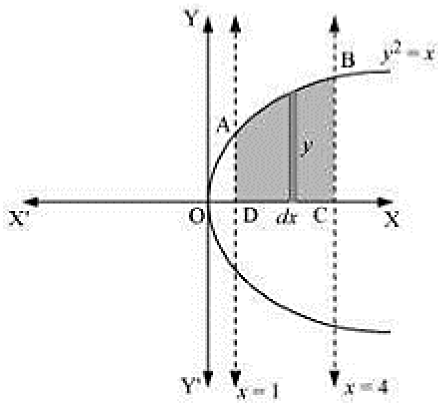
Chapter 8 – Application of Integrals

EXERCISE- 8.1

Question 1:

Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis in the first quadrant.

Solution 1:



The area of the region bounded by the curve, $y^2 = x$, the lines, $x = 1$ and $x = 4$, and the x-axis is the area ABCDA.

$$\text{Area ABCDA} = \int_1^4 \sqrt{x}\, dx$$

Area of ABCDA =

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$$

$$= \frac{2}{3}\left[(4)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right]$$

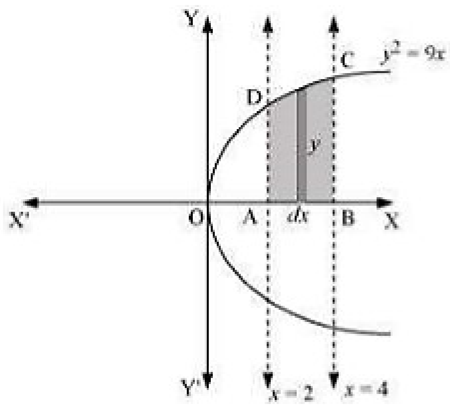
$$= \frac{2}{3}[8 - 1]$$

$$= \frac{14}{3} \text{ sq.units}$$

Question 2:

Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.

Solution 2:



The area of the region bounded by the curve, $y^2 = 9x, x = 2,$ and $x = 4$, and the x-axis is the area ABCDA.

$$\text{Area ABCDA} = \int_2^4 3\sqrt{x}\,dx$$

$$= 3\left[\frac{x^{3/2}}{3/2}\right]_2^4$$

$$= 2\left[x^{\frac{3}{2}}\right]_2^4$$

$$= 2\left[4^{3/2} - 2^{3/2}\right]$$

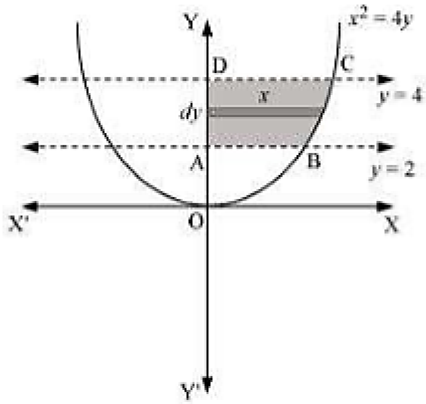
$$= 2\left[2^3 - 8^{1/2}\right] = 2\left[8 - 2\sqrt{2}\right]$$

$$= 16 - 4\sqrt{2} \text{ sq. units}$$

Question 3:

Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y-axis in the first quadrant.

Solution 3:



The area of the region bounded by the curve, $x^2 = 4y$, $y = 2$, and $y = 4$, and the y-axis is the area ABCDA.

Area of ABCDA $= \int_2^4 x dy$

$x^2 = 4y$

$x = 2\sqrt{y}$

$\int_2^4 x dy = \int_2^4 2\sqrt{y} dy = 2\int_2^4 \sqrt{y} dy$

$= 2\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$

$= \frac{4}{3}\left[(4)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right]$

$= \frac{4}{3}\left[8 - 2\sqrt{2}\right]$

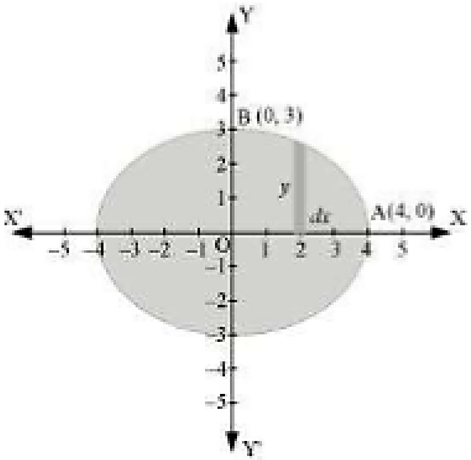
$= \left(\frac{32 - 8\sqrt{2}}{3}\right)$ sq.units

Question 4:

Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Solution 4:

The given equation of the ellipse, $\frac{x^2}{16}+\frac{y^2}{9}=1$, can be represented as



It can be observed that the ellipse is symmetrical about x-axis and y-axis.

$\therefore$ Area bounded by ellipse $=4\times$ Area of OABO

Area OABO = $\int_0^4 y dx$

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

$$\Rightarrow \frac{y^2}{9}=1-\frac{x^2}{16} \Rightarrow y^2=9\left(1-\frac{x^2}{16}\right)$$

$$y=3\sqrt{1-\frac{x^2}{16}}$$

Area OABO $=\int_0^4 3\sqrt{1-\frac{x^2}{16}}dx$

$$=\frac{3}{4}\int_0^4 \sqrt{16-x^2}dx$$

Substitute $x=4\sin\theta, \theta=\sin^{-1}\frac{x}{4}$

$dx=4\cos\theta d\theta$

when, $x=0 \;\; \theta=0$ & $x=4 \;\; \theta=\frac{\pi}{2}$

$$=\frac{3}{4}\int_0^{\pi/2}\sqrt{16-16\sin^2\theta}.4\cos\theta d\theta$$

$$=12\int_0^{\pi/2}\sqrt{1-\sin^2\theta}, \cos\theta d\theta$$

$$= 12\int_0^{\pi/2} \cos^2\theta d\theta = 12\int_0^{\pi/2} \frac{1+\cos 2\theta}{2} d\theta$$

$$= 6\int_0^{\pi/2} (1+\cos 2\theta) d\theta = 6\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= 6\left[\frac{\pi}{2} + \frac{\sin\pi}{2} - 0 - \frac{\sin 0}{2}\right] = 6\left[\frac{\pi}{2}\right] = 3\pi$$

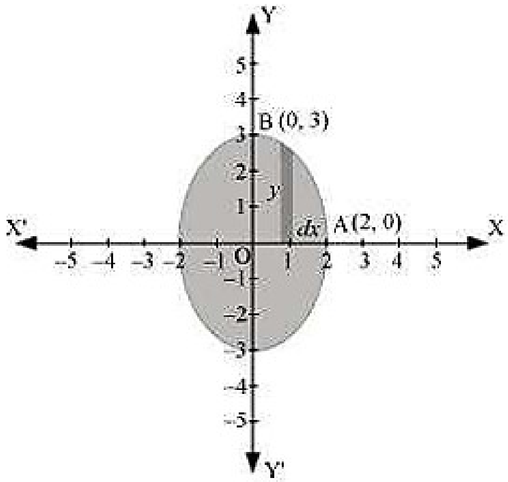
Therefore, area bounded by the ellipse $= 4 \times 3\pi = 12\pi$ sq. units

Question 5:

Find the area of the region bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Solution 5:

The given equation of the ellipse can be represented as



$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$\frac{y^2}{9} = 1 - \frac{x^2}{4}$$

$$y^2 = 9\left(1 - \frac{x^2}{4}\right)$$

$$\Rightarrow y = 3\sqrt{1 - \frac{x^2}{4}} \qquad ...(1)$$

It can be observed that the ellipse is symmetrical about x-axis and y-axis.

$\therefore$ Area bounded by ellipse $= 4 \times$ Area OABO

$\therefore$ Area of OABO $= \int_0^2 ydx$

$$= \int_0^2 3\sqrt{1-\frac{x^2}{4}}\,dx \qquad [\text{Using (1)}]$$

$$= \frac{3}{2}\int_0^2 \sqrt{4-x^2}\,dx$$

Substitute $x = 2\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{2}\right)$

$dx = 2\cos\theta\, d\theta$

when, $x = 0 \quad \theta = 0$ & $x = 2 \quad \theta = \frac{\pi}{2}$

$$\therefore \frac{3}{2}\int_0^2 \sqrt{4-x^2}\,dx = \frac{3}{2}\int_0^{\pi/2} \sqrt{4-4\sin^2\theta}.2\cos\theta d\theta$$

$$= 3\int_0^{\pi/2} \sqrt{4-4\sin^2\theta}.\cos\theta d\theta = 6\int_0^{\pi/2} \sqrt{1-\sin^2\theta}\,\cos\theta d\theta$$

$$= 6\int_0^{\pi/2} \cos 2\theta d\theta = 6\int_0^{\pi/2} \frac{1+\cos 2\theta}{2} d\theta$$

$$= \frac{6}{2}\int_0^{\pi/2} (1+\cos 2\theta) d\theta = 3\left[0+\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= 3\left[\frac{\pi}{2}+\frac{\sin\pi}{2}-0\right] = 3\times\frac{\pi}{2} = \frac{3\pi}{2}$$

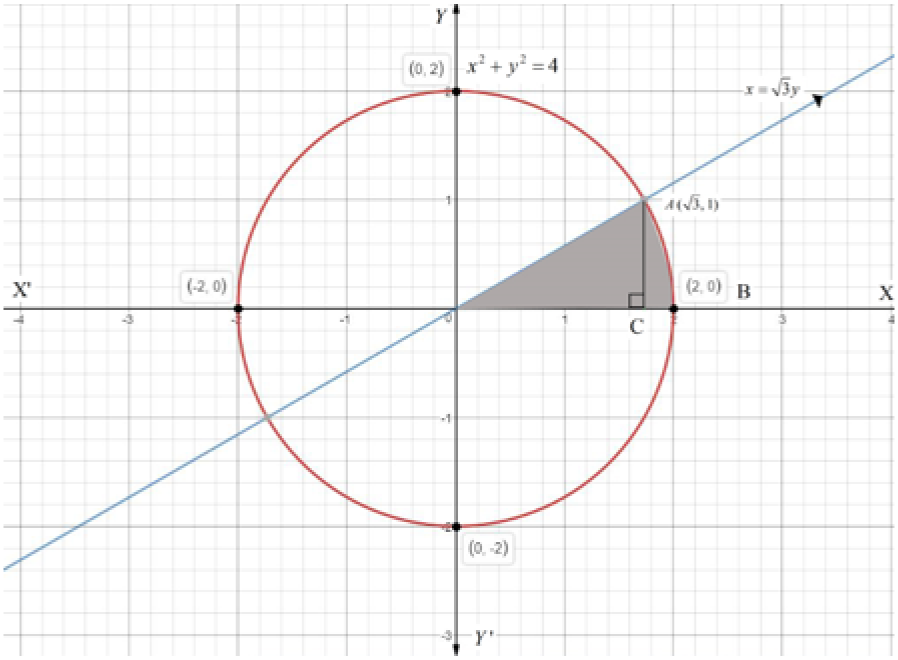
Therefore, area bounded by the ellipse $= 4\times\frac{3\pi}{2} = 6\pi$ sq. units

Question 6:

Find the area of the region in the first quadrant enclosed by x-axis, line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$

Solution 6:

The area of the region bounded by the circle, $x^2 + y^2 = 4$, $x = \sqrt{3}y$, and the x-axis is the area OAB.



Substituting $x=\sqrt{3}y$ in $x^2+y^2=4$, for finding the point of intersection.

$\therefore (\sqrt{3}y)+y^2=4 \Rightarrow y^2=1 \Rightarrow y=\pm1,\ x=\pm\sqrt{3}$

The point of intersection of the line and the circle in the first quadrant is $\left(\sqrt{3},1\right)$.

Area OABO = Area ΔOCA + Area ACBA

Area of OAC = $\frac{1}{2}\times OC\times AC=\frac{1}{2}\times\sqrt{3}\times1=\frac{\sqrt{3}}{2}$ ...(1)

Area of ABCA = $\int_{\sqrt{3}}^{2} ydx$

$\int_{\sqrt{3}}^{2} ydx=\int_{\sqrt{3}}^{2}\sqrt{4-x^2}\,dx$

$x=2\sin\theta \quad \theta=\sin^{-1}\left(\frac{x}{2}\right)$

when $x = 2 \quad \theta = \frac{\pi}{2}$

$x = \sqrt{3} \quad \theta = \frac{\pi}{3}$

$$\therefore \int_{\sqrt{3}}^{2} \sqrt{4-x}\, dx = \int_{\pi/3}^{\pi/2} \sqrt{4-4\sin^2\theta}(2\cos\theta)d\theta$$

$$= 4\int_{\pi/3}^{\pi/2} \cos^2\theta d\theta = 4\int_{\pi/3}^{\pi/2} 1 + \frac{\cos 2\theta}{2} d\theta$$

$$= 2\int_{\pi/3}^{\pi/2} (1+\cos 2\theta)d\theta = 2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/3}^{\pi/2}$$

$$= 2\left[\frac{\pi}{2} + \frac{1}{2}\sin\pi - \frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3}\right]$$

$$= 2\left[\frac{\pi}{2} - \frac{\pi}{3} - \frac{1}{2}\times\frac{\sqrt{3}}{2}\right] = 2\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right] \qquad \text{....(2)}$$

From (1) & (2)

$$\text{Area of OAB} = \frac{\sqrt{3}}{2} + 2\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right] = \frac{\pi}{3}$$

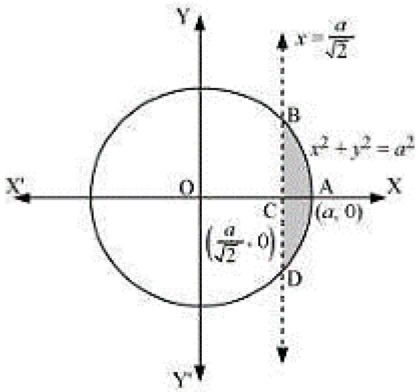
Therefore, area enclosed by x-axis, the line $x = \sqrt{3}y$, and the circle $x^2 + y^2 = 4$ in the first quadrant = $\frac{\pi}{3}$ sq. units

Question 7:

Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$.

Solution 7:

The area of the smaller part of the circle, $x^2 + y^2 = a^2$, cut off by the line $x = \frac{a}{\sqrt{2}}$, is the area ABCDA.



It can be observed that the area ABCDA is symmetrical about x-axis.

$\therefore$ Area ABCDA $= 2 \times$ Area ABCA

Area of ABCA = $\int_{\frac{a}{\sqrt{2}}}^{a} y dx$

$$\int_{a/\sqrt{2}}^{a} \sqrt{a^2 - x^2} dx$$

$$x = a\sin\theta \quad dx = a\cos\theta \, d\theta$$

$$x = \frac{a}{\sqrt{2}} \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

$$x = a, \ \theta = \sin^{-1}\left(\frac{a}{a}\right) = \frac{\pi}{2}$$

$$\therefore \int_{a/\sqrt{2}}^{a} \sqrt{a^2 - x^2} dx = \int_{\pi/4}^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta}.(a\cos\theta)d\theta$$

$$\Rightarrow a^2 \int_{\pi/4}^{\pi/2} \cos^2\theta = a^2 \int_{\pi/4}^{\pi/2} \frac{1+\cos 2\theta}{2} d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/2}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2} + \frac{\sin\pi}{2} - \frac{\pi}{4} - \frac{\sin\frac{\pi}{2}}{2}\right]$$

$$= \frac{a^2}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \frac{a^2}{4}\left[\frac{\pi}{2} - 1\right]$$

$$\Rightarrow Area \quad ABCD = 2\left[\frac{a^2}{4}\left(\frac{\pi}{2}-1\right)\right] = \frac{a^2}{2}\left(\frac{\pi}{2}-1\right)$$

Therefore, the area of smaller part of the circle, $x^2 + y^2 = a^2$, cut off by the line $x = \frac{a}{\sqrt{2}}$, is $\frac{a^2}{2}\left(\frac{\pi}{2}-1\right)$ sq. units.

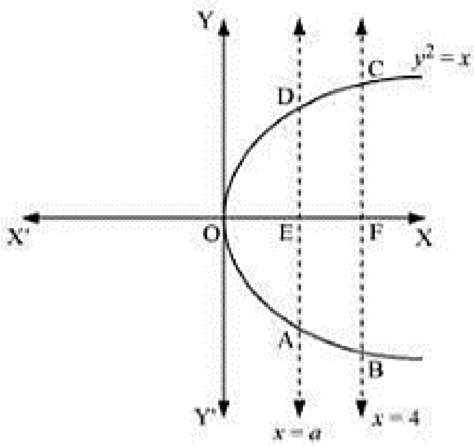
Question 8:

The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

Solution 8:

The line $x = a$, divides the area bounded by the parabola $x = y^2$ and $x = 4$ into two equal parts.

$\therefore$ Area OADO = Area ABCDA



It can be observed that the given area is symmetrical about x-axis.

Area of OEDO = ½ Area of OADO

Area of EFCDE = ½ Area of ABCDA

Therefore, Area OEDO = Area EFCDE

Area OEDO = $\int_0^a y dx$

$= \int_0^a \sqrt{x} dx$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^a$$

$$= \frac{2}{3}(a)^{\frac{3}{2}} \qquad \ldots(1)$$

Area of EFCDE $= \int_a^4 y\,dx = \int_a^4 \sqrt{x}\,dx$

$$= \left[\frac{x^{3/2}}{3/2} \right]_a^4$$

$$= \frac{2}{3}\left[4^{3/2} - a^{3/2} \right]$$

$$= \frac{2}{3}\left[8 - a^{\frac{3}{2}} \right] \qquad \ldots(2)$$

From (1) and (2), we obtain

$$\frac{2}{3}(a)^{\frac{3}{2}} = \frac{2}{3}\left[8 - (a)^{\frac{3}{2}} \right]$$

$$\Rightarrow 2.(a)^{\frac{3}{2}} = 8$$

$$\Rightarrow (a)^{\frac{3}{2}} = 4$$

$$\Rightarrow a = (4)^{\frac{2}{3}}$$

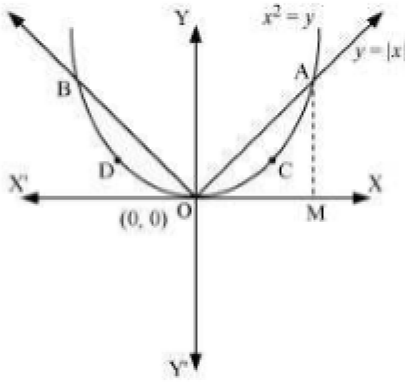
Therefore, the value of a is $(4)^{\frac{2}{3}}$.

Question 9:

Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

Solution 9:

The area bounded by the parabola, $x^2 = y$, and the line, $y = |x|$, can be represented as



The given area is symmetrical about y-axis.

∴ Area OACO = Area ODBO

The point of intersection of parabola $x^2 = y$ and line $y = x$ is A (1, 1).

Area of OACO = Area ΔOAM − Area OMACO

$$\therefore \text{Area of } \Delta OAM = \frac{1}{2} \times OM \times AM = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$$

$$\text{Area of OMACO} = \int_0^1 y\,dx = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\Rightarrow \text{Area of OACO} = \text{Area of } \Delta OAM - \text{Area of OMACO}$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}$$

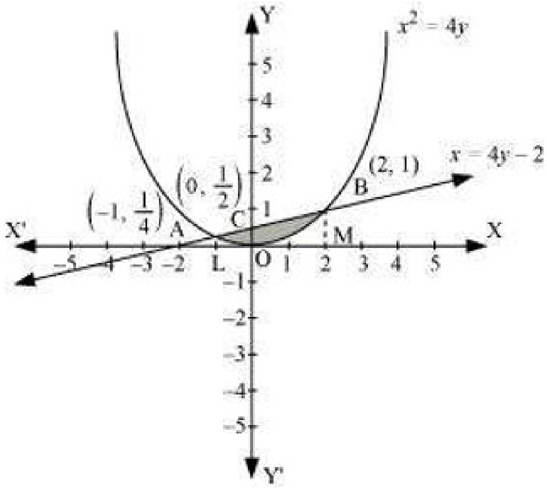
Therefore, required area = $2\left[\frac{1}{6}\right] = \frac{1}{3}$ sq. units.

Question 10:

Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

Solution 10:

The area bounded by the curve, $x^2 = 4y$, and line, $x = 4y - 2$, is represented by the shaded area OBAO.



Let A and B be the points of intersection of the line and parabola.

Substituting $x = 4y - 2$ in $x^2 = 4y$

$(4y-2)^2 = 4y$

$16y^2 - 16y + 4 = 4y$

$16y^2 - 20y + 4 = 0$

$4y^2 - 5y + 1 = 0$

$(4y-1)(y-1) = 0$

$y = 1/4, x = -1$

$y = 1, x = 2$

Coordinates of point A are $\left(-1, \frac{1}{4}\right)$.

Coordinates of point B are (2, 1).

We draw AL and BM perpendicular to x-axis.

It can be observed that,

Area OBAO = Area OBCO + Area OACO … (1)

Area OMBCO = Area under the line $x = 4y - 2$ between $x = 0$ and $x = 2$

Area OMBCO = $\int_0^2 \frac{x+2}{4} dx$

Area OMBO = Area under the curve $x^2 = 4y$ between $x = 0$ and $x = 2$

Area OMBO = $\int_0^2 \frac{x^2}{4} dx$

Then, Area OBCO = Area OMBCO – Area OMBO

$$=\int_0^2 \frac{x+2}{4}dx-\int_0^2 \frac{x^2}{4}dx$$
$$=\frac{1}{4}\left[\frac{x^2}{2}+2x\right]_0^2-\frac{1}{4}\left[\frac{x^3}{3}\right]_0^2$$
$$=\frac{1}{4}[2+4]-\frac{1}{4}\left[\frac{8}{3}\right]$$

$$=\frac{3}{2}-\frac{2}{3}$$
$$=\frac{5}{6}$$

Area OLACO = Area under the line $x=4y-2$ between $x=-1$ and $x=0$

Area OLACO = $\int_{-1}^{0}\frac{x+2}{4}dx$

Area OLAO = Area under the curve $x^2=4y$ between $x=-1$ and x =

Area OMBO = $\int_0^2 \frac{x^2}{4}dx$

Area OACO = Area OLACO – Area OLAO

$$=\int_{-1}^0 \frac{x+2}{4}dx-\int_{-1}^0 \frac{x^2}{4}dx$$
$$=\frac{1}{4}\left[\frac{x^2}{2}+2x\right]_{-1}^0-\frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^0$$
$$=\frac{1}{4}\left[\frac{0}{2}+0-\frac{(-1)^2}{2}-2(-1)\right]-\frac{1}{4}\left[\frac{0^3}{3}-\frac{(-1)^3}{3}\right]$$

$$= -\frac{1}{4}\left[\frac{(-1)}{2} + 2(-1)\right] - \left[-\frac{1}{4}\left(\frac{(-1)^3}{3}\right)\right]$$

$$= -\frac{1}{4}\left[\frac{1}{2} - 2\right] - \frac{1}{12}$$

$$= \frac{1}{2} - \frac{1}{8} - \frac{1}{12}$$

$$= \frac{7}{24}$$

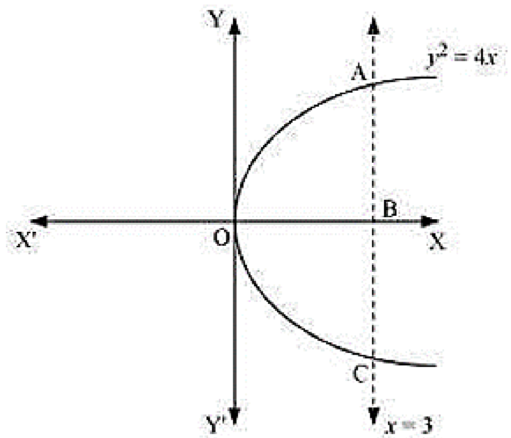
Therefore, required area $= \left(\frac{5}{6} + \frac{7}{24}\right) = \frac{9}{8}$ sq. units

Question 11:

Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

Solution 11:

The region bounded by the parabola, $y^2 = 4x$, and the line, $x = 3$, is the area OACO



The area OACO is symmetrical about x-axis.

∴ Area of OACO = 2 (Area of OABO)

Area OACO = $2\left[\int_0^3 y\,dx\right]$

$$= 2\left[\int_0^3 2\sqrt{x}dx\right]$$

$$= 4\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3$$

$$= \frac{8}{3}\left[(3)^{\frac{3}{2}}\right]$$

$$= 8\sqrt{3}$$

Therefore, the required area is $8\sqrt{3}$ sq. units.

Question 12:

Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

A. π

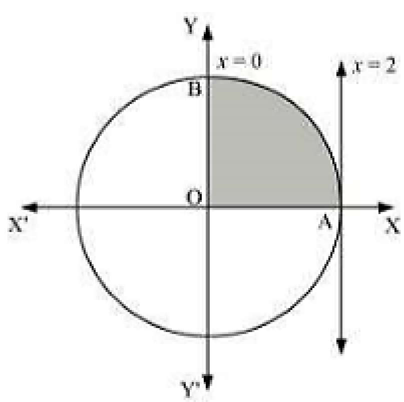
B. $\frac{\pi}{2}$

C. $\frac{\pi}{3}$

D. $\frac{\pi}{4}$

Solution 12:

The area bounded by the circle and the lines, $x = 0$ and $x = 2$, in the first quadrant is represented as



$\therefore$ Area OAB $= \int_0^2 y\,dx$

$= \int_0^2 \sqrt{4-x^2}\,dx$

$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$

$= 2\left(\frac{\pi}{2}\right)$

$= \pi$ units

Alternate Solution:

Area OABO = ½ Area of circle

Radius = 2

Area OABO $= \frac{1}{4} \times \pi \times 2^2 = \pi$ sq. units

Thus, the correct answer is A.

Question 13:

Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is

A. 2

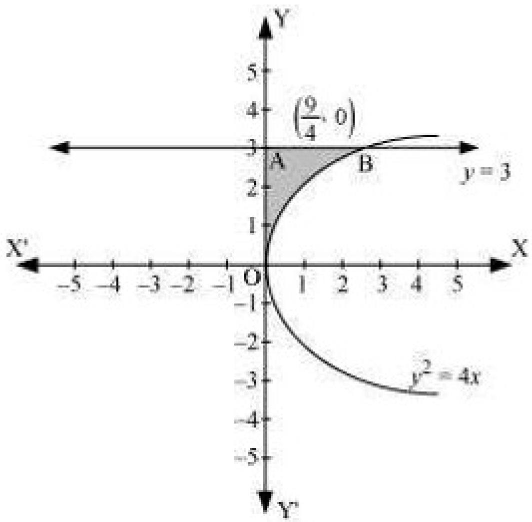
B. $\frac{9}{4}$

C. $\frac{9}{3}$

D. $\frac{9}{2}$

Solution 13:

The area bounded by the curve $y^2 = 4x$, y-axis, and $y = 3$ is represented as



$\therefore$ Area OABO $= \int_0^3 x\,dy$

$= \int_0^3 \frac{y^2}{4} dy$

$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^3$

$= \frac{1}{12}(27)$

$= \frac{9}{4}$ sq. units

Thus, the correct answer is B.

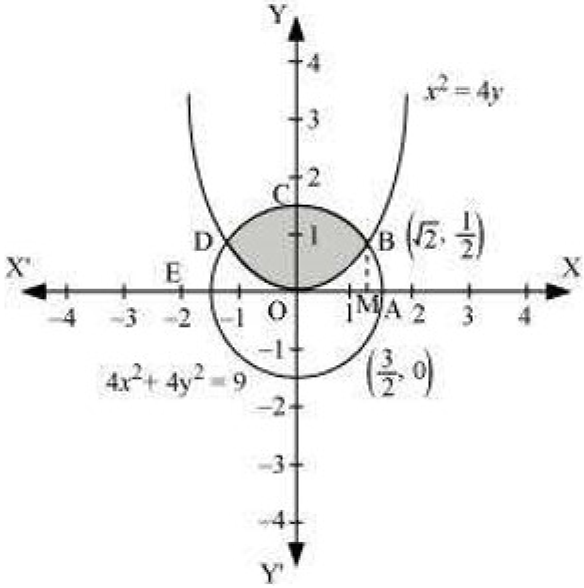
EXERCISE- 8.2

Question 1:

Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$

Solution 1:

The required area is represented by the shaded area OBCDO.



Solving the given equation of circle, $4x^2 + 4y^2 = 9$, and parabola, $x^2 = 4y$,

$4(4y) + 4y^2 = 9$

$4y^2 + 16y - 9 = 0$

$$y = \frac{-16 \pm \sqrt{16^2 - 4*4*(-9)}}{2*4}$$

$y = ½ \text{ or } -4.5$

The value of y cannot be negative as $x^2 = 4y$.

We obtain the point of intersection as B $\left(\sqrt{2}, \frac{1}{2}\right)$ and D $\left(-\sqrt{2}, \frac{1}{2}\right)$.

It can be observed that the required area is symmetrical about y-axis.

∴ Area OBCDO = 2 × Area OBCO

We draw BM perpendicular to OA.

Therefore, the coordinates of M are $\left(\sqrt{2}, 0\right)$.

Therefore, Area OBCO = Area OMBCO – Area OMBO

Area OMBCO = Area under the circle $4x^2+4y^2=9$ between $x=0$ & $x=\sqrt{2}$

Area OMBCO = $\int_0^{\sqrt{2}}\sqrt{\frac{9-4x^2}{4}}dx$

$=\int_0^{\sqrt{2}}\sqrt{\frac{9}{4}-x^2}dx$

substitute $x=\frac{3}{2}\sin\theta,\ \text{dx}=\frac{3}{2}\cos\theta d\theta$

$\int\sqrt{\frac{9}{4}-\frac{9}{4}\sin^2\theta}.\frac{3}{2}\cos\theta d\theta$

$=\frac{9}{4}\int\sqrt{1-\sin^2\theta}\cos\theta d\theta=\frac{9}{4}\int\cos^2\theta d\theta$

$=\frac{9}{4}\int\frac{1+\cos2\theta}{2}d\theta$

$=\frac{9}{4}\int\frac{1}{2}+\frac{\cos2\theta}{2}d\theta$

$=\frac{9}{8}\left[0+\frac{\sin2\theta}{2}\right]$

$=\frac{9}{8}\left[\sin^{-1}\frac{2x}{3}+\frac{\cancel{2}.\sin\theta\cos\theta}{\cancel{2}}\right]$

$=\frac{9}{8}\left[\sin^{-1}\frac{2x}{3}+\frac{2x}{3}\sqrt{1-\frac{4x^2}{9}}\right]$

$=\frac{9}{8}\left[\sin^{-1}\frac{2x}{3}+\frac{2x}{9}\sqrt{9-4x^2}\right]$

Applying the limits

$=\frac{9}{8}\left[\sin^{-1}\frac{2x}{3}+\frac{2x}{9}\sqrt{9-4x^2}\right]_0^{\sqrt{2}}$

$=\frac{9}{8}\left[\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)+\frac{2\sqrt{2}}{9}\right]$

Area OMBCO $= \frac{1}{4}\left[\sqrt{2} + \frac{9}{2}\sin^{-1}\frac{2\sqrt{2}}{3}\right]$ ...(1)

Area OMBCO = Area under $x^2 = 4y$ between $x = 0$ and $x = \sqrt{2}$

Area OMBO $= \int_0^{\sqrt{2}} \frac{x^2}{4}\,dx$

Area OMBO $= \frac{1}{4}\left[\frac{x^3}{3}\right]_0^{\sqrt{2}}$

Area OMBO$= \frac{1}{12}\left[2\sqrt{2}\right] = \frac{\sqrt{2}}{6}$ ...(2)

From (1) and (2)

Area OBCO =

$$= \frac{\sqrt{2}}{4} + \frac{9}{8}\sin^{-1}\frac{2\sqrt{2}}{3} - \frac{\sqrt{2}}{6}$$

$$= \frac{\sqrt{2}}{12} + \frac{9}{8}\sin^{-1}\frac{2\sqrt{2}}{3}$$

$$= \frac{1}{2}\left(\frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right)$$

Therefore, the required area OBCDO is

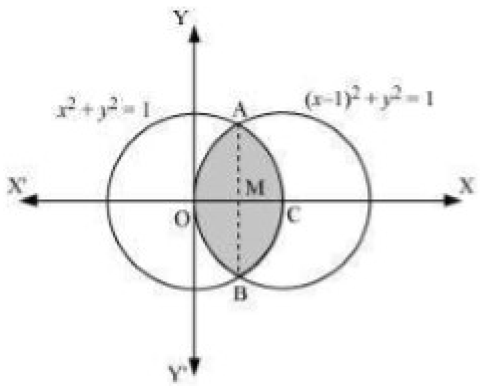
$$\left(2 \times \frac{1}{2}\left[\frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right]\right) = \left[\frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right] \text{sq. units}$$

Question 2:

Find the area bounded by curves $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$

Solution 2:

The area bounded by the curves, $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$, is represented by the shaded area as



On solving the equations, $(x-1)^2+y^2=1$ and $x^2+y^2=1$,

$(x-1)^2+1-x^2=1$

$-2x+1=0$

$x=1/2$

$y^2=1-x^2=1-\frac{1}{4}=\frac{3}{4}$

we obtain the point of intersection as A $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and B $\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

It can be observed that the required area is symmetrical about x-axis.
∴ Area OBCAO = 2 × Area OCAO
We join AB, which intersects OC at M, such that AM is perpendicular to OC.

The coordinates of M are $\left(\frac{1}{2},0\right)$

$\Rightarrow Area\ OCAO = Area\ OMAO + Area\ MCAM$

$$=\left[\int_0^{\frac{1}{2}}\sqrt{1-(x-1)^2}\,dx+\int_{\frac{1}{2}}^{1}\sqrt{1-x^2}\,dx\right]$$

$$=\left[\frac{x-1}{2}\sqrt{1-(x-1)^2}+\frac{1}{2}\sin^{-1}(x-1)\right]_0^{\frac{1}{2}}+\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]_{\frac{1}{2}}^{1}$$

$$=\left[-\frac{1}{4}\sqrt{1-\left(-\frac{1}{2}\right)^2}+\frac{1}{2}\sin^{-1}\left(\frac{1}{2}-1\right)-\frac{1}{2}\sin^{-1}(-1)\right]+\left[\frac{1}{2}\sin^{-1}(+1)-\frac{1}{4}\sqrt{1-\left(+\frac{1}{2}\right)^2}-\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$=\left[-\frac{\sqrt{3}}{8}+\frac{1}{2}\left(-\frac{\pi}{6}\right)-\frac{1}{2}\left(-\frac{\pi}{2}\right)\right]+\left[\frac{1}{2}\left(\frac{\pi}{2}\right)-\frac{\sqrt{3}}{8}-\frac{1}{2}\left(\frac{\pi}{6}\right)\right]$$

$$=\left[-\frac{\sqrt{3}}{4}-\frac{\pi}{12}+\frac{\pi}{4}+\frac{\pi}{4}-\frac{\pi}{12}\right]$$

$$=\left[-\frac{\sqrt{3}}{4}-\frac{\pi}{6}+\frac{\pi}{2}\right]$$

$$=\left[\frac{2\pi}{6}-\frac{\sqrt{3}}{4}\right]$$

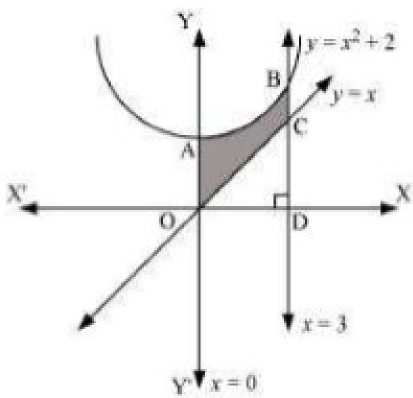
Therefore, required area OBCAO = $2\times\left(\frac{2\pi}{6}-\frac{\sqrt{3}}{4}\right)=\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$ sq. units

Question 3:

Find the area of the region bounded by the curves $y=x^2+2$, $y=x$, $x=0$ and $x=3$.

Solution 3:

The area bounded by the curves, $y=x^2+2$, $y=x$, $x=0$, and $x=3$, is represented by the shaded area OCBAO as



Then, Area OCBAO = Area ODBAO – Area ODCO

$$=\int_0^3\left(x^2+2\right)dx-\int_0^3 xdx$$

$$=\left[\frac{x^3}{3}+2x\right]_0^3-\left[\frac{x^2}{2}\right]_0^3$$

$=[9+6]-\left[\frac{9}{2}\right]$

$=15-\frac{9}{2}$

$=\frac{21}{2}\text{sq. units}$

Question 4:

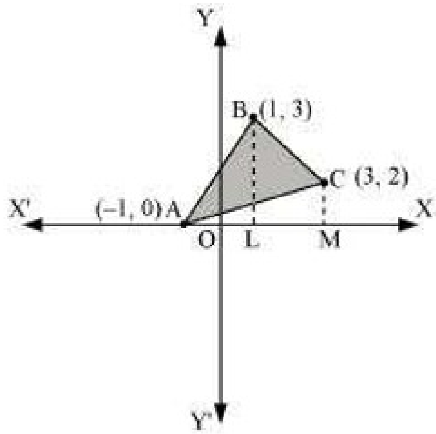
Using integration find the area of the region bounded by the triangle whose vertices are (–1, 0), (1, 3) and (3, 2).

Solution 4:

BL and CM are drawn perpendicular to x-axis.

It can be observed in the following figure that,

Area (ΔACB) = Area (ALBA) + Area (BLMCB) - Area (AMCA) ...(1)



Equation of line segment AB is

$y-0=\frac{3-0}{1+1}(x+1)$

$y=\frac{3}{2}(x+1)$

$\therefore \text{Area}(\text{ALBA})=\int_{-1}^{1}\frac{3}{2}(x+1)dx=\frac{3}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{1}=\frac{3}{2}\left[\frac{1}{2}+1-\frac{1}{2}+1\right]=3\text{ sq. units}$

Equation of line segment BC is

$$y-3=\frac{2-3}{3-1}(x-1)$$

$$y=\frac{1}{2}(-x+7)$$

$$\therefore \text{Area}(\text{BLMCB})=\int_1^3 \frac{1}{2}(-x+7)\,dx=\frac{1}{2}\left[-\frac{x^2}{2}+7x\right]_1^3=\frac{1}{2}\left[-\frac{9}{2}+21+\frac{1}{2}-7\right]=5 \text{ sq. units}$$

Equation of line segment AC is

$$y-0=\frac{2-0}{3+1}(x+1)$$

$$y=\frac{1}{2}(x+1)$$

$$\therefore \text{Area}(\text{AMCA})=\frac{1}{2}\int_{-1}^3 (x+1)\,dx=\frac{1}{2}\left[\frac{x^2}{2}+x\right]_{-1}^3=\frac{1}{2}\left[\frac{9}{2}+3-\frac{1}{2}+1\right]=4 \text{ sq. units}$$

Therefore, from equation (1), we obtain

Area $(\Delta ABC)=(3+5-4)=4$ sq. units

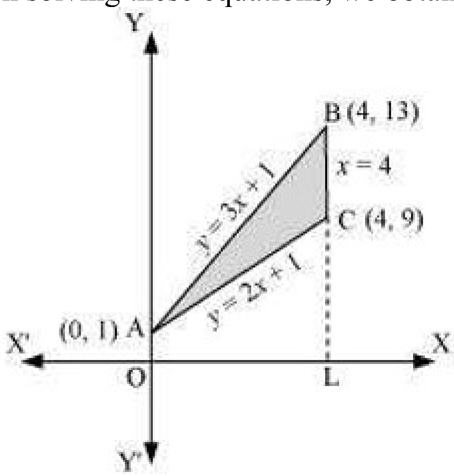
Question 5:

Using integration find the area of the triangular region whose sides have the equations $y=2x+1$, $y=3x+1$ and $x=4$.

Solution 5:

The equations of sides of the triangle are $y=2x+1$, $y=3x+1$ and $x=4$.

On solving these equations, we obtain the vertices of triangle as A(0, 1), B(4, 13), and C (4, 9).



It can be observed that,

Area (ΔACB) = Area (OLBAO) –Area (OLCAO)

$$= \int_0^4 (3x+1)\,dx - \int_0^4 (2x+1)\,dx$$

$$= \left[\frac{3x^2}{2} + x\right]_0^4 - \left[\frac{2x^2}{2} + x\right]_0^4$$

$$= (24+4) - (16+4)$$

$$= 28 - 20$$

$$= 8 \text{ sq. units}$$

Question 6:

Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

A. $2(\pi - 2)$

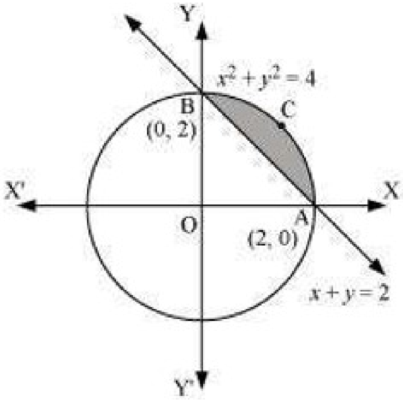
B. $\pi - 2$

C. $2\pi - 1$

D. $2(\pi + 2)$

Solution 6:

The smaller area enclosed by the circle, $x^2 + y^2 = 4$, and the line, $x + y = 2$, is represented by the shaded area ACBA as



It can be observed that,

Area ACBA = Area OACBO – Area (ΔOAB)

$$= \int_0^2 \sqrt{4-x^2}\,dx - \int_0^2 (2-x)\,dx$$

$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 - \left[2x - \frac{x^2}{2}\right]_0^2$$

$$= \left[2.\frac{\pi}{2}\right] - [4-2]$$

$$= (\pi - 2) \text{sq. units}$$

Thus, the correct answer is B.

Question 7:

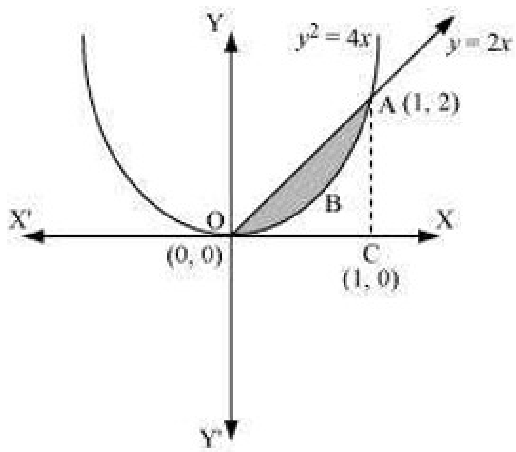
Area lying between the curves $y^2 = 4x$ and $y = 2x$ is

A. $\frac{2}{3}$ C. $\frac{1}{4}$

B. $\frac{1}{3}$ D. $\frac{3}{4}$

Solution 7:

The area lying between the curves, $y^2 = 4x$ and $y = 2x$, is represented by the shaded area OBAO as



The points of intersection of these curves are O (0, 0) and A (1, 2).
We draw AC perpendicular to x-axis such that the coordinates of C are (1, 0).
∴ Area OBAO = Area (ΔOCA) – Area (OCABO)

$$= \int_0^1 2x dx - \int_0^1 2\sqrt{x} dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^1 - 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$$

$$= \left|1 - \frac{4}{3}\right|$$

$$= \left|-\frac{1}{3}\right|$$

$$= \frac{1}{3} \text{ sq.units}$$

Thus, the correct answer is B.

Miscellaneous Exercise

Question 1:

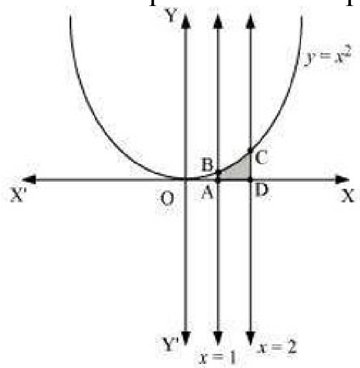
Find the area under the given curves and given lines:

(i) $y = x^2, x = 1, x = 2$ and x-axis

(ii) $y = x^4, x = 1, x = 5$ and x-axis

Solution 1:

i. The required area is represented by the shaded area ADCBA as



Area ADCBA = $\int_1^2 y dx$

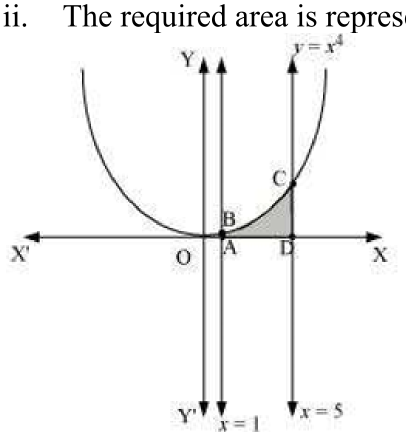
$= \int_1^2 x^2 dx$

$= \left[\frac{x^3}{3}\right]_1^2$

$= \frac{8}{3} - \frac{1}{3}$

$= \frac{7}{3}$ sq.units

ii. The required area is represented by the shaded area ADCBA as



Area of ADCBA = $\int_1^5 x^4 dx$

$= \left[\frac{x^5}{5}\right]_1^5$

$= \frac{(5)^5}{5} - \frac{1}{5}$

$= (5)^4 - \frac{1}{5}$

$= 625 - \frac{1}{5}$

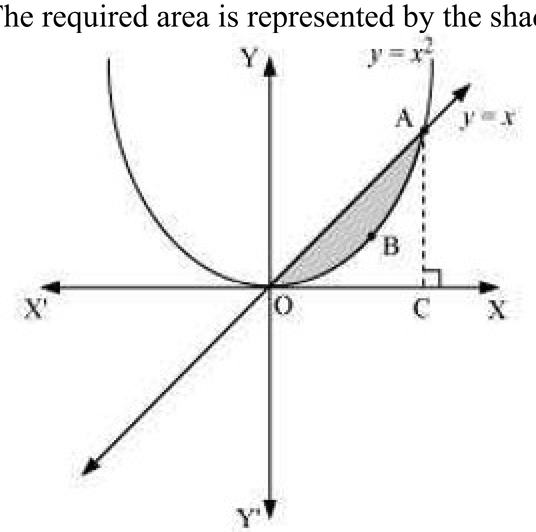
$= 624.8$ sq.units

Question 2:

Find the area between the curves $y = x$ and $y = x^2$

Solution 2:

The required area is represented by the shaded area OBAO as



The points of intersection of the curves, $y = x$ and $y = x^2$, is A (1, 1).

We draw AC perpendicular to x-axis.

∴ Area (OBAO) = Area (ΔOCA) – Area (OCABO) … (1)

$= \int_0^1 x dx - \int_0^1 x^2 dx$

$= \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1$

$$= \frac{1}{2} - \frac{1}{3}$$

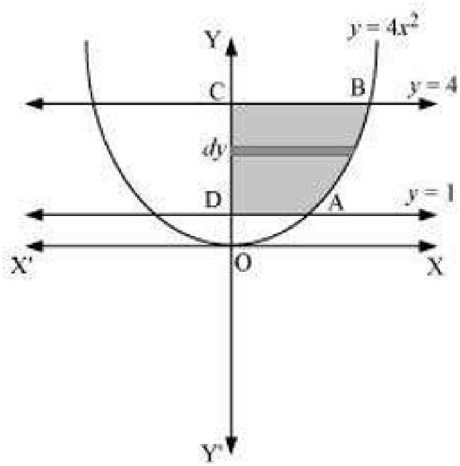
$$= \frac{1}{6} \text{ sq.units}$$

Question 3:

Find the area of the region lying in the first quadrant and bounded by $y = 4x^2$, $x = 0$, $y = 1$ and $y = 4$

Solution 3:

The area in the first quadrant bounded by $y = 4x^2$, $x = 0$, $y = 1$, and $y = 4$ is represented by the shaded area ABCDA as



$$\text{Area ABCDA} = \int_1^4 x dy$$

$$= \int_1^4 \frac{1}{2}\sqrt{y} dy$$

$$= \frac{1}{2}\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$$

$$= \frac{1}{3}\left[(4)^{\frac{3}{2}} - 1\right]$$

$= \frac{1}{3}[8-1]$

$= \frac{7}{3}$ sq.units

Question 4:

Sketch the graph of $y = |x+3|$ and evaluate $\int_{-6}^{0} |x+3|\,dx$

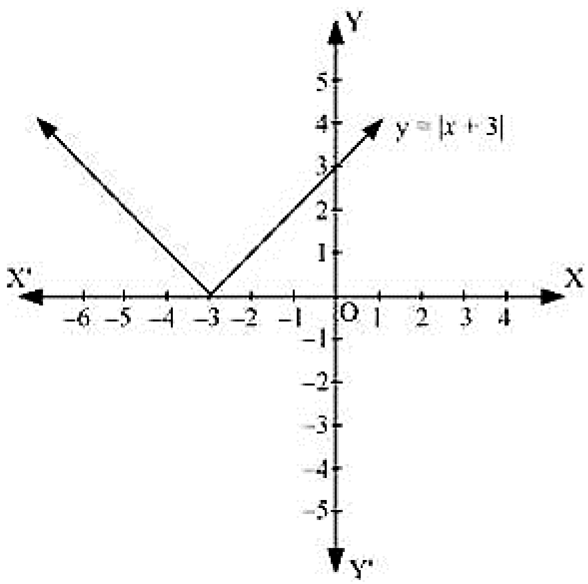
Solution 4:

The given equation is $y = |x+3|$

The corresponding values of x and y are given in the following table.

x	– 6	– 5	– 4	– 3	– 2	– 1	0
y	3	2	1	0	1	2	3

On plotting these points, we obtain the graph of $y = |x+3|$ as follows.



It is known that, $(x+3) \le 0$ for $-6 \le x \le -3$ and $(x+3) \ge 0$ for $-3 \le x \le 0$

$$\therefore \int_{-6}^{0} |(x+3)|\,dx = -\int_{-6}^{-3} (x+3)\,dx + \int_{-3}^{0} (x+3)\,dx$$

$$= -\left[\frac{x^2}{2} + 3x\right]_{-6}^{-3} + \left[\frac{x^2}{2} + 3x\right]_{-3}^{0}$$

$$= -\left[\left(\frac{(-3)^2}{2}+3(-3)\right)-\left(\frac{(-6)^2}{2}+3(-6)\right)\right]+\left[0-\left(\frac{(-3)^2}{2}+3(-3)\right)\right]$$

$$= -\left[-\frac{9}{2}\right]-\left[-\frac{9}{2}\right]$$

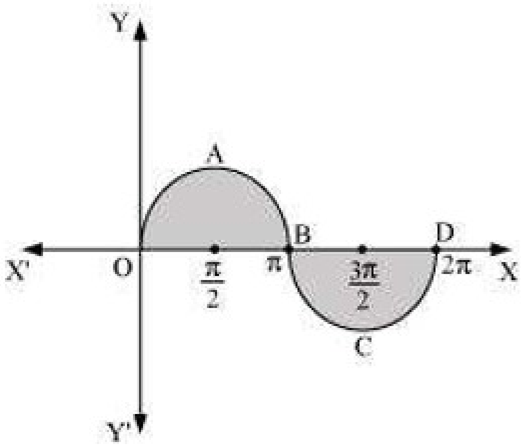
$$= 9$$

Question 5:

Find the area bounded by the curve y = sin x between x = 0 and $x = 2\pi$

Solution 5:

The graph of y = sin x can be drawn as



∴ Required area = Area OABO + Area BCDB

$$= \int_0^{\pi} \sin x dx + \left|\int_{\pi}^{2\pi} \sin x dx\right|$$

$$= [-\cos x]_0^{\pi} + \left|[-\cos x]_{\pi}^{2\pi}\right|$$

$$= [-\cos\pi + \cos 0] + |-\cos 2\pi + \cos\pi|$$

$$= 1 + 1 + |(-1-1)|$$

$$= 2 + |-2|$$

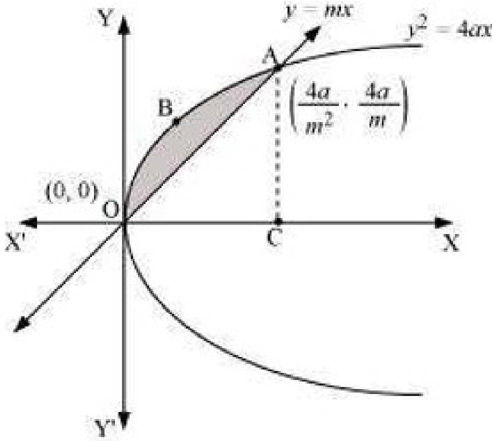
$$= 2 + 2 = 4 \text{ sq.units}$$

Question 6:

Find the area enclosed between the parabola $y^2 = 4ax$ and the line $y = mx$.

Solution 6:

The area enclosed between the parabola, $y^2 = 4ax$ and the line $y = mx.$, is represented by the shaded area OABO as



$y^2 = 4ax,\ y = mx$

$m^2x^2 = 4ax$

$m^2x^2 - 4ax = 0$

$x(m^2x - 4a) = 0$

$x = 0 \text{ or } x = 4/m^2$

The points of intersection of both the curves are (0, 0) and $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$.

We draw AC perpendicular to x-axis.

∴ Area OABO = Area OCABO – Area (ΔOCA)

$= \int_0^{\frac{4a}{m^2}} 2\sqrt{ax}dx - \int_0^{\frac{4a}{m^2}} mxdx$

$= 2\sqrt{a}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{4a}{m^2}} - m\left[\frac{x^2}{2}\right]_0^{\frac{4a}{m^2}}$

$= \frac{4}{3}\sqrt{a}\left(\frac{4a}{m^2}\right)^{\frac{3}{2}} - \frac{m}{2}\left[\left(\frac{4a}{m^2}\right)^2\right]$

$= \frac{32a^2}{3m^3} - \frac{m}{2}\left(\frac{16a^2}{m^3}\right)$

$= \frac{32a^2}{3m^3} - \frac{8a^2}{m^3}$

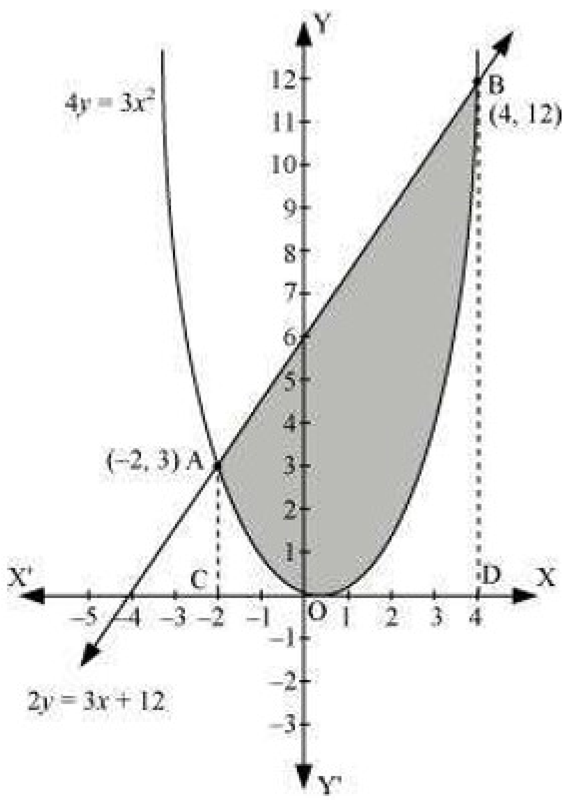
$= \frac{8a^2}{3m^3}$ sq.units

Question 7:

Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$.

Solution 7:

The area enclosed between the parabola $4y = 3x^2$ and the line $2y = 3x + 12$, is represented by the shaded area OBAO as



From the given equation of line, we have

$2y = 3x + 12$

$y = \frac{3x+12}{2}$ ...(1)

From the given equation of parabola, we have

$4y = 3x^2$

$4\left(\frac{3x+12}{2}\right) = 3x^2 \qquad [From\ (1)]$

$6x + 24 = 3x^2$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x = 4,\ x = -2$

The points of intersection of the given curves are A (–2, 3) and (4, 12).

We draw AC and BD perpendicular to x-axis.

∴ Area OBAO = Area CDBAC – (Area ODBO + Area OACO)

$= \int_{-2}^{4} \frac{1}{2}(3x+12)dx - \int_{-2}^{4} \frac{3x^2}{4}dx$

$= \frac{1}{2}\left[\frac{3x^2}{2} + 12x\right]_{-2}^{4} - \frac{3}{4}\left[\frac{x^3}{3}\right]_{-2}^{4}$

$= \frac{1}{2}[24 + 48 - 6 + 24] - \frac{1}{4}[64 + 8]$

$= \frac{1}{2}[90] - \frac{1}{4}[72]$

$= 45 - 18$

$= 27$ sq. units

Question 8:

Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$

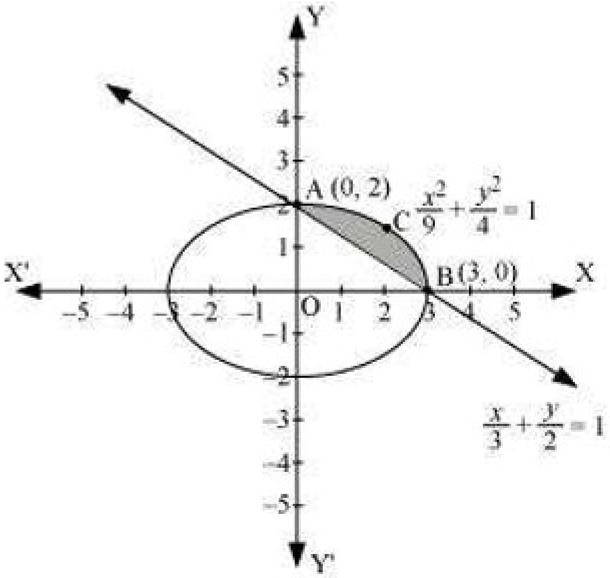
Solution 8:

The area of the smaller region bounded by the ellipse, $\frac{x^2}{9} + \frac{y^2}{4} = 1$, and the line, $\frac{x}{3} + \frac{y}{2} = 1$, is represented by the shaded region BCAB as

∴ Area BCAB = Area (OBCAO) – Area (OBAO)

$= \int_0^3 2\sqrt{1 - \frac{x^2}{9}}dx - \int_0^3 2\left(1 - \frac{x}{3}\right)dx$

$= \frac{2}{3}\left[\int_0^3 \sqrt{9 - x^2}dx\right] - \frac{2}{3}\int_0^3 (3-x)dx$



$$=\frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3-\frac{2}{3}\left[3x-\frac{x^2}{2}\right]_0^3$$

$$=\frac{2}{3}\left[\frac{9}{2}\left(\frac{\pi}{2}\right)\right]-\frac{2}{3}\left[9-\frac{9}{2}\right]$$

$$=\frac{2}{3}\left[\frac{9\pi}{4}-\frac{9}{2}\right]$$

$$=\frac{2}{3}\times\frac{9}{4}(\pi-2)$$

$$=\frac{3}{2}(\pi-2)\text{ sq.units}$$

Question 9:

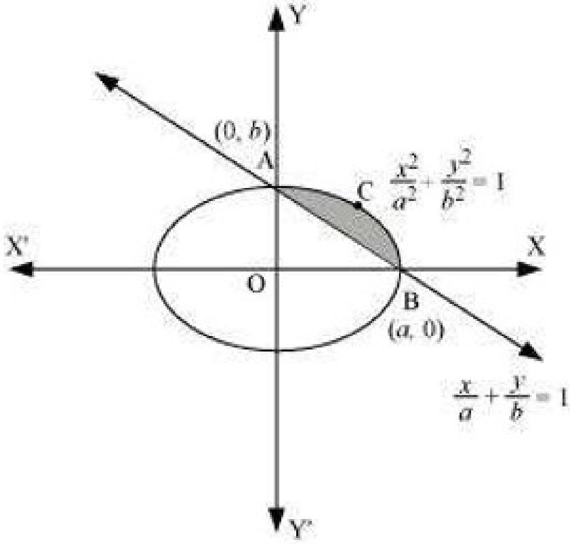
Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$

Solution 9:

The area of the smaller region bounded by the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the line, $\frac{x}{a}+\frac{y}{b}=1$, is represented by the shaded region BCAB as

∴ Area BCAB = Area (OBCAO) – Area (OBAO)

$$=\int_0^a b\sqrt{1-\frac{x^2}{a^2}}dx-\int_0^a b\left(1-\frac{x}{a}\right)dx$$



$$= \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx - \frac{b}{a}\int_0^a (a-x)\,dx$$

$$= \frac{b}{a}\left[\left\{\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right\}_0^a - \left\{ax-\frac{x^2}{2}\right\}_0^a\right]$$

$$= \frac{b}{a}\left[\left\{\frac{a^2}{2}\left(\frac{\pi}{2}\right)\right\} - \left\{a^2-\frac{a^2}{2}\right\}\right]$$

$$= \frac{b}{a}\left[\frac{a^2\pi}{4}-\frac{a^2}{2}\right]$$

$$= \frac{ba^2}{2a}\left[\frac{\pi}{2}-1\right]$$

$$= \frac{ab}{2}\left[\frac{\pi}{2}-1\right]$$

$$= \frac{ab}{4}(\pi-2)$$

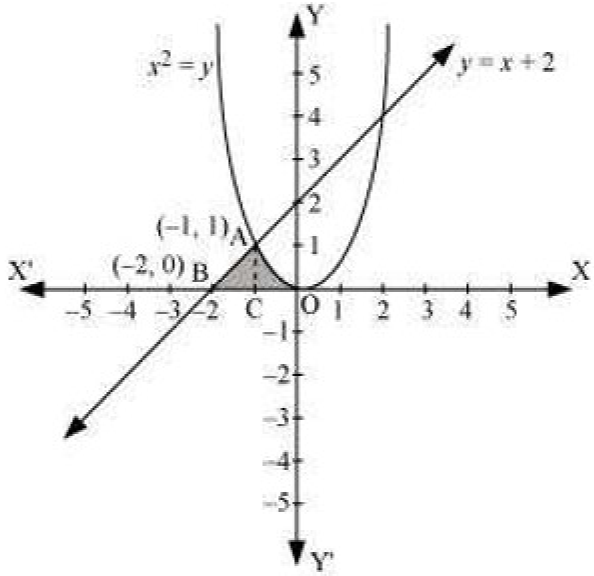
$\therefore$ The required area is $\frac{ab}{4}(\pi-2)$ sq. units

Question 10:

Find the area of the region enclosed by the parabola $x^2 = y$, the line $y = x + 2$ and x axis

Solution 10:

The area of the region enclosed by the parabola, $x^2 = y$, the line, $y = x + 2$, and x-axis is represented by the shaded region OABCO as



$y = x + 2, x^2 = y$

$\therefore x^2 = x + 2$

$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x-2)(x+1) = 0$

$\Rightarrow x = 2 \text{ or } x = -1$

The point of intersection of the parabola, $x^2 = y$, and the line, $y = x + 2$, is A (–1, 1).

$\therefore$ Area OABCO = Area (BCAB) + Area COAC

$= \int_{-2}^{-1} (x+2)\,dx + \int_{-1}^{0} x^2 dx$

$= \left[\frac{x^2}{2} + 2x\right]_{-2}^{-1} + \left[\frac{x^3}{3}\right]_{-1}^{0}$

$= \left[\frac{(-1)^2}{2} + 2(-1) - \frac{(-2)^2}{2} - 2(-2)\right] + \left[-\frac{(-1)^3}{3}\right]$

$= \left[\frac{1}{2} - 2 - 2 + 4 + \frac{1}{3}\right]$

$= \frac{5}{6}$ sq. units

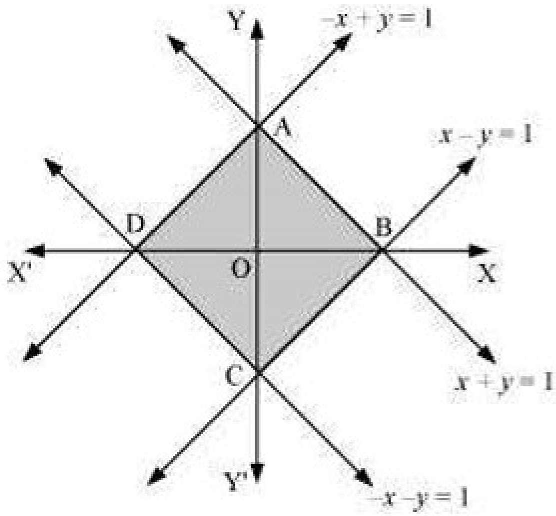
Question 11:

Using the method of integration find the area bounded by the curve $|x|+|y|=1$

[Hint: the required region is bounded by lines $x+y=1, x-y=1, -x+y=1$ and $-x-y=11$]

Solution 11:

The area bounded by the curve, $|x|+|y|=1$, is represented by the shaded region ADCBA as



The curve intersects the axes at points A (0, 1), B (1, 0), C (0, –1), and D (–1, 0).
It can be observed that the given curve is symmetrical about x-axis and y-axis.

$\therefore$ Area ADCBA $= 4 \times$ Area OBAO

$= 4\int_0^1 (1-x)\,dx$

$= 4\left(x-\frac{x^2}{2}\right)_0^1$

$= 4\left[1-\frac{1}{2}\right]$

$= 4\left(\frac{1}{2}\right)$

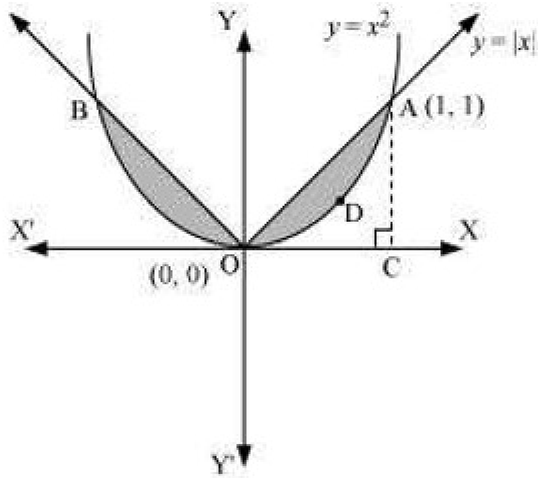
$= 2$ sq. units

Question 12:

Find the area bounded by curves $\{(x, y) : y \geq x^2$ and y=|x|$\}$

Solution 12:

The area bounded by the curves, $\{(x,y): y \geq x^2 \text{ and } y=|x|\}$, is represented by the shaded region as



It can be observed that the required area is symmetrical about y-axis.
Required area = 2 [Area (OCAO) – Area (OCADO)]

$$= 2\left[\int_0^1 x dx - \int_0^1 x^2 dx\right]$$

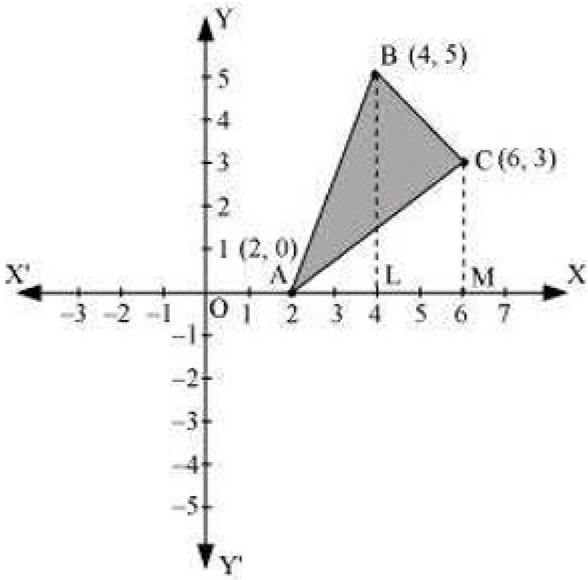
$$= 2\left[\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right]$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= 2\left[\frac{1}{6}\right] = \frac{1}{3} \text{ sq. units}$$

Question 13:
Using the method of integration find the area of the triangle ABC, coordinates of whose vertices are A (2, 0), B (4, 5) and C (6, 3)

Solution 13:
The vertices of ΔABC are A (2, 0), B (4, 5), and C (6, 3).



Equation of line segment AB is

$$y-0=\frac{5-0}{4-2}(x-2)$$

$$y=\frac{5}{2}(x-2) \qquad ...(1)$$

Equation of line segment BC is

$$y-5=\frac{3-5}{6-4}(x-4)$$

$$2y-10=-2x+8$$

$$2y=-2x+18$$

$$y=-x+9 \qquad ...(2)$$

Equation of line segment CA is

$$y-3=\frac{0-3}{2-6}(x-6)$$

$$-4y+12=-3x+18$$

$$4y=3x-6$$

$$y=\frac{3}{4}(x-2) \qquad ...(3)$$

Area (ΔABC) = Area (ABLA) + Area (BLMCB) – Area (ACMA)

$$=\int_2^4\frac{5}{2}(x-2)\,dx+\int_4^6(-x+9)\,dx-\int_2^6\frac{3}{4}(x-2)\,dx$$

$$=\frac{5}{2}\left[\frac{x^2}{2}-2x\right]_2^4+\left[\frac{-x^2}{2}+9x\right]_4^6-\frac{3}{4}\left[\frac{x^2}{2}-2x\right]_2^6$$

$$=\frac{5}{2}[8-8-2+4]+[-18+54+8-36]-\frac{3}{4}[18-12-2+4]$$

$$=5+8-\frac{3}{4}(8)$$

$$=13-6$$

$$=7 \text{ sq. units}$$

Question 14:

Using the method of integration find the area of the region bounded by lines:

$2x+y=4,\ 3x-2y=6$ and $x-3y+5=0$

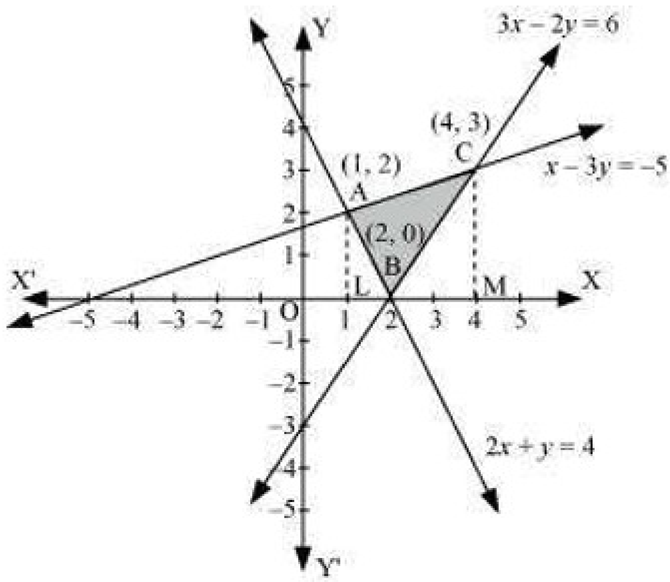
Solution 14:

The given equations of lines are

$2x+y=4$ ...(1)

$3x-2y=6$ ...(2)

And, $x-3y+5=0$ …(3)



The area of the region bounded by the lines is the area of ΔABC. AL and CM are the perpendiculars on x-axis.

Area (ΔABC) = Area (ALMCA) – Area (ALBA) – Area (CMBC)

$$=\int_1^4\left(\frac{x+5}{3}dx\right)-\int_1^2(4-2x)\,dx-\int_2^4\left(\frac{3x-6}{2}\right)dx$$

$$=\frac{1}{3}\left[\frac{x^2}{2}+5x\right]_1^4-\left[4x-x^2\right]_1^2-\frac{1}{2}\left[\frac{3x^2}{2}-6x\right]_2^4$$

$$=\frac{1}{3}\left[8+20-\frac{1}{2}-5\right]-[8-4-4+1]-\frac{1}{2}[24-24-6+12]$$

$$=\left(\frac{1}{3}\times\frac{45}{2}\right)-(1)-\frac{1}{2}(6)$$

$$=\frac{15}{2}-1-3$$

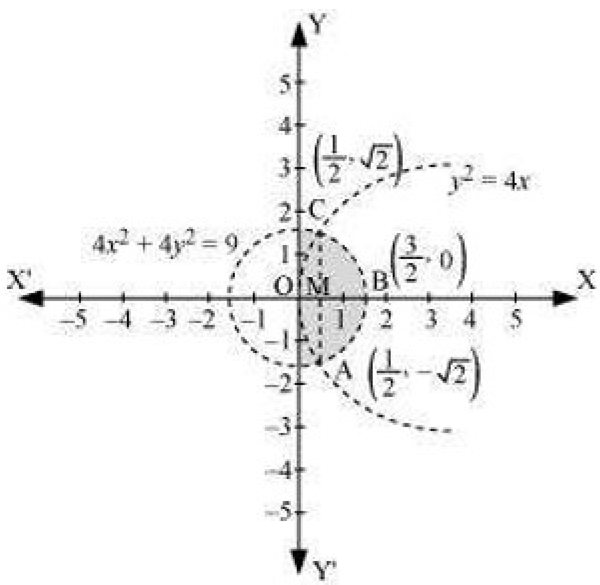
$$=\frac{15}{2}-4=\frac{15-8}{2}=\frac{7}{2}\text{ sq. units}$$

Question 15:

Find the area of the region $\left\{(x,y): y^2 \le 4x,\ 4x^2+4y^2 \le 9\right\}$

Solution 15:

The area bounded by the curves, $\left\{(x,y): y^2 \le 4x,\ 4x^2+4y^2 \le 9\right\}$ is represented as



$y^2 = 4x$

$\Rightarrow 4x^2 + 4(4x) = 9$

$\Rightarrow 4x^2 + 16x - 9 = 0$

$\Rightarrow 4x^2 + 18x - 2x - 9 = 0$

$\Rightarrow 2x(2x+9) - (2x+9) = 0$

$\Rightarrow (2x-1)(2x+9) = 0$

$\therefore x = \frac{1}{2}$ & $y = \pm\sqrt{4x} = \pm\sqrt{2}$

The points of intersection of both the curves are $\left(\frac{1}{2}, \sqrt{2}\right)$ and $\left(\frac{1}{2}, -\sqrt{2}\right)$.

The required area is given by OABCO.
It can be observed that area OABCO is symmetrical about x-axis.
$\therefore$ Area OABCO $= 2 \times$ Area OBCO

Area OBCO = Area OMCO + Area MBCM

$$= \int_0^{\frac{1}{2}} 2\sqrt{x}dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{2}\sqrt{9-4x^2}dx$$

$$= \int_0^{\frac{1}{2}} 2\sqrt{x}dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{2}\sqrt{(3)^2-(2x)^2}dx$$

Let $2x = t$, $dx = \frac{dt}{2}$

When $x = \frac{1}{2}, t = 1$ and when $x = \frac{3}{2}, t = 3$

$$\Rightarrow \int_0^{\frac{1}{2}} 2\sqrt{x}dx + \frac{1}{4}\int_1^3 \sqrt{9-t^2}dt$$

$$\Rightarrow 2\left[\frac{2}{3}x^{3/2}\right]_0^{1/2} + \frac{1}{4}\left[\frac{t}{2}\sqrt{9-t^2} + \frac{9}{2}\sin^{-1}\left(\frac{t}{3}\right)\right]_1^3$$

$$\Rightarrow \frac{4}{3}\left[\frac{1}{2\sqrt{2}}\right] + \frac{1}{4}\left[\frac{9}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\sqrt{8} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$\Rightarrow \frac{2}{3\sqrt{2}} + \frac{9\pi}{16} - \frac{1}{2\sqrt{2}} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)$$

$$\Rightarrow \frac{1}{6\sqrt{2}} + \frac{9\pi}{16} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)$$

Area of OABCD = 2 Area of OBCO

$\therefore$ Required area $= 2\left[\dfrac{1}{6\sqrt{2}} + \dfrac{9\pi}{16} - \dfrac{9}{8}\sin^{-1}\left(\dfrac{1}{3}\right)\right]$

$= \dfrac{9\pi}{8} + \dfrac{1}{3\sqrt{2}} - \dfrac{9}{4}\sin^{-1}\left(\dfrac{1}{3}\right)$ sq. units

Question 16:

Area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$ is

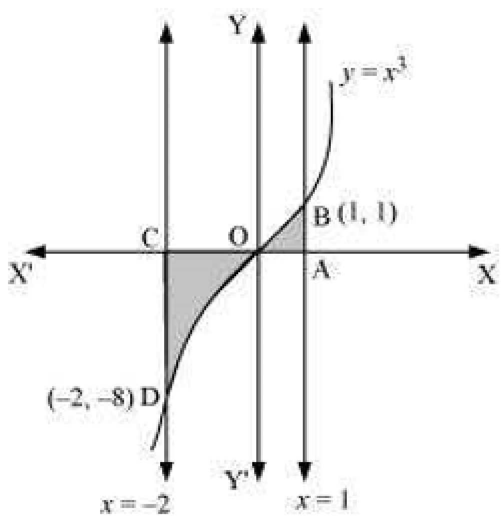
A. – 9

B. $-\dfrac{15}{4}$

C. $\dfrac{15}{4}$

D. $\dfrac{17}{4}$

Solution 16:



Required area $= \int_{-2}^{1} y\,dx$

$= \int_{-2}^{1} x^3\,dx$

$= \left[\dfrac{x^4}{4}\right]_{-2}^{1}$

$\left[\frac{1}{4}-\frac{(-2)^4}{4}\right]$

$\left(\frac{1}{4}-4\right)=-\frac{15}{4}$

$\therefore$ Area $=\left|\frac{-15}{4}\right|=\frac{15}{4}$ sq. units

Thus, the correct answer is C.

Question 17:

The area bounded by the curve $y=x|x|$, x-axis and the ordinates $x=-1$ and $x=1$ is given by

[Hint: $y=x^2$ if $x>0$ and $y=-x^2$ if $x<0$]

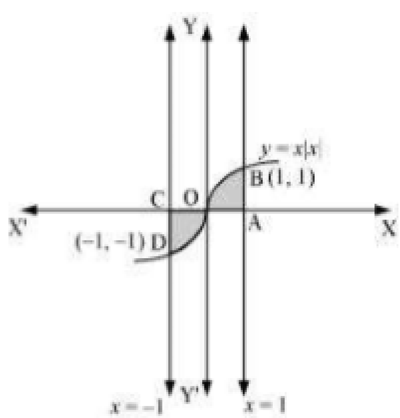
A.0

B. $\frac{1}{3}$

C. $\frac{2}{3}$

D. $\frac{4}{3}$

Solution 17:



Required area $=\int_{-1}^{1} y dx$

$= \int_{-1}^{1} x|x|\,dx = \left|\int_{-1}^{0} x^2 dx\right| + \left|\int_{0}^{1} x^2 dx\right|$

$= \left|\int_{-1}^{0} -x^2 dx\right| + \left|\int_{0}^{1} x^2 dx\right|$

$= \left|\left[\frac{-x^3}{3}\right]_{-1}^{0}\right| + \left|\left[\frac{x^3}{3}\right]_{0}^{1}\right|$

$= \left|-\left(-\frac{1}{3}\right)\right| + \frac{1}{3}$

$= \frac{2}{3}$ sq. units

Thus, the correct answer is C.

Question 18:

The area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$ is

A. $\frac{4}{3}\left(4\pi - \sqrt{3}\right)$

B. $\frac{4}{3}\left(4\pi + \sqrt{3}\right)$

C. $\frac{4}{3}\left(8\pi - \sqrt{3}\right)$

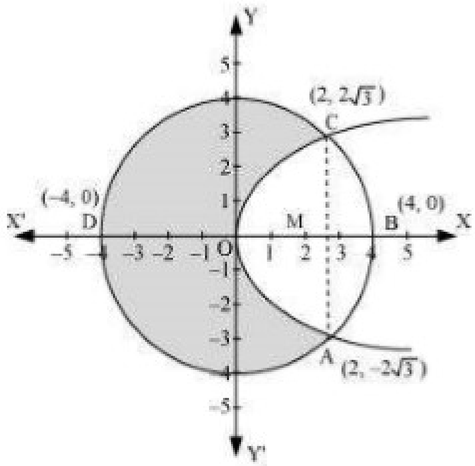
D. $\frac{4}{3}\left(8\pi + \sqrt{3}\right)$

Solution 18:

The given equations are

$x^2 + y^2 = 16$ ...(1)

$y^2 = 6x$ ...(2)



$y^2 = 6x$

$\Rightarrow x^2 + 6x = 16$

$\Rightarrow x^2 + 6x - 16 = 0$

$\Rightarrow (x+8)(x-2) = 0$

$\Rightarrow x = -8 \ \& \ x = 2$

when $x = 2 \quad y = \pm\sqrt{15} = \pm 2\sqrt{3}$

Area bounded by the circle and parabola (unshaded portion)

= Area (OCBAC)

= 2[Area (OBCO)]

= 2[Area(OMCO) + Area (BMCB)]

$$= 2\left[\int_0^2 \sqrt{16xdx} + \int_2^4 \sqrt{16-x^2}\,dx\right]$$

$$= 2\sqrt{6}\int_0^2 \sqrt{x}dx + 2\int_2^4 \sqrt{16-x^2}\,dx$$

$$= 2\sqrt{6}\times\frac{2}{3}\left[(x)^{3/2}\right]_0^2 + 2\left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_2^4$$

$$= 2\sqrt{6}\times\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^2 + 2\left[8.\frac{\pi}{2} - \sqrt{16-4} - 8\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \frac{4\sqrt{6}}{3}\left(2\sqrt{2}\right) + 2\left[4\pi - \sqrt{12} - 8\frac{\pi}{6}\right]$$

$$= \frac{16\sqrt{3}}{3} + 8\pi - 4\sqrt{3} - \frac{8}{3}\pi$$

$=\frac{4}{3}\left[4\sqrt{3}+6\pi-3\sqrt{3}-2\pi\right]$

$=\frac{4}{3}\left[\sqrt{3}+4\pi\right]$

$=\frac{4}{3}\left[4\pi+\sqrt{3}\right]$ sq. units

Area of circle = $\pi(r)^2$

$=\pi(4)^2$

$=16\pi$ sq. units

$\therefore$ Area of the circle $x^2+y^2=16$ exterior to the parabola $y^2=6x$

$=$ Area of circle $-\frac{4}{3}\left[4\pi+\sqrt{3}\right]$

$=\pi(4)^2-\frac{4}{3}\left[4\pi+\sqrt{3}\right]$

$=\frac{4}{3}\left[4\times3\pi-4\pi-\sqrt{3}\right]$

$=\frac{4}{3}\left(8\pi-\sqrt{3}\right)$ sq. units

Thus, the correct answer is C.

Question 19:

The area bounded by the y-axis, y = cos x and y = sin x when $0\le x\le\frac{\pi}{2}$

A. $2\left(\sqrt{2}-1\right)$

B. $\sqrt{2}-1$

C. $\sqrt{2}+1$

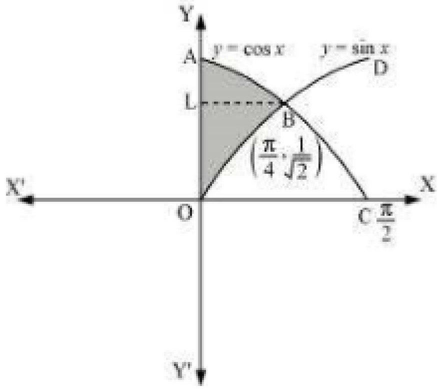
D. $\sqrt{2}$

Solution 19:

The given equations are

y = cos x … (1)

And, y = sin x … (2)



Required area = Area (ABLA) + area (OBLO)

$$= \int_{\frac{1}{\sqrt{2}}}^{1} x\,dy + \int_{0}^{\frac{1}{\sqrt{2}}} x\,dy$$

$$= \int_{\frac{1}{\sqrt{2}}}^{1} \cos^{-1} y\,dy + \int_{\frac{1}{\sqrt{2}}}^{1} \sin^{-1} y\,dy$$

Integrating by parts, we obtain

$$= \left[y\cos^{-1} y - \sqrt{1-y^2} \right]_{\frac{1}{\sqrt{2}}}^{1} + \left[y\sin^{-1} y + \sqrt{1-y^2} \right]_{0}^{\frac{1}{\sqrt{2}}}$$

$$= \left[\cos^{-1}(1) - \frac{1}{\sqrt{2}}\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) + \sqrt{1-\frac{1}{2}} \right] + \left[\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \sqrt{1-\frac{1}{2}} - 1 \right]$$

$$= \frac{-\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - 1$$

$$= \frac{2}{\sqrt{2}} - 1$$

$$= \sqrt{2} - 1 \text{ sq. units}$$

$\therefore$ The correct answer is option B.