the maximum value of the function $\sin x + \cos x$?

Solution 9:

Let $f(x)=\sin x+\cos x$

$\therefore f'(x)=\cos x-\sin x$

$f'(x)=0 \Rightarrow \sin x=\cos x \Rightarrow \tan x=1 \Rightarrow x=\frac{\pi}{4},\frac{5\pi}{4}...,$

$f'(x)=-\sin x-\cos x=-(\sin x+\cos x)$

Now, $f''(x)$ will be negative when $(\sin x+\cos x)$ is positive i.e., when $\sin x$ and $\cos x$ are both positive. Also, we know that $\sin x$ and $\cos x$ both are positive in the first quadrant. Then, $f''(x)$ will be negative when $x \in \left(0,\frac{\pi}{2}\right)$.

Thus, we consider $x=\frac{\pi}{4}$.

$$f''\left(\frac{\pi}{4}\right)=-\left(\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\right)=-\left(\frac{2}{\sqrt{2}}\right)=-\sqrt{2}<0$$

By second derivative test, f will be the maximum at $x=\frac{\pi}{4}$ and the maximum value of f is

$$f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}\text{x}\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}.$$

Question 10:

Find the maximum value of $2x^3-24x+107$ in the interval $[1,3]$. Find the maximum value of the same function in $[-3,-1]$.

Solution 10:

Let $f(x)=2x^3-24x+107$

$\therefore f'(x)=6x^2-24=6(x^2-4)$

Now,

$f'(x)=0 \Rightarrow 6(x^2-4)=0 \Rightarrow x^2=4 \Rightarrow x=\pm 2$

We first consider the interval $[1,3]$.

Then, we evalutat the value of f at the critical point $x \in [1,3]$ and at the end points of the interval [1, 3]

$f(2)=2(8)-24(2)+107=16-48+107=75$

$f(1)=2(1)-24(1)+107=2-24+107=85$

$f(3)=2(27)-24(3)+107=54-72+107=89$

Hence, the absolute maximum value of $f(x)$ in the interval $[1,3]$ is 89 occurring at $x=3$.

Next, we consider the interval $[-3,-1]$.

Evaluate the value of f at the critical pont $x=-2\in[-3,-1]$ and at the end points of the interval [1, 3].

$f(-3)=2(-27)-24(-3)+107=54+72+107=125$

$f(-1)=2(-1)-24(-1)+107=2+24+107=129$

$f(-2)=2(-8)-24(-2)+107=-16+48+107=139$

Hence, the absolute maximum value of $f(x)$ in the interval $[-3,-1]$ is 139 occurring at $x=-2$.

Question 11:

It is given that at $x=1$, the function x^4-62x^2+ax+9 attains its maximum value, on the interval $[0,2]$. Find the value of a.

Solution 11:

Let $f(x)=x^4-62x^2+ax+9$.

$\therefore f'(x)=4x^2-124x+a$

It is given that function f attains its maximum value on the interval $[0,2]$ at $x=1$.

$\therefore f'(1)=0$

$\Rightarrow 4-124+a=0$

$\Rightarrow a=120$

Hence, the value of a is 120.

Question 12:

Find the maximum and minimum values of $x+\sin 2x$ on $[0,2\pi]$.

Solution 12:

Let $f(x)=x+\sin 2x$.

$\therefore f'(x)=1+2\cos 2x$

Now, $f'(x)=0 \Rightarrow \cos 2x=-\frac{1}{2}=-\cos\frac{\pi}{3}=\cos\left(\pi-\frac{\pi}{3}\right)=\cos\frac{2\pi}{3}$

$2x=2\pi\pm\frac{2\pi}{3}, n\in\mathbf{Z}$

$\Rightarrow x=\pi\pm\frac{\pi}{3}, n\in\mathbf{Z}$

$\Rightarrow x=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}\in[0,2\pi]$

Then, we evaluate the value of f at critical points $\Rightarrow x=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$ and the end points of the interval $[0,2\pi]$.

$$f\left(\frac{\pi}{3}\right)=\frac{\pi}{3}+\sin\frac{2\pi}{3}=\frac{\pi}{3}+\frac{\sqrt{3}}{2}$$

$$f\left(\frac{2\pi}{3}\right)=\frac{2\pi}{3}+\sin\frac{4\pi}{3}=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$$

$$f\left(\frac{4\pi}{3}\right)=\frac{4\pi}{3}+\sin\frac{8\pi}{3}=\frac{4\pi}{3}+\frac{\sqrt{3}}{2}$$

$$f\left(\frac{5\pi}{3}\right)=\frac{5\pi}{3}+\sin\frac{10\pi}{3}=\frac{5\pi}{3}-\frac{\sqrt{3}}{2}$$

$$f(0)=0+\sin 0=0$$

$$f(2\pi)=2\pi+\sin 4\pi=2\pi+0=2\pi$$

Hence, we can conclude that the absolute maximum value of $f(x)$ in the interval $[0,2\pi]$ is 2π occurring at $x=2\pi$ and the absolute minimum value of $f(x)$ in the interval $[0,2\pi]$ is 0 occurring at $x=0$.

Question 13:

Find two numbers whose sum is 24 and whose product is as large as possible.

Solution 13:

Let one number be x. Then, the other number is $(24-x)$.

Let $p(x)$ denote the product of the two numbers. Thus, we have:

$$P(x)=x(24-x)=24x-x^2$$

$$\therefore P'(x)=24-2x$$

$$P''(x)=-2$$

Now,

$$P'(x)=0 \Rightarrow x=12$$

Also,

$$P''(12)=-2<0$$

By second derivative test, $x=12$ is the point of local maxima of P. Hence, the product of the numbers is the maximum when the numbers are 12 and $24-12=12$.

Question 14:

Find two positive numbers x and y such that $x+y=60$ and xy^3 is maximum.

Solution 14:

The two numbers are x and y such that $x+y=60$.

$$y=60-x$$

Let $f(x)=xy^3$

$\Rightarrow f(x) = x(60-x)^3$

$\therefore f'(x) = (60+x)^3 - 3x(60-x)^2$

$= (60+x)^3[60-x-3x]$

$= (60+x)^3(60-4x)$

And, $f''(x) = -2(60-x)(60-4x) - 4(60-x)^2$

$= -2(60-x)[60-4x+2(60-x)]$

$= -2(60-x)(180-6x)$

$= -12(60-x)(30-x)$

Now, $f'(x) = 0 \Rightarrow x = 60$ or $x = 15$

When $x = 60$, $f''(x) = 0.$

When, $x = 15$, $f''(x) = -12(60-15)(30-15) = 12 \times 45 \times 15 < 0.$

By second derivative test, $x = 15$ is a point of local maxima of f. Thus , function xy^3 is maximum when $x = 15$ and $y = 60 - 15 = 45.$

Hence, the required numbers are 15 and 45.

Question 15:

Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum.

Solution 15:

Let one number be x. Then, the other number is $y = (35-x).$

Let $p(x) = x^2y^5$. Then, we have:

$P(x) = x^2(35-x)^5$

$\therefore P'(x) = 2x(35-x)^5 - 5x^2(35-x)^4$

$= x(35-x)^4[2(35-x) - 5x]$

$= x(35-x)^4(70-7x)$

$= 7x(35-x)^4(10-x)$

And, $P''(x) = 7(35-x)^4(10-x) + 7x\left[-(35-5)^4 - 4(35-x)^3(10-x)\right]$

$= 7(35-x)^4(10-x) - 7x(35-x)^4 - 28x(35-x)^3(10-x)$

$= 7(35-x)^3[(35-x)(10-x) - x(35-x) - 4x(10-x)]$

$= 7(35-x)^3\left[350 - 45x + x^2 - 35x + x^2 - 40x + 4x^2\right]$

$= 7(35-x)^3(6x^2 - 120x + 350)$

Now, $P'(x)=0 \Rightarrow x=0, x=35, x=10$

When, $x=35$, $f'(x)=f(x)=0$ and $y=35-35=0$. This will make the product x^2y^5 equal to 0.

When $x=0, y=35-0=35$ and the product x^2y^2 will be 0.

$x=0$ and $x=35$ cannot be the possible values of x.

When $x=10$, we have:

$P''(x)=7(35-10)^3(6\text{x}100\text{-}120\text{x}10\text{+}350)$

$=7(25)^3(-250)<0$

By second derivative test, $P(x)$ will be the maximum when $x=10$ and $y=35-10=25$.

Hence, the required numbers are 10 and 25.

Question 16:

Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

Solution 16:

Let one number be x. Then, the other number is $(16-x)$.

Let the sum of the cubes of these numbers be denoted by $S(x)$. Then,

$S(x)=x^3+(16-x)^3$

$\therefore S'(x)=3x^2-3(16-x)^2, S''(x)=6x+6(16-x)$

Now, $S'(x)=0 \Rightarrow 3x^2-3(16-x)^2=0$

$\Rightarrow x^2-(16-x)^2=0$

$\Rightarrow x^2-256-x^2+32x=0$

$\Rightarrow x=\frac{256}{32}=8$

Now, $S''(8)=6(8)+6(16-8)=48+48=96>0$

By second derivative test, $x = 8$ is the point of local minima of S.

Hence, the sum of the cubes of the numbers is the minimum when the numbers are 8 and 16 –8 = 8.

Question 17:

A square piece of tin od side 18 cm is to made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Solution 17:

Let the side of the square to be cut off be x cm.Then, the length and the breath of the box will be $(18 – 2x)$ cm each and the height of the box is x cm

Therefore, the volume V(x) of the box is given by,

$V(x) = x(18-2x)^2$

$\therefore V'(x) = (18-2x)^2 - 4x(18-2x)$

$= (18-2x)[18-2x-4x]$

$= (18-2x)[18-6x]$

$= 6\times 2(9-x)(3-x)$

$= 12(9-x)(3-x)$

And, $V'(x) = 12[-(9-x)-(3-x)]$

$= -12(9-x+3-x)$

$= -12(12-2x)$

$= -24(6-x)$

Now, $v'(x) = 0 \Rightarrow x = 9$ or $x = 3$

If x = 9, then the length and the breadth will become 0.

$\therefore x \neq 9$

$\Rightarrow x = 3$

Now, $V''(3) = -24(6-3) = -72 < 0$

$\therefore$ By second derivative test, x = 3 is the point of maxima of *V*.

Hence, if we remove a square of side 3 cm from each corner of the square tin and make a box from the remaining sheet, then the volume of the box obtained is the largest possible.

Question 18:

A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Solution 18:

Let the side of the square to be cut be *x* cm. Then, the height of the box is *x*, the length is 45 – 2*x*, ad the breadth is 24 – 2*x*

Therefore, the volume *V*(*x*) of the box is given by,

$V(x) = x(45 - 2x)(24 - 2x)$

$= x(1080 - 90x - 48x + 4x^2)$

$= 4x^3 - 138x^2 + 1080x$

$V'(x) = 12x^2 - 276 + 1080$

$= 12(x^2 - 23x + 90)$

$= 12(x - 18)(x - 5)$

$V''(x) = 24x - 276 = 12(2x - 23)$

Now, $V'(x) = 0 \Rightarrow x = 18$ and $x = 5$

It is not possible to cut off a square of side 18 cm from each corner of the rectangular sheet, Thus *x* cannot be equal to 18.

x = 5

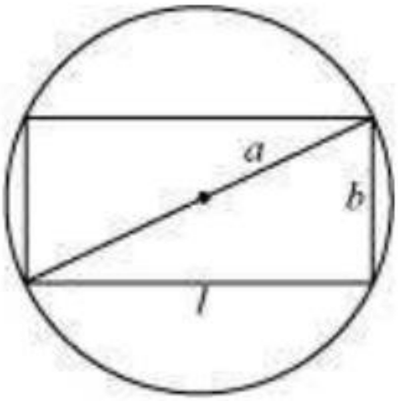
Now, $V''(5) = 12(10 - 23) = 12(-13) = -156 < 0$

$\therefore$ By second derivative test x = 5 is the point of maxima.

Hence, the side of the square to be cut off to make the volume of the box maximum possible is 5 cm.

Question 19:
Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

Solution 19:
Let a rectangle of length l and breadth b be inscribed in the given circle of radius a.
Then, the diagonal passes through the center and is of length $2a$ cm.



Now, by applying the Pythagoras theorem, we have:

$(2a)^2 = l^2 + b^2$

$\Rightarrow b^2 = 4a^2 - l^2$

$\Rightarrow b = \sqrt{4a^2 - l^2}$

Area of the rectangle, $A = l\sqrt{4a^2 - l^2}$

$\therefore \frac{dA}{dl} = \sqrt{4a^2 - l^2} + l\frac{1}{2\sqrt{4a^2 - l^2}}(-2l) = \sqrt{4a^2 - l^2} + \frac{1^2}{\sqrt{4a^2 - l^2}}$

$= \frac{4a^2 - 2l^2}{\sqrt{4a^2 - l^2}}$

$\frac{d^2A}{dl^2} = \frac{\sqrt{4a^2 - l^2}(-4l) - (4a^2 - 2l^2)\frac{(-2l)}{2\sqrt{4a^2 - l^2}}}{(4a^2 - l^2)}$

$= \frac{(4a^2 - l^2)(-4l) + 1(4a^2 - 2l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$

$= \frac{-12a^2 l + 2l^3}{(4a^2 - l^2)^{\frac{3}{2}}} = \frac{-2l(6a^2 - l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$

Now, $\frac{dA}{dl} = 0$ gives $4a^2 = 2l^2 \Rightarrow l = \sqrt{2}a$

$\Rightarrow b = \sqrt{4a^2 - 2a^2} = \sqrt{2a^2} = \sqrt{2}a$

Now, when $l = \sqrt{2}a$,

$$\frac{d^2A}{dl^2}=\frac{-2(\sqrt{2a})(-6a^2-2a^2)}{2\sqrt{2a^3}}=\frac{-8\sqrt{2a^3}}{2\sqrt{2a^3}}=-4<0$$

$\therefore$ By the second derivative test, when $l=\sqrt{2}a$, then the area of the rectangle is the maximum.

Since $l=b=\sqrt{2}a$, the rectangle is a square.

Hence, it has been proved that of all the rectangles inscribed in the given fixed circle, the square has the maximum area.

Question 20:

Show that the right circular cylinder of given surface and maximum volume is such that is heights is equal to the diameter of the base.

Solution 20:

Let r and h be the radius and height of the cylinder respectively.

Then, the surface area (S) of the cylinder is given by,

$$S=2\pi r^2+2\pi rh$$

$$\Rightarrow h=\frac{S-2\pi r^2}{2\pi r}$$

$$=\frac{S}{2\pi}\left(\frac{1}{r}\right)-r$$

Let V be the volume of the cylinder. Then,

$$V=\pi r^2h=\pi r^2\left[\frac{S}{2\pi}\left(\frac{1}{r}\right)-r\right]=\frac{Sr}{2}=-\pi r^3=\frac{S}{2\pi}\left(\frac{1}{r}\right)-r$$

Then, $\frac{dV}{dr}=\frac{S}{2}-3\pi r^2$, $\frac{d^2V}{dr^2}=-6\pi r$

Now, $\frac{dV}{dr}=0\Rightarrow\frac{S}{2}=3\pi r^2\Rightarrow r^2=\frac{S}{6\pi}$

When $r^2=\frac{S}{6\pi}$, then $\frac{d^2V}{dr^2}=-6\pi\left(\sqrt{\frac{S}{6\pi}}\right)<0$.

By Second derivative test, the volume is the maximum when $r^2=\frac{S}{6\pi}$.

Now, when $r^2=\frac{S}{6\pi}$, then $h=\frac{6\pi r^2}{2\pi}\left(\frac{1}{r}\right)-r=3r-r=2r$

Hence, the volume is the maximum when the height is twice the radius i.e., when the height is equal to the diameter.

Question 21:

Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimeters, find the dimensions of the can which has the minimum surface area?

Solution 21:

Let *r* and *h* be the radius and height of the cylinder respectively.
Then, volume (*V*) of the cylinder is given by,
$V = \pi r^2 h = 100$ (given)

$\therefore h = \dfrac{100}{\pi r^2}$

Surface area (*S*) of the cylinder is given by,

$$S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + \frac{200}{r}$$

$$\frac{dS}{dr} = 4\pi r - \frac{200}{r^2},\ \frac{d^2S}{dr^2} = 4\pi + \frac{400}{r^3}$$

$$\frac{dS}{dr} = 0 \Rightarrow 4\pi r = \frac{200}{r^2}$$

$$\Rightarrow r^3 = \frac{200}{4\pi} = \frac{50}{\pi}$$

$$\Rightarrow r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

Now, it is observed that when $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$, $\frac{d^2S}{dr^2} > 0$.

By second derivative test, the surface area is the minimum when the radius of the cylinder is $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

When r = $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$, h = $2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

Hence the required dimensions of the can which has them minimum surface area is given $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ and height = $2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

Question 22:

A Wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the circle is minimum?

Solution 22:

Let a piece of length *l* be cut from the given wire to make a square.
Then, the other piece of wire to be made into a circle is of length (28 – *l*)*m*.

Now, side of square= $\dfrac{l}{4}$

Let r be the radius of the circle. Then, $2\pi r = 28 - l \Rightarrow r = \frac{1}{2\pi}(28-l)$.

The combined areas of the square and the circle (A) is given by,

A = (side of the square) $\pi^2 + r^2$

$$= \frac{l^2}{16} + \pi\left[\frac{1}{2\pi}(28-l)\right]^2$$

$$= \frac{l^2}{16} + \frac{1}{4\pi}(28-l)^2$$

$$\therefore \frac{dA}{dl} = \frac{2l}{16} + \frac{2}{4\pi}(28-l)(-1) = \frac{l}{8} - \frac{1}{2\pi}(28-l)$$

$$\frac{d^2A}{dl^2} = \frac{l}{8} + \frac{1}{2\pi} > 0$$

Now, $\frac{dA}{dl} = 0 \Rightarrow \frac{l}{8} - \frac{1}{2\pi}(28-l) = 0$

$$\Rightarrow \frac{\pi l - 4(28-l)}{8\pi} = 0$$

$$(\pi + 4)l - 112 = 0$$

$$\Rightarrow l = \frac{112}{\pi + 4}$$

Thus, when $l = \frac{112}{\pi+4}$, $\frac{d^2A}{dl^2} > 0$

$\therefore$ By second derivative test, the area (A) is the minimum when $l = \frac{112}{\pi+4}$.

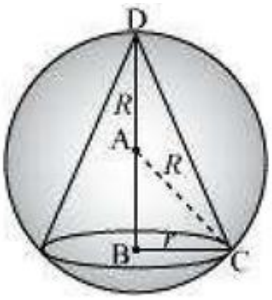
Hence, the combined area is the minimum when the length of the wire in making the square is $\frac{112}{\pi+4}$ cm while the length of the wire in making the circle is $28 - \frac{112}{\pi+4} = \frac{28\pi}{\pi+4}$ cm.

Question 23:

Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

Solution 23:

Let r and h be the radius and height of the cone respectively inscribed in a sphere of radius R.



Let V be the volume of the cone.

Then, $V = \frac{1}{3}\pi r^2 h$

Height of the cone is given by,

$h = R + AB = R + \sqrt{R^2 - r^2}$ [ABC is a right triangle]

$V = \frac{1}{3}\pi r^2 (R + \sqrt{R^2 - r^2})$

$= \frac{1}{3}\pi r^2 R + \frac{1}{3}\pi r^2 \sqrt{R^2 - r^2}$

$\frac{dV}{dr} = \frac{2}{3}\pi r R + \frac{2}{3}\pi r \sqrt{R^2 - r^2} + \frac{1}{3}\pi r^2 \cdot \frac{(-2r)}{2\sqrt{R^2 - r^2}}$

$= \frac{2}{3}\pi r R + \frac{2}{3}\pi r \sqrt{R^2 - r^2} - \frac{1}{3}\pi \cdot \frac{r^3}{\sqrt{R^2 - r^2}}$

$= \frac{2}{3}\pi r R + \frac{2\pi r (R^2 - r^2) - \pi r^3}{3\sqrt{R^2 - r^2}}$

$= \frac{2}{3}\pi r R + \frac{2\pi r R^2 - 3\pi r^3}{3\sqrt{R^2 - r^2}}$

$$\frac{d^2V}{dr^2} = \frac{2\pi R}{3} + \frac{3\sqrt{R^2 - r^2}(2\pi R^2 - 9\pi r^2) - (2\pi r R^2 - 3\pi r^3)\cdot \frac{(-2r)}{2\sqrt{R^2 - r^2}}}{9(R^2 - r^2)}$$

$$= \frac{2}{3}\pi r R + \frac{9(R^2 - r^2)(2\pi R^2 - 9\pi r^2) + 2\pi r^2 R^2 + 3\pi r^4}{27(R^2 - r^2)^{\frac{3}{2}}}$$

Now, $\frac{dV}{dr} = 0 \Rightarrow \pi \frac{2}{3} r R = \frac{3\pi r^3 - 2\pi R^2}{3\sqrt{R^2 - r^2}}$

$\Rightarrow 2R = \frac{3\pi r^3 - 2\pi R^2}{\sqrt{R^2 - r^2}} = 2R\sqrt{R^2 - r^2} = 3r^2 - 2R^2$

$\Rightarrow 4R^2 (R^2 - r^2) = (3r^2 - 2R^2)^2$

$\Rightarrow 4R^4 - 4R^2 r^2 = 9r^4 + 4R^4 - 12r^2 R^2$

$\Rightarrow 9r^4 = 8R^2 r^2$

$\Rightarrow r^2 = \frac{8}{9}R^2$

when $r^2 = \frac{8}{9}R^2$, then $\frac{d^2V}{dr^2} < 0$

By second derivative test, the volume of the cone is the maximum when $r^2 = \frac{8}{9}R^2$.

When $r^2 = \frac{8}{9}R^2$, $h = R + \sqrt{R^2 - \frac{8}{9}R^2} = R + \sqrt{\frac{1}{9}R^2} = R + \frac{R}{3} = \frac{4}{3}R$

Therefore,

$$=\frac{1}{3}\pi\left(\frac{8}{9}R^2\right)\left(\frac{4}{3}R\right)$$

$$=\frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$$

$$=\frac{8}{27}\times \text{(Volume of the sphere)}$$

Hence, the volume of the largest cone that can be inscribed in the sphere is $\frac{8}{27}$ the volume of the sphere.

Question 24:

Show that he right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

Solution 24:

Let *r* and *h* be the radius and the height (altitude) of the cone respectively.

Then, the volume (*V*) of the cone is given as:

$$V=\frac{1}{3\pi}\pi r^2 h \Rightarrow h=\frac{3V}{r^2}$$

The surface area (*S*) of the cone is given by,

$S = \pi rl$ (Where *l* is the slant height)

$$=\pi r\sqrt{r^2=h^2}$$

$$=\pi r\sqrt{r^2+\sqrt{\frac{9V}{\pi^2 r^4}}^{\,2}}=\pi\frac{r\sqrt{9^2r^2+V^2}}{\pi r^2}$$

$$=\frac{1}{r}\sqrt{\pi^2 r^6+9V^2}$$

$$\frac{dS}{dr}=\frac{r\cdot\frac{6\pi^2r^5}{2\sqrt{\pi^2r^6+9V^2}}-\sqrt{\pi^2r^6+9V^2}}{r^2}$$

$$=\frac{3\pi^2r^6-\pi^2r^6-9V^2}{r^2\sqrt{\pi^2r^6+9V^2}}$$

$$=\frac{2\pi^2r^6-9V^2}{r^2\sqrt{\pi^2r^6+9V^2}}$$

Now, $\frac{dS}{dr}=0\Rightarrow 2\pi^2r^6=9V^2\Rightarrow r^6=\frac{9V^2}{2\pi^2}$

Thus, it can be easily verified that when $r^6=\frac{9V^2}{2\pi^2}$, $\frac{d^2S}{dr^2}>0$.

By second derivative test, the surface area of the cone is the least when $r^6=\frac{9V^2}{2\pi^2}$

When, $r^6 = \frac{9V^2}{2\pi^2}$, $h = \frac{3V}{\pi r^2} = \frac{3V}{\pi r^2}\left(\frac{2\pi^2 r^6}{9}\right)^{\frac{1}{2}} = \frac{3}{\pi r^2}\frac{\sqrt{2}\pi r^3}{3} = \sqrt{2}r$

Hence, for a given volume, the right circular cone of the least curved surface has an altitude equal to $\sqrt{2}$ times the radius of the base.

Question 25:

Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

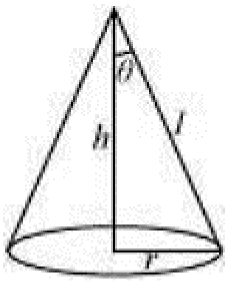
Solution 25:

Let θ be the semi-vertical angle of the cone.

It is clear that $\theta \in \left[0, \frac{\pi}{2}\right]$.

Let r, h and l be the radius, height, and the slant height of the cone respectively.

The slant height of the cone is given as constant.



Now $r = l\sin\theta$ and $h = l\cos\theta$

The volume (V) of the cone is given by,

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi(l^2\sin^2\theta)(l\cos\theta)$$

$$= \frac{1}{3}\pi l^3\sin^2\theta\cos\theta$$

$$\therefore \frac{dV}{d\theta} = \frac{l^3\pi}{3}[\sin^2\theta(-\sin\theta) + \cos\theta(2\sin\theta\cos\theta)]$$

$$= \frac{l^3\pi}{3}[-\sin^3 + 2\sin\theta\cos^2\theta]$$

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}[-3\sin^2\theta\cos\theta + 2\cos^3\theta - 4\sin^2\theta\cos\theta]$$

$$= \frac{l^3\pi}{3}[2\cos^2\theta - 7\sin^2\theta\cos\theta]$$

Now, $\frac{dV}{d\theta} = 0$

$\Rightarrow \sin^3\theta = 2\sin\theta\cos^2\theta$

$\Rightarrow \tan^2\theta = 2$

$\Rightarrow \tan\theta = \sqrt{2}$

$\Rightarrow \theta = \tan^{-1}\sqrt{2}$

Now, when $\theta = \tan^{-1}\sqrt{2}$, then $\tan^2\theta = 2$ or $\sin^2\theta = 2\cos^2\theta$

Then, we have:

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}[2\cos^3\theta - 14\cos^3\theta] = -4\pi l^3\cos^3\theta < 0 \quad \text{for } \theta \in \left[0, \frac{\pi}{2}\right]$$

By second derivative test, the volume (V) is the maximum when $\theta = \tan^{-1}\sqrt{2}$.

Hence, for a given slant height, the semi-vertical angle of the cone of the maximum volume is $\tan^{-1}\sqrt{2}$

Question 26:

The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is

(A) $(2\sqrt{2}, 4)$ **(B)** $(2\sqrt{2}, 0)$ **(C)** (0, 0) **(D)** (2, 2)

Solution 26:

The given curve is $x^2 = 2y$.

For each value of *x*, the position of the point will be $\left(x, \frac{x^2}{2}\right)$.

The distance d(x) between the points $\left(x, \frac{x^2}{2}\right)$ and (0, 5) is given by,

$$d(x) = \sqrt{(x-0)^2 + \left(\frac{x^2}{2} - 5\right)^2} = \sqrt{x^2 + \frac{x^4}{4} + 25 - 5x^2} = \sqrt{\frac{x^4}{4} - 4x^2 + 25}$$

$$\therefore d'(x) = \frac{(x^3 - 8x)}{2\sqrt{\frac{x^4}{4} - 4x^2 + 25}} = \frac{(x^3 - 8x)}{\sqrt{x^4 - 16x^2 + 100}}$$

Now, $d'(x) = 0 \Rightarrow x^3 - 8x = 0$

$\Rightarrow x(x^2 - 8) = 0$

$\Rightarrow x = 0, \pm 2\sqrt{2}$

And $$d''(x) = \frac{\sqrt{x^2 - 16x^2 + 100}(3x^2 - 8) - (x^3 - 8x)\frac{4x^3 - 32x}{2\sqrt{x^4 - 16x^2 + 100}}}{(x^2 - 16x^2 + 100)}$$

$$= \frac{(x^4 - 16x^2 + 100)(3x^2 - 8) - 2(x^3 - 8x)(x^3 - 8x)}{(x^4 - 16x^2 + 100)^{\frac{3}{2}}}$$

$$= \frac{(x^4 - 16x^2 + 100)(3x^2 - 8) - 2(x^3 - 8x)^2}{(x^4 - 16x^2 + 100)^{\frac{3}{2}}}$$

When, $x = 0$ then $d''(x) = \frac{36(-8)}{6^3} < 0$

When, $x = \pm 2\sqrt{2},\ d(x) > 0$

By second derivative test, d(x) is the minimum at $x = \pm 2\sqrt{2}$.

When, $x = \pm 2\sqrt{2},\ y = \frac{(2\sqrt{2})^2}{2} = 4$.

Hence, the point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is $(\pm 2\sqrt{2},\ 4)$.

The correct answer is **A.**

Question 27:

For all real values of *x*, the minimum value of $\frac{1 - x + x^2}{1 + x + x^2}$ is

(A) 0 **(B)** 1 **(C)** 3 **(D)** $\frac{1}{3}$

Solution 27:

Let $f(x) = \frac{1 - x + x^2}{1 + x + x^2}$

$$f'(x) = \frac{(1 + x + x^2)(-1 + 2x) - (1 - x + x^2)(1 + 2x)}{(1 + x + x^2)^2}$$

$$= \frac{2x^2 - 2}{(1 + x + x^2)^2} = \frac{2(x^2 - 1)}{(1 + x + x^2)^2}$$

$\therefore\ f'(x) = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

Now, $f''(x) = \frac{2[(1 + x + x^2)(2x) - (x^2 - 1)(2)(1 + x + x^2)(1 + 2x)]}{(1 + x + x^2)^4}$

$$= \frac{4(1 + x + x^2)[(1 + x + x^2)x - (x^2 - 1)(1 + 2x)]}{(1 + x + x^2)^4}$$

$$= 4\frac{[x + x^2 + x^3 - x^2 - 2x^3 + 1 + 2x]}{(1 + x + x^2)^3}$$

$$= 4\frac{(1 + 3x - x^3)}{(1 + x + x^2)^3}$$

And, $f''(1) = \frac{4(1 + 3 - 1)}{(1 + 1 + 1)^3} = \frac{4(3)}{(3)^3} = \frac{4}{9} > 0$

And, $f''(-1) = \frac{4(1 - 3 + 1)}{(1 + 1 + 1)^3} = 4(-1) = 4 < 0$

By second derivative test *f* is the minimum at x = 1 and the minimum value is given by

$$f(1)=\frac{1-1+1}{1+1+1}=\frac{1}{3}$$

The correct answer is **D.**

Question 28:

The maximum value of $[x(x+1)+1]^{\frac{1}{3}}$, $0\le x\le 1$ is

(A) $\left(\frac{1}{3}\right)^{\frac{1}{3}}$ **(B)** $\frac{1}{2}$ **(C) 1** **(D) 0**

Solution 28:

Let $f(x)=[x(x+1)+1]^{\frac{1}{3}}$.

$$\therefore\ f'(x)=\frac{2x+1}{3[x(x+1)+1]^{\frac{2}{3}}}$$

Now, $f(x)=0\Rightarrow x=-\frac{1}{2}$

But, x = -1/2 is not part of the interval [0,1]

Then, we evaluate the value of *f* at the end points of the interval [0, 1] {i.e., at x = 0 and x =1}.

$$f(0)=[0(0-1)+1]^{\frac{1}{3}}=1$$

$$f(0)=[1(1-1)+1]^{\frac{1}{3}}=1$$

Hence, we can conclude that the maximum value of *f* in the interval [0, 1] is 1.

The correct answer is **C.**

Miscellaneous Exercise

Question 1:

Using differentials, find the approximate value of each of the following.

(a) $\left(\frac{17}{81}\right)^{\frac{1}{4}}$

(b) $(33)^{\frac{1}{5}}$

Solution 1:

(a) Consider $y=x^{\frac{1}{4}}$. Let $x=\frac{16}{81}$ and $\Delta x=\frac{1}{81}$.

Then, $\Delta y=(x+\Delta x)^{\frac{1}{4}}-x^{\frac{1}{4}}$

$=\left(\frac{17}{81}\right)^{\frac{1}{4}}-\left(\frac{16}{81}\right)^{\frac{1}{4}}$

$=\left(\frac{17}{81}\right)^{\frac{1}{4}}-\frac{2}{3}$

$=\left(\frac{17}{81}\right)^{\frac{1}{4}}-\frac{2}{3}+\Delta y$

Now, dy is approximately equal to Δy and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left(\text{as } y=x^{\frac{1}{4}}\right)$$

$$=\frac{1}{4\left(\frac{16}{81}\right)^{\frac{3}{4}}}\left(\frac{1}{81}\right)=\frac{27}{4\times8}\times\frac{1}{81}=\frac{1}{32\times3}=\frac{1}{96}=0.010$$

Hence, the approximate value of $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ is

$\frac{2}{3}+0.010=0.667+0.010$

= 0.677.

(b) Consider $y=x^{\frac{1}{5}}$. Let x =32 and $\Delta x=1$.

Then, $\Delta y=(x+\Delta x)^{\frac{1}{5}}-x^{\frac{1}{5}}=(33)^{\frac{1}{5}}-(32)^{\frac{1}{5}}=(33)^{-\frac{1}{5}}-\frac{1}{2}$

$\therefore (33)^{\frac{1}{5}}=\frac{1}{2}+\Delta y$

Now, dy is approximately equal to Δy and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{-1}{5(x)^{\frac{6}{5}}}(\Delta x) \qquad \left(\text{as } y=x^{\frac{1}{5}}\right)$$

$$=\frac{1}{5(2)^{6}}(1)=\frac{1}{320}=-0.003$$

Hence, the approximate value of $(33)^{\frac{1}{5}}$ is $\frac{1}{2}+(-0.003)=0.5-0.003=0.497$.

Question 2:

Show that the function given by $f(x)=\frac{\log x}{x}$ has maximum at x = e.

Solution 2:

The given function is $f(x)=\frac{\log x}{x}$

$$f'(x)=\frac{x\left(\frac{1}{x}\right)-\log x}{x^2}=\frac{1-\log x}{x^2}$$

Now, f'(x) = 0

⇒ 1 – log x = 0

⇒ log x = 1

⇒ log x = log e

⇒ x = e

Now, $f''(x)=\frac{x^2\left(-\frac{1}{x}\right)-(1-\log x)(2x)}{x^4}$

$$=\frac{-x-2x(1-\log x)}{x^4}$$

$$=\frac{-3+2\log x}{x^3}$$

Now, $f''(e)=\frac{-3+2\log e}{e^3}=\frac{-3+2}{e^3}=\frac{-1}{e^3}<0$

Therefore, by second derivative test *f* is the maximum at x = e.

Question 3:

The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?

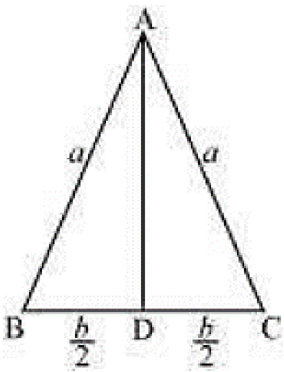
Solution 3:

Let ΔABC be isosceles where BC is the base of fixed length *b*.

Let the length of the two equal sides of ΔABC be *a*.

AD ⊥ BC.

Draw



Now, in ΔADC, by applying the Pythagoras theorem, we have:

$$AD=\sqrt{a^2-\frac{b^2}{4}}$$

Area of triangle $(A)=\frac{1}{2}b\sqrt{a^2-\frac{b^2}{4}}$

The rate of change of the area with respect to time (t) is given by,

$$\frac{dA}{dt}=\frac{1}{2}b\frac{2a}{2\sqrt{a^2-\frac{b^2}{4}}}\frac{da}{dt}=\frac{ab}{\sqrt{4a^2-b^2}}\frac{da}{dt}$$

It is given that the two equal sides of the triangle are decreasing at the rate of 3 cm per second.

$$\frac{da}{dt}=-3\text{ cm/S}$$

$$\therefore\frac{da}{dt}=\frac{-3ab}{\sqrt{4a^2-b^2}}$$

Then, when a = b,we have:

$$\frac{dA}{dt}=\frac{-3ab^2}{\sqrt{4b^2-b^2}}=\frac{-3b^2}{\sqrt{3b^2}}=-\sqrt{3}b$$

Hence, if two equal sides are equal to the base, then the area of the triangle is decreasing at the rate of $\sqrt{3}b$ cm^2/s.

Question 4:

Find the equation of the normal to curve $y^2=4x$ at the point (1.2).

Solution 4:

The equation of the given curve is $y^2=4x$.

Differentiating with respect to x, we have:

$$2y\frac{dy}{dx}=4$$

$$\Rightarrow\frac{dy}{dx}=\frac{4}{dy}=\frac{2}{y}$$

$$\therefore\left.\frac{dy}{dx}\right]_{(1,2)}=\frac{2}{2}=1$$

Now, the slope of the normal at point (1, 2) is $\frac{-1}{\left.\frac{dy}{dx}\right]_{(1,2)}}=\frac{-1}{1}=-1$.

Equation of the normal at (1, 2) is y – 2 = –1(x – 1).

y – 2 = –x + 1

x + y – 3 = 0

Question 5:

Show that the normal at any point θ to the curve

$x=a\cos\theta+a\theta\sin\theta, y=a\sin\theta-a\theta\cos\theta$ is at a constant distance from the origin.

Solution 5:

We have x = a cos θ + aθ sin θ

$$\therefore \frac{dx}{d\theta} = -a\sin\theta + a\sin\theta + a\theta\cos\theta = a\theta\cos\theta$$

$$y = a\sin\ \theta - a\theta\cos\theta$$

$$\therefore \frac{dy}{d\theta} = a\cos\theta - a\cos\theta + a\theta sin\theta = a\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = \frac{a\theta sin\theta}{a\theta\cos\theta} = \tan\theta$$

Slope of the normal at any point $\theta\ \frac{1}{\tan\theta}$.

The equation of the normal at a given point (x,y)is given by,

$$y = a\sin\theta + a\theta\cos\theta = \frac{-1}{\tan\theta}(x - a\cos\theta - a\theta\sin\theta)$$

$$\Rightarrow y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta = -x\cos\theta + a\cos^2\theta + \sin\theta\cos\theta$$

$$\Rightarrow x\cos\theta + y\sin\theta - a(\sin^2\theta + \cos^2\theta) = 0$$

$$\Rightarrow z\cos\theta + y\sin\theta - a = 0$$

Now, the perpendicular distance of the normal from the origin is

$\frac{|-a|}{\sqrt{\cos^2\theta + \sin^2\theta}} = \frac{|-a|}{\sqrt{1}} = |-a|$, which is independent of θ.

Hence, the perpendicular distance of the normal from the origin is constant.

Question 6:

Find the intervals in which the function given f by

$$f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$$

Is **(i)** increasing **(ii)** decreasing

Solution 6:

$$f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$$

$$\therefore f'(x) = \frac{(2+\cos x)(4x\cos x - 2 - \cos x + x\sin x) - (4\sin x - 2x - x\cos x)(-\sin x)}{2 + \cos x}$$

$$= \frac{(2+\cos x)(3\cos x - 2 + x\sin x) - \sin x(4\sin x - 2x - x\cos x)}{(2+\cos x)^2}$$

$$= \frac{6\cos x - 4 + 2x\sin x + 3\cos^2 x - 2\cos x + \sin x\cos x + 4\sin^2 x - 2\sin^2 x - 2x\sin x - x\sin x\cos x}{(2+\cos x)^2}$$

$$= \frac{4\cos x - \cos^2 x}{(2+\cos x)^2} = \frac{\cos x(4 - \cos x)}{(2+\cos x)^2}$$

= x

Now, $f'(x) = 0$

$\Rightarrow$ cos x = 0, cos x = 4

But, cos x $\neq$ 4
cos x = 0

$\Rightarrow x=\frac{\pi}{2},\frac{3\pi}{2}$

Now, $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$ divides (0, 2π) into three disjoint intervals i.e.,

$\left(0,\frac{\pi}{2}\right)$, $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, and $\left(\frac{3\pi}{2},\ 2\pi\right)$

In intervals $\left(0,\frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2},\ 2\pi\right)$, $f'(x)>0$.

Thus, f(x) is increasing for 0 < x < $\frac{x}{2}$ and $\frac{3\pi}{2}$ < x < 2π.

In the interval, $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, $f'(x)<0$.

Thus, f(x) is decreasing for $\frac{\pi}{2}$ < x < $\frac{3\pi}{2}$.

Question 7:

Find the intervals in which the function *f* given by f(x) = x^3 + $\frac{1}{x^3}$, x $\neq$ 0 is

(i) increasing **(ii)** decreasing

Solution 7:

f(x) = x^3 + $\frac{1}{x^3}$

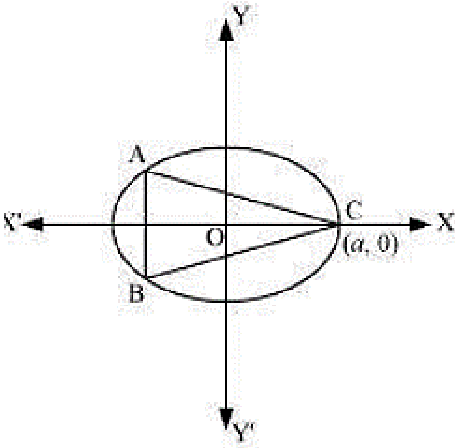
$\therefore f'(x)=3x^2-\frac{3}{x^4}=\frac{3x^6-3}{x^4}$

Then, f'(x) = 0 $\Rightarrow$ $3x^6 - 3 = 0$ $\Rightarrow$ x^6 = x = ±1.
Now, the points x = 1 and x = –1 divide the real line into three disjoint intervals i.e., $(-\infty, 1)$, $(-1, 1)$ and $(1, \infty)$.
In intervals $(-\infty, 1)$ and $(1, \infty)$ i.e., when x < –1 and x > 1, $f'(x)>0$.
Thus, when x < –1 and x > 1, f is increasing.
In interval $(-1, 1)$ i.e., when $-1 < x < 1$, $f'(x)<0$.
Thus, when –1 < x < 1, *f* is decreasing.

Question 8:

Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis.

Solution 8:



The given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let the major axis be along the x-axis.

Let ABC, be the triangle inscribed in the ellipse where vertex C is at $(a, 0)$.

Since the ellipse is symmetrical with respect to the x-axis and y-axis, we can assume the coordinates of A to be $(-x_1, y_1)$.

Now, we have $y_1=\pm\frac{b}{a}\sqrt{a^2-x_1^2}$.

Coordinates of A are $\left(-x_1, \frac{b}{a}\sqrt{a^2-x_1^2}\right)$ and the coordinates of B are $\left(x_1, -\frac{b}{a}\sqrt{a^2-x_1^2}\right)$.

As the point $(-x_1, y_1)$, lies on the ellipse, the area of triangle ABC (A) is given by,

$$A=\frac{1}{2}\left|a\left(\frac{2b}{a}\sqrt{a^2-x_1^2}\right)+(-x_1)\left(-\frac{b}{a}\sqrt{a^2-x_1^2}\right)+(-x_1)\left(-\frac{b}{a}\sqrt{a^2-x_1^2}\right)\right| \quad \ldots\ldots (1)$$

$$\Rightarrow A=ba\sqrt{a^2-x_1^2}+x_1\frac{b}{a}\sqrt{a^2-x_1^2}$$

$$\therefore \frac{dA}{dx_1}=\frac{-2xb}{2\sqrt{a^2-x_1^2}}+\frac{b}{a}\sqrt{a^2-x_1^2}-\frac{2bx_1^2}{a^2\sqrt{a^2-x_1^2}}$$

$$=\frac{b}{2\sqrt{a^2-x_1^2}}[-x_1a+(a^2-x_1^2)-x_1^2]$$

$$=\frac{b(-2x_1^2-x_1^2+a^2)}{2\sqrt{a^2-x_1^2}}$$

Now, $\frac{dA}{dx_1}=0$

$$\Rightarrow -2{x_1}^2 - x_1 a + a^2 = 0$$

$$\Rightarrow x_1 = \frac{a \pm \sqrt{a^2 - 4(-2)(a^2)}}{2(-2)}$$

$$= \frac{a \pm \sqrt{9a^2}}{-4}$$

$$= \frac{a \pm 3a}{-4}$$

$$\Rightarrow x_1 = -a, \frac{a}{2}$$

But x_1 cannot be equal to a.

$$\therefore x_1 = \frac{a}{2} \Rightarrow y_1 = \frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}} = \frac{ba}{2a}\sqrt{3} = \frac{\sqrt{3}b}{2}$$

Now, $$\frac{d^2A}{d{x_1}^2} = \frac{b}{a}\left\{\frac{\sqrt{a^2 - {x_1}^2}(-4x_1 - a) - (-2{x_1}^2 - x_1 a + a^2)\frac{-2x_1}{2\sqrt{a^2 - {x_1}^2}}}{a^2 - {x_1}^2}\right\}$$

$$= \frac{b}{a}\left\{\frac{(a^2 - {x_1}^2)(-4x_1 - a) + x_1(-2{x_1}^2 - x_1 a + a^2)}{(a^2 - {x_1}^2)^{\frac{2}{3}}}\right\}$$

$$= \frac{b}{a}\left\{\frac{2x^3 - 3a^2 x - a^3)}{(a^2 - {x_1}^2)^{\frac{3}{2}}}\right\}$$

Also, when $x_1 = \frac{a}{2}$, then

$$\frac{d^2A}{d{x_1}^2} = \frac{b}{a}\left\{\frac{2\frac{a^3}{8} - 3\frac{a^3}{2} - a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\} = \frac{b}{a}\left\{\frac{\frac{a^3}{4} - \frac{3}{2}a^3 - a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}$$

$$= \frac{b}{a}\left\{\frac{\frac{9}{4}a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\} < 0$$

Thus, the area is the maximum when $x_1 = \frac{a}{2}$.

Maximum area of the triangle is given by,

$$A = b\sqrt{a^2 - \frac{a^2}{4}} + \left(\frac{a}{2}\right)\frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}}$$

$$= ab\frac{\sqrt{3}}{2} + \left(\frac{a}{2}\right)\frac{b}{a}\frac{a\sqrt{3}}{2}$$

$$= \frac{ab\sqrt{3}}{2} + \frac{ab\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}ab$$

Question 9:

A tank with rectangular base and rectangular sides, open at the top is to constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs Rs. 70 per sq. meters for the base and Rs. 45 per sq. meters for sides. What is the cost of least expensive tank?

Solution 9:

Let l, b and h represent the length, breadth, and height of the tank respectively.

Then, we have height (h) = 2 m

Volume of the tank = 8 m^3

Volume of the tank = lbh

$8 = l \times b \times 2$

$$\Rightarrow lb = 4 \Rightarrow b = \frac{4}{l}$$

Now, area of the base = lb = 4

Area, of the 4 walls (A) = $2h(l + b)$

$$\therefore A = 4\left(l + \frac{4}{l}\right)$$

$$\Rightarrow \frac{dA}{dl} = 4\left(l - \frac{4}{l^2}\right)$$

Now, $\frac{dA}{dl} = 0$

$$\Rightarrow l - \frac{4}{l^2} = 0$$

$$\Rightarrow l^2 = 4$$

$$\Rightarrow l = \pm 2$$

However, the length cannot be negative.

Therefore, we have l = 4.

$$b = \frac{4}{l} = \frac{4}{2} = 2$$

Now, $\frac{d^2A}{dl^2} = \frac{32}{l^3}$

When, $l = 2$

$$\frac{d^2A}{dl^2} = \frac{32}{8} = 4 > 0$$

Thus, by second derivative test, the area is the minimum when $l = 2$.
We have l = b = h = 2.
Cost of building the base = Rs. 70 × (lb) = Rs. 70(4) = Rs. 280
Cost of building the walls = Rs. 2h (l + h) × 45 = Rs. 90(2)(2 + 2) = Rs. 8(90) = Rs. 720
Required total cost = Rs. (280 + 720) = Rs. 1000
Hence, the total cost of the tank will be Rs. 1000.

Question 10:
The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their area is least when the side of square is double the radius of the circle.

Solution 10:
Let r be the radius of the circle and a be the side of the square.
Then, we have:
$2\pi r + 4a = k$ (where k is constant)

$$\Rightarrow a = \frac{k - 2\pi r}{4}$$

The sum of the areas of the circle and the square (A) is given by,

$$A = \pi r^2 + a^2 = \pi r^2 + \frac{(k - 2\pi r)^2}{16}$$

$$\therefore \frac{dA}{dr} = 2\pi r + \frac{2(k - 2\pi r)(2\pi)}{16} = 2\pi r = -\frac{\pi(k - 2\pi r)}{4}$$

Now, $\frac{dA}{dr} = 0$

$$\Rightarrow 2\pi r = \frac{\pi(k - 2\pi r)}{4}$$

$$8r = k - 2\pi r$$

$$\Rightarrow (8 + 2\pi)r = k$$

$$\Rightarrow r = \frac{k}{8 + 2\pi} = \frac{k}{2(4 + \pi)}$$

Now, $\frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$

Where $r = \frac{k}{2(4 + \pi)}, \frac{d^2A}{dr^2} > 0$

The sum of the areas is least when $r = \frac{k}{2(4 + \pi)}$

where $r = \frac{k}{2(4 + \pi)}, a = \frac{k - 2\pi\left[\frac{k}{2(4 + \pi)}\right]}{4} = \frac{8k + 2\pi k - 2\pi k}{2(4 + \pi) \times 4} = \frac{k}{4 + \pi} = 2r$

Hence, it has been proved that the sum of their areas is least when the side of the square is double the radius of the circle.

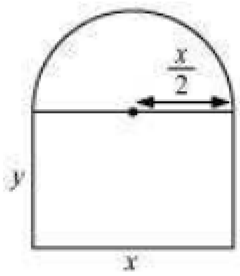
Question 11:

A window is in the form of rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Solution 11:

Let x and y be the length and breadth of the rectangular window.

Radius of the semicircular opening $= \frac{x}{2}$



It is given that the perimeter of the window is 10 m.

$$\therefore x + 2y + \frac{\pi x}{2} = 10$$

$$\Rightarrow x\left(1 + \frac{\pi}{2}\right) + 2y = 10$$

$$\Rightarrow 2y = 10 - x\left(1 + \frac{\pi}{2}\right)$$

$$\Rightarrow y = 5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)$$

Area of the window (A) is given by,

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2$$

$$= x\left[5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)\right] + \frac{\pi}{8}x^2$$

$$= 5x - x^2\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi}{8}x^2$$

$$\therefore \frac{dA}{dx} = 5 - 2x\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi}{4}x$$

$$\frac{d^2A}{dx^2} = -\left(1 - \frac{\pi}{2}\right) + \frac{\pi}{4} = -1 - \frac{\pi}{4}$$

Now, $\frac{dA}{dx} = 0$

$$\Rightarrow 5 - x\left(1+\frac{\pi}{2}\right)+\frac{\pi}{4}x = 0$$

$$\Rightarrow 5 - x - \frac{\pi}{4}x = 0$$

$$\Rightarrow x\left(1+\frac{\pi}{4}\right) = 5$$

$$\Rightarrow x = \frac{5}{\left(1+\frac{\pi}{4}\right)} = \frac{20}{\pi+4}$$

Thus, when $x = \frac{20}{\pi+4}$ then $\frac{d^2A}{dx^2} < 0$.

Therefore, by second derivative test, the area is the maximum when length $x = \frac{20}{\pi+4}$ m.

Now,

$$y = 5 - \frac{20}{\pi+4}\left(\frac{2+\pi}{4}\right) = 5 - \frac{5(2+\pi)}{\pi+4} = \frac{10}{\pi+4} \text{ m}$$

Hence, the required dimensions of the window to admit maximum light is given by length = $\frac{20}{\pi+4}$ m and breadth = $\frac{10}{\pi+4}$ m.

Question 12:

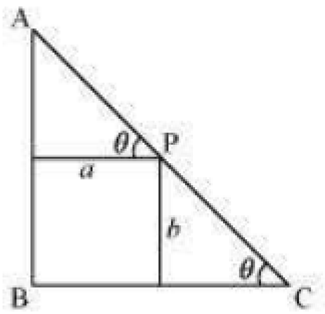
A point of the hypotenuse of a triangle is at distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

Solution 12:

Let ΔABC be right-angles at B. Let AB = x and BC = y.

Let P be a point on the hypotenuse of the triangle such that P is at a distance of a and b from the sides AB and BC respectively.

Let $\angle c = \theta$



We have, $AC = \sqrt{x^2+y^2}$

Now,

PC = b cosec θ

And, AP = a sec θ

AC = AP + PC

AC = b cosec θ + a sec θ … (1)

$$\therefore \frac{d(AC)}{d\theta} = -b\cos ec\theta\cot\theta + a\sec\theta\tan\theta$$

$$\therefore \frac{d(AC)}{d\theta} = 0$$

$$\Rightarrow a\sec\theta\tan\theta = b\text{cosec}\,\theta\cot\theta$$

$$\Rightarrow \frac{a}{\cos\theta}\frac{\sin\theta}{\cos\theta} = \frac{b}{\sin\theta}\frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow a\sin^3\theta = b\cos^3\theta$$

$$\Rightarrow (a)^{\frac{1}{3}}\sin\theta = (b)^{\frac{1}{3}}\cos\theta$$

$$\Rightarrow \tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

$$\sin\theta = \frac{(b)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}} \text{ and } \cos\theta = \frac{(a)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}} \qquad \ldots(2)$$

It can be clearly shown that $\frac{d^2(AC)}{d\theta^2} < 0$ when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Therefore, by second derivative test, the length of the hypotenuse is the maximum when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Now, when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$, we have:

$$\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

[Using (1) and (2)]

$$AC = \frac{b\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{b^{\frac{1}{3}}} + \frac{a\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{a^{\frac{1}{3}}}$$

$$= a\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}\left(b^{\frac{2}{3}} + a^{\frac{2}{3}}\right)$$

$$= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

Hence, the maximum length of the hypotenuses is = $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

Question 13:

Find the points at which the function *f* given by f(x) = $(x-2)^4(x+1)^3$ has

(i) local maxima **(ii)** local minima **(iii)** point of inflexion

Solution 13:

The given function is f(x) = $(x-2)^4(x+1)^3$

$f'(x)=4(x-2)^3(x+1)^3+3(x+1)^2(x-2)^4$

$=(x-2)^3(x+1)^2[4(x+1)+3(x-2)]$

$=(x-2)^3(x+1)^2(7x-2)$

Now, f'(x) = 0 $\Rightarrow$ x = -1 and $x=\frac{2}{7}$ or x = 2

Now, for values of *x* close to $\frac{2}{7}$ and to the left of $\frac{2}{7}$, f(x) > 0. Also, for values of *x* close to $\frac{2}{7}$ and to the right of $\frac{2}{7}$, f'(x) > 0.

Thus, $x=\frac{2}{7}$ is the point of local minima.

Now, as the value of *x* varies through -1 f'(x) does not changes its sign.

Thus, x = -1 is the point of inflexion.

Question 14:

Find the absolute maximum and minimum values of the function *f* given by

f(x) = $\cos^2 x + \sin x$, $x\in[0,\pi]$

Solution 14:

f(x) = $\cos^2 x + \sin x$

$f'(x)=2\cos x(-\sin x)+\cos x$

$= -2\sin x \cos x + \cos x$

Now, f'(x) = 0

$\Rightarrow 2\sin x\cos x=\cos x\Rightarrow \cos x(2\sin x-1)=0$

$\Rightarrow \sin x=\frac{1}{2}$ or $\cos x=0$

$\Rightarrow x=\frac{\pi}{6}$ or $\frac{\pi}{2}$ as $x\in[0,\pi]$

Now, evaluating the value of *f* at critical points x = $\frac{\pi}{2}$ and x = $\frac{\pi}{6}$ and at the end points of the interval $[0,\pi]$ (i.e., at *x* = 0 and *x* = *n*), we have:

$$f\left(\frac{\pi}{6}\right)=\cos^2\frac{\pi}{6}+\sin\frac{\pi}{6}=\left(\frac{\sqrt{3}}{2}\right)^2+\frac{1}{2}=\frac{5}{4}$$

$$f(0)=\cos^2 0+\sin 0=1+0=1$$

$$f(\pi)=\cos^2\pi+\sin\pi=(-1)^2+0=1$$

$$f\left(\frac{\pi}{2}\right)=\cos^2\frac{\pi}{2}+\sin\frac{\pi}{2}=0+1=1$$

Hence, the absolute maximum value of f is $\frac{5}{4}$ occurring at x = $\frac{\pi}{6}$ and the absolute minimum value of f is 1 occurring at x = 0, x = $\frac{\pi}{2}$, and π.

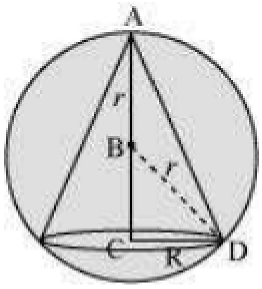
Question 15:

Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Solution 15:

A sphere of fixed radius (r) is given.

Let R and h be the radius and the height of the cone respectively.



The volume (V) of the cone is given by,

$$V=\frac{1}{3}\pi R^2 h$$

Now, from the right triangle BCD, we have:

$$BC=\sqrt{r^2-R^2}$$

$$h=r+\sqrt{r^2-R^2}$$

$$\therefore V=\frac{1}{3}\pi R^2(r+\sqrt{r^2-R^2})=\frac{1}{3}\pi R^2 r+\frac{1}{3}\pi R^2\sqrt{r^2-R^2}$$

$$\therefore \frac{dV}{dR}=\frac{2}{3}\pi Rr+\frac{2\pi}{3}\pi R\sqrt{r^2-R^2}+\frac{R^2}{3}\cdot\frac{(-2R)}{2\sqrt{r^2-R^2}}$$

$$=\frac{2}{3}\pi Rr+\frac{2\pi}{3}\pi R\sqrt{r^2-R^2}-\frac{R^3}{3\sqrt{r^2-R^2}}$$

$$=\frac{2}{3}\pi Rr+\frac{2\pi Rr(r^2-R^2)-\pi R^3}{3\sqrt{r^2-R^2}}$$

$$=\frac{2}{3}\pi Rr+\frac{2\pi Rr^2-3\pi Rr^3}{3\sqrt{r^2-R^2}}$$

Now, $\frac{d^2V}{dR^2}=0$

$$\Rightarrow \frac{2\pi rR}{3}=\frac{3\pi R^2-2\pi Rr^2}{3\sqrt{r^2-R^2}}$$

$$\Rightarrow 2r\sqrt{r^2-R^2}=3R^2-2r^2$$

$$\Rightarrow 4r^2(r^2-R^2)=(3R^2-2r^2)^2$$

$$\Rightarrow 14r^4-4r^2R^2=9R^4+4r^4-12R^2r^2$$

$$\Rightarrow 9R^4-8R^2r^2=0$$

$$\Rightarrow 9R^2=8r^2$$

$$\Rightarrow R^2=\frac{8r^2}{9}$$

Now, $$\frac{d^2V}{dR^2}=\frac{2\pi r}{3}+\frac{3\sqrt{r^2-R^2}(2\pi r^2-9\pi R^2)-(2\pi R^3-3\pi R^3)(-6R)\frac{1}{2\sqrt{r^2-R^2}^2}}{9(r^2-R^2)}$$

$$=\frac{2\pi r}{3}+\frac{3\sqrt{r^2-R^2}(2\pi r^2-9\pi R^2)-(2\pi R^3-3\pi R^3)(3R)\frac{1}{2\sqrt{r^2-R^2}^2}}{9(r^2-R^2)}$$

Now, when $R^2=\frac{8r^2}{9}$, it can be shown that $\frac{d^2V}{dR^2}<0$.

The volume is the maximum when $R^2=\frac{8r^2}{9}$.

When $R^2=\frac{8r^2}{9}$, height of the cone = $r+\sqrt{r^2-\frac{8R^2}{9}}=r\sqrt{\frac{r^2}{9}}=r+\frac{r}{3}=\frac{4r}{3}$.

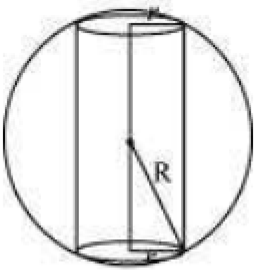
Hence, it can be seen that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Question 17:

Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$, also find the maximum volume.

Solution 17:

A sphere of fixed radius (R) is given.
Let r and h be the radius and the height of the cylinder respectively.



Form the given figure, we have $h = 2\sqrt{R^2 - r^2}$
The volume (V) of the cylinder is given by,

$$V = \pi r^2 h = 2\pi r^2 \sqrt{R^2 - r^2}$$

$$\therefore \frac{dV}{dr} = 4\pi r\sqrt{R^2 - r^2} + \frac{2\pi r^2(-2r)}{2\sqrt{R^2 - r^2}}$$

$$= 4\pi r\sqrt{R^2 - r^2} - \frac{2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r(R^2 - r^2) - 2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi rR^2 - 6\pi r^3}{\sqrt{R^2 - r^2}}$$

Now, $\frac{dV}{dr} = 0 \Rightarrow 4\pi rR^2 - 6\pi r^3 = 0$

$$\Rightarrow r^2 = \frac{2R^2}{3}$$

Now, $$\frac{d^2V}{dr^2} = \frac{\sqrt{R^2 - r^2}(4\pi R^2 - 18\pi r^2) - (4\pi rR^2 - 6\pi r^3)\frac{(-2r)}{2\sqrt{R^2 - r^2}}}{(R^2 - r^2)}$$

$$= \frac{(R^2 - r^2)(4\pi R^2 - 18\pi r^2) + r(4\pi rR^2 - 6\pi r^3)}{(R^2 - r^2)^{\frac{3}{2}}}$$

$$= \frac{4\pi R^4 - 22\pi r^2R^2 + 12\pi r^4 + 4\pi r^2R^2}{(R^2 - r^2)^{\frac{3}{2}}}$$

Now, it can be observed that at $r^2 = \frac{2R^2}{3}$, $\frac{d^2V}{dr^2} < 0$.

The volume is the maximum when $r^2 = \frac{2R^2}{3}$

When $r^2 = \frac{2R^2}{3}$ the height of the cylinder is $2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}$.

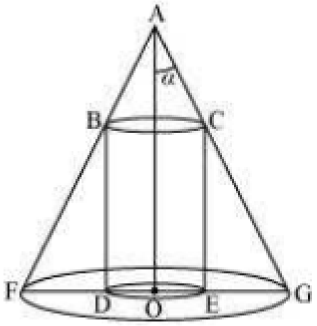
Hence, the volume of the cylinder is the maximum when the height of the cylinder is $\frac{2R}{\sqrt{3}}$.

Question 18:

Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle a is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^2 \tan^2 a$.

Solution 18:

The given right circular cone of fixed height (h) and semi-vertical angle (a) can be drawn as:



Here, a cylinder of radius *R* and height *H* is inscribed in the cone.

Then, ∆GAO = a, OG = r, OA = h, OE = R, and CE = H.

We have,

$r = h \tan a$

Now, since ∆AOG is similar to ∆CEG, we have:

$$\frac{AO}{OG} = \frac{CE}{EG}$$

$$\Rightarrow \frac{h}{r} = \frac{H}{r - R} \qquad [EG = OG - OE]$$

$$\Rightarrow H = \frac{h}{r}(r - R) = \frac{h}{h \tan a}(h \tan a - R) = \frac{1}{\tan a}(h \tan a - R)$$

Now, the volume (V) of the cylinder is given by,

$$V = \pi R^2 H = \frac{\pi R^2}{\tan a}(h \tan a - R) = \pi R^2 h - \frac{\pi R^3}{\tan a}$$

$$\frac{dV}{dR} = 2\pi Rh \frac{3\pi R^2}{\tan a}$$

Now, $\frac{dV}{dR} = 0$

$\Rightarrow 2\pi Rh = \dfrac{3\pi R^2}{\tan a}$

$\Rightarrow 2h \tan a = 3R$

$\Rightarrow R = \dfrac{2h}{3}\tan a$

Now, $\dfrac{d^2V}{dR^2} = 2\pi Rh - \dfrac{6\pi R}{\tan a}$

And, for $R = \dfrac{2h}{3}\tan a$ we have:

$\dfrac{d^2V}{dR^2} = 2\pi h - \dfrac{6\pi}{\tan a}\left(\dfrac{2h}{3}\tan a\right) = 2\pi h - 4\pi h = -2\pi h < 0$

By second derivative test, the volume of the cylinder is the greatest when $R = \dfrac{2h}{3}\tan a$.

When $R = \dfrac{2h}{3}\tan a$, $H = \dfrac{1}{\tan a}\left(h\tan a - \dfrac{2h}{3}\tan a\right) = \dfrac{1}{\tan a}\left(\dfrac{h\tan a}{3}\right) = \dfrac{h}{3}$

Thus, the height of the cylinder is one-third the height of the cone when the volume of the cylinder is the greatest.

Now, the maximum volume of the cylinder can be obtained as:

$\pi\left(\dfrac{2h}{3}\tan a\right)^2\left(\dfrac{h}{3}\right) = \pi\left(\dfrac{4h^2}{9}\tan^2 a\right)^2\left(\dfrac{h}{3}\right) = \dfrac{4}{27}\pi h^3 \tan^2 a$

Hence, the given result is proved.

Question 19:

A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic mere per hour. Then the depth of the wheat is increasing at the rate of

(A) 1 m/h **(B) 0.**1 m/h **(C)** 1.1 m/h **(D)** 0.5 m/h

Solution 19:

Let *r* be the radius of the cylinder.

Then, volume (V) of the cylinder is given by,

$V = \pi(\text{radius})^2 \times \text{height}$

$= \pi(10)^2 h$ (radius = 10 m)

$= 100\,\pi h$

Differentiating with respect to time *t*, we have:

$\dfrac{dV}{dt} = 100\pi\dfrac{dh}{dt}$

The tank is being filled with wheat at the rate of 314 cubic meters per hour.

$\dfrac{dV}{dt} = 314 \text{ m}^3/\text{h}$

Thus, we have:

$$314 = 100\pi \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{314}{100(3.14)} = \frac{314}{314} = 1$$

Hence, the depth of wheat is increasing at the rate of 1 m/h.
The correct answer is **A.**

Question 20:
The slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point (2, –1) is
(A) $\frac{22}{7}$ **(B)** $\frac{6}{7}$ **(C)** $\frac{7}{6}$ **(D)** $\frac{-6}{7}$

Solution 20:
The given curve is $x = t^2 + 3t - 8$, and $y = 2t^2 - 2t - 5$

$$\frac{dx}{dt} = 2t + 3 \text{ and } \frac{dy}{dt} = 4t - 2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{4t-2}{2t+3}$$

The given points is (2, –1).
At x = 2, we have:
$t^2 + 3t - 8 = 2$
$\Rightarrow t^2 + 3t - 10 = 0$
$\Rightarrow (t - 2)(t + 5) = 0$
$\Rightarrow t = 2$ or $t = -5$
At y = –1, we have
$2t^2 - 2t - 5 = -1$
$\Rightarrow 2t^2 - 2t - 4 = 0$
$\Rightarrow 2(t^2 - t - 2) = 0$
$\Rightarrow (t - 2)(t + 1) = 0$
$\Rightarrow t = 2$ or $t = -1$
The common value of *t* is 2.
Hence, the slope of the tangent to the given curve at point (2, –1) is

$$\left.\frac{dy}{dx}\right]_{t=2} = \frac{4(2)-2}{2(2)+3} = \frac{8-2}{4+3} = \frac{6}{7}$$

The correct answer is **B.**

Question 21:
The line $y = mx + 1$ is tangent to the given curve $y^2 = 4x$ if the value on *m* is
(A) 1 **(B)** 2 **(C)** 3 **(D)** $\frac{1}{2}$

Solution 21:

The equation of the tangent to the given curve is y = mx + 1
Now, substituting y = mx + 1 in $y^2 = 4x$, we get:
$\Rightarrow (mx + 1)^2 = 4x$
$\Rightarrow m^2x^2 + 1 + 2mx - 4x = 0$
$\Rightarrow m^2x^2 + x(2m - 4) + 1 = 0$ ....(i)
Since a tangent touches the curve at one point, the toots of equation (i) must be equal.
Therefore, we have:
Discriminant = 0
$(2m - 4)^2 - 4(m^2)(1) = 0$
$\Rightarrow 4m^2 + 16 - 16m - 4m^2 = 0$
$\Rightarrow 16 - 16m = 0$
$\Rightarrow m = 1$
Hence, the required value of *m* is 1.
The correct answer is **A.**

Question 22:
The normal at the point (1, 1) on the curve $2y + x^2 = 3$ is
(A) x + y = 0 **(B)** x – y = 0 **(C)** x + y + 1 = 0 **(D)** x – y = 1

Solution 22:
The equation of the given curve is $2y + x^2 = 3$
Differentiating with respect to x, we have:

$$\frac{2dy}{dx} + 2x = 0$$

$$\Rightarrow \frac{dy}{dx} = -x$$

$$\therefore \left.\frac{dy}{dx}\right]_{(1,1)} = -1$$

The slope of the normal to the given curve at point (1, 1) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(1,1)}} = 1$$

Hence, the equation of the normal to the given curve at (1, 1) is given as:
$\Rightarrow y - 1 = 1(x - 1)$
$\Rightarrow y - 1 = x - 1$
$\Rightarrow x - y = 0$
The correct answer is **B.**

Question 23:
The normal to the curve $x^2 = 4y$ passing (1, 2) is
A) x + y = 3 **(B)** x – y = 3 **(C)** x + y = 1 **(D)** x – y = 1

Solution 23:

The equation of the given curve is $x^2 = 4y$.

Differentiating with respect to x, we have:

$$2x = 4 \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{2}$$

The slope of the normal to the given curve at point (h, k) is given by,

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(h,k)}} = -\frac{2}{h}$$

Equation of the normal at point (h, k) is given as:

$$y - k = \frac{-2}{h}(x - h)$$

Now, it is given that the normal passes through the point (1, 2).

Therefore, we have:

$$2 - k = \frac{-2}{h}(1 - h) \text{ or } k = 2 + \frac{2}{h}(1 - h) \qquad \ldots(i)$$

Since (h, k) lies on the curves $x^2 = 4y$, we have $h^2 = 4k$

$$\Rightarrow k = \frac{h^2}{4}$$

From equation (i), we have:

$$\frac{h^2}{4} = 2 + \frac{2}{h}(1 - h)$$

$$\Rightarrow \frac{h^3}{4} = 2h + 2 - 2h = 2$$

$$\Rightarrow h^3 = 8$$

$$\Rightarrow h = 2$$

$$\therefore k = \frac{h^2}{4} \Rightarrow k = 1$$

Hence, the equation of the normal is given as:

$$\Rightarrow y - 1 = \frac{-2}{2}(x - 2)$$

$$\Rightarrow y - 1 = (x - 2)$$

$$\Rightarrow x + y = 3$$

The correct answer is **A.**

Question 24:

The points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes are

(A) $\left(4, \pm\frac{8}{3}\right)$ **(B)** $\left(4, \frac{-8}{3}\right)$ **(C)** $\left(4, \pm\frac{3}{8}\right)$ **(D)** $\left(\pm 4, \frac{8}{3}\right)$

Solution 24:

The equation of the given curve is $9y^2 = x^2$

Differentiating with respect to x, we have:

$$9(2y)\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{6y}$$

The slope of the normal to the given curve at point (x_1, y_1) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(x_1,y_1)}} = -\frac{6y_1}{x_1^{\,2}}$$

The equation of the normal to the curve at (x_1, y_1) is

$$y - y_1 = \frac{-6y_1}{x_1^{\,2}}(x - x_1)$$

$$\Rightarrow x_1^{\,2}y - x_1^{\,2}y_1 = -6xy_1 + 6x_1y_1$$

$$\Rightarrow 6x_1y_1 - x_1^{\,2}y = 6x_1y_1 + x_1^{\,2}y_1$$

$$\Rightarrow \frac{6xy_1}{6x_1y_1 + x_1^{\,2}y_1} + \frac{x^2y}{6x_1y_1 + x^2y_1} = 1$$

$$\Rightarrow \frac{x}{\frac{x_1(6+x_1)}{6}} + \frac{y}{\frac{y_1(6+x_1)}{x_1}} = 1$$

It is given that the normal makes equal intercepts with the axes.

Therefore, we have:

$$\therefore \frac{x_1(6+x_1)}{6} = \frac{y_1(6+x_1)}{x_1}$$

$$\Rightarrow \frac{x_1}{6} = \frac{y_1}{x_1}$$

$$\Rightarrow x_1^{\,2} = 6y_1 \qquad ....(i)$$

Also, the point (x_1, y_1) lies on the curve, so we have

$9y_1^2 = x_1^3$ ...(*ii*)

From (i) and (ii), we have:

$$9\left(\frac{x_1^{\,2}}{6}\right)^2 = x_1^{\,3} \Rightarrow \frac{x_1^{\,4}}{4} = x_1^{\,3} \Rightarrow x_1 = 4$$

From (iii), we have:

$$9{y_1}^2 = (4)^3 = 64$$

$$\Rightarrow {y_1}^2 = \frac{64}{9}$$

$$\Rightarrow y_1 = \pm\frac{8}{3}$$

Hence, the required points are $\left(4, \pm\frac{8}{3}\right)$.

The correct answer is **A.**