differential equations(cbse) solutions

Solutions

1. Given differential equation is

$$x^2 \frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^4$$

Since, highest order derivative occurring in the differential equation is $\frac{d^2y}{dx^2}$ therefore order is 2 and as given equation can be expressed as a polynomial in derivatives so its degree is 1, which is the power of $\frac{d^2y}{dx^2}$. **(1)**

2. Do same as Q. No. 1.

[**Ans.** Order = 2 and degree = 2]

3. Since highest order derivative occuring in the differential equation is $\frac{d^2y}{dx^2}$, therefore order is 2 and as the differential equation is not a polynomial in derivatives, therefore its degree is not defined. **(1)**

4. Given, $y = ae^{2x} + 5$ ...(i)

Differentiating w.r.t. x, we get

$$y' = ae^{2x} \cdot 2 \Rightarrow ae^{2x} = \frac{y'}{2} \Rightarrow y - 5 = \frac{y'}{2}$$

[from Eq. (i)]

$$\Rightarrow \quad 2y - 10 = y' \Rightarrow y' - 2y + 10 = 0,$$

which is the required equation. **(1)**

5. Given equation of family of curves

$V = \frac{A}{r} + B$, where A and B are arbitrary constants.

On differentiating both sides w.r.t. r, we get

$$\frac{dV}{dr} = \frac{-A}{r^2} + 0 \Rightarrow \frac{dV}{dr} = \frac{-A}{r^2} \quad \text{...(i)}$$ **(1/2)**

Now, again differentiating both sides w.r.t. r, we get

$$\frac{d^2V}{dr^2} = \frac{2A}{r^3}$$

$$\Rightarrow \quad \frac{d^2V}{dr^2} = \frac{2}{r^3} \times \left(-r^2 \frac{dV}{dr}\right) \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad \frac{d^2V}{dr^2} = -\frac{2}{r}\frac{dV}{dr}$$

Thus, the required differential equation is

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0.$$ **(1/2)**

6. The degree of the differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions sign.

Given differential equation is $\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0$

$\Rightarrow \quad 3\left(\frac{dy}{dx}\right)^{3-1}\frac{d}{dx}\left(\frac{dy}{dx}\right)=0$ **(1/2)**

$\Rightarrow \quad 3\left(\frac{dy}{dx}\right)^{2}\frac{d^2y}{dx^2}=0$

Here, order = 2 and degree = 1

$\therefore$ Sum of the order and degree = 2 + 1 = 3 **(1/2)**

7. Given differential equation is

$$\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^3+x^4=0.$$

Here, we see that the highest order derivative is $\frac{d^2y}{dx^2}$, whose degree is 2.

Here, order = 2 and degree = 2 **(1/2)**

$\therefore$ Sum of the order and degree = 2 + 2 = 4 **(1/2)**

8. Given equation of family of curves is $xy = C\cos x$. ...(i)

On differentiating both sides w.r.t. x, we get

$$1\cdot y+x\frac{dy}{dx}=C(-\sin x)$$

$\Rightarrow \quad y+x\frac{dy}{dx}=-\left(\frac{xy}{\cos x}\right)\sin x$ [from Eq. (i)]

$\therefore \quad y+x\frac{dy}{dx}+xy\tan x=0$ **(1)**

9. Given differential equation is

$$\left(\frac{dy}{dx}\right)^4+3x\frac{d^2y}{dx^2}=0.$$

Here, highest order derivative is d^2y/dx^2, whose degree is one. So, the degree of differential equation is 1. **(1)**

10. Do same as Q. No. 9. **[Ans. 3]**

11. Do same as Q. No. 9. **[Ans. 1]**

12. Given family of curves is $y=mx$. ...(i)

On differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx}=m$$

On putting $m=\frac{dy}{dx}$ in Eq. (i), we get

$$y=x\frac{dy}{dx}$$

which is the required differential equation. **(1)**

13. Do same as Q. No. 9. **[Ans. 1]**

14. Given, $y=e^{2x}(a+bx)$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=e^{2x}\frac{d}{dx}(a+bx)+(a+bx)\frac{d}{dx}e^{2x}$$

$\Rightarrow \quad \frac{dy}{dx}=e^{2x}(b)+(a+bx)\,2\cdot e^{2x}$

$\Rightarrow \quad y'=b\cdot e^{2x}+2\cdot e^{2x}(a+bx)$

$\Rightarrow \quad y'=2y+be^{2x}$ ...(ii) **(1)**

Again differentiating w.r.t. x, we get

$$y''=2y'+2be^{2x} \quad \text{...(iii)}$$

On multiplying Eq. (ii) by 2 and then subtracting from Eq. (iii), we get

$$y''-2y'=2y'-4y$$

$$y''=2y'+2y'-4y$$

$$y''-4y'+4y=0,$$

which is the required equation. **(1)**

15. Given equation of family of curves is

$$y=A\cdot e^{2x}+B\cdot e^{-2x} \quad \text{...(i)}$$

Differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx}=2A\cdot e^{2x}-2B\cdot e^{-2x} \quad \text{...(ii)}$$ **(1/2)**

Again differentiating eq. (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2}=4Ae^{2x}+4Be^{-2x}$$

$$=4(Ae^{2x}+Be^{-2x})$$ **(1)**

$\Rightarrow \quad \frac{d^2y}{dx^2}=4y \Rightarrow \frac{d^2y}{dx^2}-4y=0,$

which is the required equation. **(1/2)**

16. We have, $y=ae^{bx+5}$ ...(i)

On differentiating both sides w.r.t. x, we get

$\frac{dy}{dx}=a\,e^{bx+5}\cdot b\Rightarrow\frac{dy}{dx}=by$ [using Eq. (i)] ...(ii) **(1)**

Again, on differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2}=b\frac{dy}{dx}$$

$\Rightarrow \quad \frac{d^2y}{dx^2}=\left(\frac{1}{y}\cdot\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)$ [using Eq. (ii)]

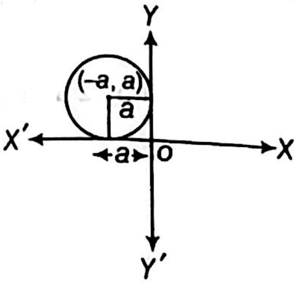
$\Rightarrow \quad \frac{d^2y}{dx^2}=\frac{1}{y}\left(\frac{dy}{dx}\right)^2$

$\Rightarrow \quad y\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2=0,$

which is the required differential equation. **(1)**

17. The equation of family of circles in second quadrant, which touch the coordinate axes, is $(x+a)^2 + (y-a)^2 = a^2$, where a is radius of circle. Differentiate it one time and eliminate the arbitrary constant a.

Let a be the radius of family of circles in the second quadrant, which touch the coordinate axes.



Then, coordinates of centre of circle = $(-a, a)$. **(1)**

We know that, equation of circle whose centre (h, k) and radius r is given by

$$(x-h)^2 + (y-k)^2 = r^2$$

Here, $(h, k) = (-a, a)$ and $r = a$

∴ Equation of family of such circles is

$$(x+a)^2 + (y-a)^2 = a^2 \quad \ldots\text{(i)} \ \textbf{(1)}$$

On differentiating both sides w.r.t. x, we get

$$2(x+a) + 2(y-a)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + a + (y-a)\cdot y' = 0 \qquad \left[\because \frac{dy}{dx} = y'\right]$$

$$\Rightarrow x + yy' = -a + ay' \Rightarrow a = \frac{x+yy'}{-1+y'}$$ **(1)**

On putting above value of a in Eq. (i), we get

$$\left[x + \frac{x+yy'}{y'-1}\right]^2 + \left[y - \frac{x+yy'}{y'-1}\right]^2 = \left(\frac{x+yy'}{y'-1}\right)^2$$

$$\Rightarrow \left[\frac{xy' - x + x + yy'}{y'-1}\right]^2 + \left[\frac{yy' - y - x - yy'}{y'-1}\right]^2 = \left(\frac{x+yy'}{y'-1}\right)^2$$

On multiplying both sides by $(y'-1)^2$, we get

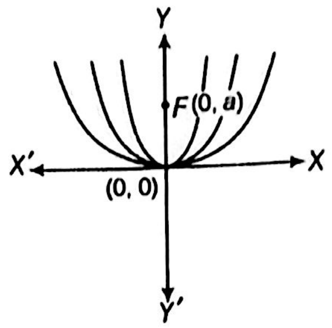
$$(xy' + yy')^2 + (y+x)^2 = (x+yy')^2$$

$$\Rightarrow (x+y)^2(y')^2 + (x+y)^2 = (x+yy')^2$$

$$\therefore \quad (x+y)^2[(y')^2 + 1] = (x+yy')^2$$

which is the required differential equation. **(1)**

18. We know that, equation of parabola having vertex at origin and axis along positive Y-axis is $x^2 = 4ay$, where a is the parameter. ...(i) **(1)**



On differentiating Eq. (i) w.r.t. 'x', we get

$$2x = 4ay' \qquad \left[\text{where } y' = \frac{dy}{dx}\right]$$

$$\Rightarrow \quad 4a = \frac{2x}{y'} \quad \ldots\text{(ii)} \ \textbf{(1½)}$$

On substituting the value of $4a$ from Eq. (ii) to Eq. (i), we get

$$x^2 = \frac{2x}{y'}y$$

$\Rightarrow xy' - 2y = 0$, which is the required differential equation. **(1½)**

19. The equation of family of circles touching Y-axis at origin is given by $(x-a)^2 + y^2 = a^2$, where a is radius of circle. Differentiate this equation once, as one arbitrary constant is present in the equation and eliminate a.

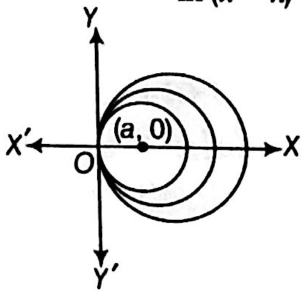
Lat a be the radius of family of circles which touch Y-axis at origin.

∴ Centre of circle = $(a, 0)$ **(1)**

Now, equation of family of circles with centre $(a, 0)$ and radius a is

$$(x-a)^2 + y^2 = a^2$$

[putting $(h, k) = (a, 0)$ and $r = a$ in $(x-h)^2 + (y-k)^2 = r^2$]



(1)

$$\Rightarrow \quad x^2 + a^2 - 2ax + y^2 = a^2$$

$$\Rightarrow \quad x^2 - 2ax + y^2 = 0 \quad \ldots\text{(i)}$$

On differentiating both sides w.r.t. x, we get

$$2x - 2a + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad a = x + y\frac{dy}{dx}$$ **(1)**

On putting above value of a in Eq. (i), we get

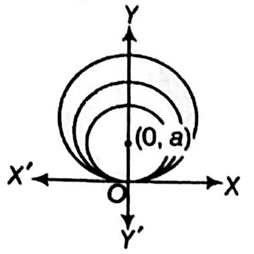
$$x^2 + y^2 - 2\left(x + y\frac{dy}{dx}\right)x = 0$$

$$\Rightarrow \quad x^2 + y^2 - 2x^2 - 2xy\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2xy\frac{dy}{dx} + x^2 - y^2 = 0$$

which is the required differential equation. **(1)**

20. Let a be the radius of family of circles which touch X-axis at origin.



$\therefore$ Centre of circle = $(0, a)$

Now, equation of family of such circles is

$$x^2 + (y - a)^2 = a^2 \quad \textbf{(1)}$$

[putting $(h, k) = (0, a)$ and $r = a$ in $(x - h)^2 + (y - k)^2 = r^2$]

$$\Rightarrow \quad x^2 + y^2 - 2ay = 0 \quad \ldots(i)$$

On differentiating both sides w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + y\frac{dy}{dx} - a\frac{dy}{dx} = 0 \quad \text{[divide by 2]} \quad \textbf{(1)}$$

$$\Rightarrow \quad x + yy' - ay' = 0 \quad \left[\text{where, } y' = \frac{dy}{dx}\right]$$

$$\Rightarrow \quad a = \frac{x + yy'}{y'} \quad \textbf{(1)}$$

On putting above value of a in Eq. (i), we get

$$x^2 + y^2 - 2y\left(\frac{x + yy'}{y'}\right) = 0$$

$$\Rightarrow \quad x^2y' + y^2y' - 2xy - 2y^2y' = 0$$

$$\Rightarrow \quad x^2y' - 2xy - y^2y' = 0$$

$$\Rightarrow \quad y'(x^2 - y^2) = 2xy$$

$$\therefore \quad y' = \frac{2xy}{x^2 - y^2} \quad \text{or} \quad \frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

which is the required differential equation. **(1)**

21. Do same as Q. No. 17.

Hint Equation of family of circles in the first quadrant which touch the coordinate axes is

$$(x - a)^2 + (y - a)^2 = a^2$$

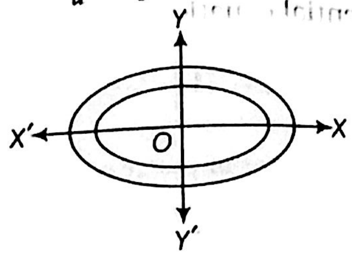
$$\left[\textbf{Ans. } (x - y)^2[(y')^2 + 1] = (x + yy')^2\right]$$

22. The equation of family of ellipses having foci on X-axis and centre at origin is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b.$

Differentiate this equation two times and eliminate two arbitrary constants a and b to get the required result.

We know that, the equation of family of ellipse having foci on X-axis and centre at origin is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } a > b \quad \ldots(i) \textbf{(1)}$$



On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \quad \left[\text{put } \frac{dy}{dx} = y'\right]$$

$$\Rightarrow \quad \frac{x}{a^2} = \frac{-yy'}{b^2}$$

$$\Rightarrow \quad \frac{yy'}{x} = \frac{-b^2}{a^2} \quad \ldots(ii) \textbf{(1)}$$

Again, on differentiating both sides of Eq. (ii) w.r.t. x, we get

$$\frac{\left[x \cdot \frac{d}{dx}(yy') - yy' \cdot \frac{d}{dx}(x)\right]}{x^2} = 0$$

$\left[\text{using quotient rule of derivative in LHS and } \frac{d}{dx}\left(\frac{-b^2}{a^2}\right) = 0\right]$

$$\Rightarrow \quad x\left[y \cdot \frac{d}{dx}(y') + y' \cdot \frac{d}{dx}(y)\right] - yy' \cdot 1 = 0 \quad \textbf{(1)}$$

$$\Rightarrow \quad x[yy'' + y'y'] - yy' = 0$$

$$\left[\because \frac{d}{dx}(y') = y'' \text{ and } \frac{d}{dx}(y) = y'\right]$$

$$\Rightarrow \quad xyy'' + x(y')^2 - yy' = 0$$

$$\therefore \quad xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

which is the required differential equation. **(1)**

Solutions

1. > First, write the given differential equation in the form of $\frac{dy}{dx} + Py = Q$. Then, determine integrating factor by using formula, $\text{IF} = e^{\int P dx}$.

Given differential equation can be rewritten as

$$\frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, here $P = \frac{1}{\sqrt{x}}$ and $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$.

$\therefore$ Integrating Factor, $\text{IF} = e^{\int P dx} = e^{\int \frac{1}{\sqrt{x}} dx} = e^{2\sqrt{x}}$ [1]

2. Given differential equation is

$$(1 + y^2) + (2xy - \cot y)\frac{dy}{dx} = 0$$

The above equation can be rewritten as

$$(\cot y - 2xy)\frac{dy}{dx} = 1 + y^2$$

$$\Rightarrow \quad \frac{\cot y - 2xy}{(1 + y^2)} = \frac{dx}{dy}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\cot y}{1 + y^2} - \frac{2xy}{1 + y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{2y}{1+y^2} \cdot x = \frac{\cot y}{1+y^2}$$ **(1/2)**

which is a linear differential equation of the form $\frac{dx}{dy} + Px = Q$, here $P = \frac{2y}{1+y^2}$ and $Q = \frac{\cot y}{1+y^2}$.

Now, integrating factor $= e^{\int P\,dy} = e^{\int \frac{2y}{1+y^2} dy}$

Put $1 + y^2 = t$

$$\Rightarrow \quad 2y\,dy = dt$$

$$\therefore \quad \text{IF} = e^{\int \frac{dt}{t}} = e^{\log|t|} = t = 1 + y^2$$ **(1/2)**

3. Given differential equation is

$$\frac{dy}{dx} = 2^{-y}$$

On separating the variables, we get

$$2^y\,dy = dx$$

On integrating both sides, we get

$$\int 2^y\,dy = \int dx$$

$$\Rightarrow \quad \frac{2^y}{\log 2} = x + C_1$$

$$\Rightarrow \quad 2^y = x\log 2 + C_1 \log 2$$

$$\therefore \; 2^y = x\log 2 + C,$$

where $C = C_1 \log 2$ **(1)**

4. Given differential equation is

$$\frac{dy}{dx} = x^3 e^{-2y}$$

On separating the variables, we get

$$e^{2y}dy = x^3 dx$$

On integrating both sides, we get

$$\int e^{2y}dy = \int x^3 dx$$

$$\Rightarrow \quad \frac{e^{2y}}{2} = \frac{x^4}{4} + C_1 = 2e^{2y} = x^4 + 4C_1$$

$$\therefore \quad 2e^{2y} = x^4 + C, \text{ where } C = 4C_1$$ **(1)**

5. The given differential equation is

$$\frac{dy}{dx} = e^{x+y} \Rightarrow \frac{dy}{dx} = e^x \cdot e^y$$

$$\Rightarrow \; dy = e^x \cdot e^y dx \Rightarrow e^{-y}dy = e^x dx$$ **(1)**

$$\Rightarrow \int e^{-y}dy = \int e^x dx$$

$$\Rightarrow \quad -e^{-y} = e^x + C$$

which is the required solution. **(1)**

6. Given equation is $\cos\left(\frac{dy}{dx}\right) = a$

which can be rewritten as $\frac{dy}{dx} = \cos^{-1} a$ **(1)**

$$\Rightarrow \quad dy = \cos^{-1} a\,dx$$

$$\Rightarrow \quad \int dy = \int \cos^{-1} a\,dx$$

$$\Rightarrow \quad y = \cos^{-1} a \cdot x + C$$

which is the required solution. **(1)**

7. We have, $(x+1)\frac{dy}{dx} = 2e^{-y} - 1$

$$\Rightarrow \quad (x+1)\,dy = (2e^{-y} - 1)\,dx$$

$$\Rightarrow \quad \frac{1}{x+1}dx = \frac{1}{2e^{-y}-1}dy$$

[separating the variables]

$$\Rightarrow \quad \int \frac{1}{x+1}dx = \int \frac{1}{2e^{-y}-1}dy$$

$$\Rightarrow \quad \int \frac{1}{x+1}dx = \int \frac{e^y}{2-e^y}dy$$ **(1)**

$$\Rightarrow \quad \int \frac{1}{x+1}dx = -\int \frac{e^y}{e^y-2}dy$$

$$\Rightarrow \quad \log|x+1| = -\log|e^y - 2| + \log C$$

$$\Rightarrow \quad \log|x+1| + \log|e^y - 2| = \log C$$

$$\Rightarrow \quad \log|(x+1)(e^y-2)| = \log C$$

$$\Rightarrow \quad [(x+1)(e^y-2)] = C \quad \ldots(\text{i})$$ **(1½)**

It is given that $y(0) = 0$ *i.e.*, $y = 0$ when $x = 0$.
Putting $x = 0$ and $y = 0$ in Eq. (i), we get

$$|(0+1)(1-2)| = C \Rightarrow C = -1$$ **(1/2)**

Putting $C = -1$ in Eq. (i), we get

$$|(x+1)(e^y-2)| = -1$$

$$\Rightarrow \; (x+1)(e^y-2) = \pm 1$$

$$\Rightarrow \quad e^y - 2 = -\frac{1}{x+1} \Rightarrow e^y = \left(2 - \frac{1}{x+1}\right)$$

$$\Rightarrow y = \log\left(2 - \frac{1}{x+1}\right),$$

which is the required solution. **(1)**

8. Given differential equation is

$$x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$$

$$\Rightarrow \quad (y + \sqrt{x^2+y^2})\,dx = x\,dy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} \quad \ldots(\text{i})$$ **(1/2)**

which is a homogeneous differential equation as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

in Eq. (i), we get

$$v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sqrt{1+v^2} \Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x} \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int\frac{dv}{\sqrt{1+v^2}} = \int\frac{dx}{x}$$

$$\Rightarrow \log|v + \sqrt{1+v^2}| = \log|x| + C$$

$$\left[\because \int\frac{dx}{\sqrt{a^2+x^2}} = \log|x + \sqrt{x^2+a^2}| \text{ and } \int\frac{dx}{x} = \log|x|\right]$$

$$\Rightarrow \log\left|\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right| = \log|x| + C \quad \left[\text{put } v = \frac{y}{x}\right] \textbf{(1)}$$

$$\Rightarrow \log\left|\frac{y+\sqrt{x^2+y^2}}{x}\right| - \log|x| = C$$

$$\Rightarrow \log\frac{\left|\frac{y+\sqrt{x^2+y^2}}{x}\right|}{x} = C$$

$$\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$\Rightarrow \frac{y+\sqrt{x^2+y^2}}{x^2} = e^C \quad \left[\begin{matrix}\text{if } \log y = x,\\ \text{then } y = e^x\end{matrix}\right]$$

$$\Rightarrow y + \sqrt{x^2+y^2} = x^2 \cdot e^C$$

$\therefore\ y + \sqrt{x^2+y^2} = Ax^2,$ ...(iii) **(1)**

where $A = e^C$

Now, as $y = 0$, when $x = 1$

$\therefore \quad 0 + \sqrt{1^2+0^2} = A\cdot 1 \Rightarrow A = 1$

Put the value of A, in Eq. (iii), we get

$$y + \sqrt{x^2+y^2} = x^2,$$

which is the required solution **(1/2)**

9. Given, differential equation is

$$(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$$

which is the equation of the form

$$\frac{dy}{dx} + Py = Q$$

where $P = \frac{2x}{1+x^2}$ and $Q = \frac{4x^2}{1+x^2}$ **(1)**

Now, $\text{IF} = e^{\int\frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$ **(1)**

The general solution is

$$y\cdot(1+x^2) = \int (1+x^2)\frac{4x^2}{(1+x^2)}dx + C$$

$$\Rightarrow \quad (1+x^2)\,y = \int 4x^2\,dx + C$$

$$\Rightarrow \quad (1+x^2)\,y = \frac{4x^3}{3} + C$$

$$\Rightarrow \quad y = \frac{4x^3}{3(1+x^2)} + C(1+x^2)^{-1} \quad ...(i)\ \textbf{(1½)}$$

Now, $y(0) = 0$

$$\Rightarrow \quad 0 = \frac{4\cdot 0^3}{3(1+0^2)} + C(1+0^2)^{-1} \Rightarrow C = 0$$

Put the value of C in Eq. (i), we get

$$y = \frac{4x^3}{3(1+x^2)},$$

which is the required solution. **(1/2)**

10. $\frac{dy}{dx} - \frac{2x}{1+x^2}y = x^2 + 2$ ...(i)

This is a linear differential equation with

$P = \frac{-2x}{1+x^2}$ and $Q = x^2 + 2$ **(1)**

$$\therefore \quad \text{IF} = e^{\int\frac{-2x}{x^2+1}dx}$$

$$= e^{-\int\frac{2x}{x^2+1}dx} = e^{-\log(x^2+1)} = \frac{1}{x^2+1} \quad \textbf{(1)}$$

$$\therefore \quad y\cdot\frac{1}{(x^2+1)} = \int (x^2+2)\cdot\frac{1}{(x^2+1)}dx + C$$

[using $y\cdot(\text{IF}) = \int Q\cdot(\text{IF})\,dx + C$]

$$\Rightarrow \quad \frac{y}{x^2+1} = \int\frac{(x^2+1)+1}{x^2+1}dx + C \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{y}{x^2+1} = \int 1\,dx + \int\frac{1}{x^2+1}dx + C$$

$$\Rightarrow \quad \frac{y}{x^2+1} = x + \tan^{-1}x + C$$

$$\Rightarrow y = x(x^2+1) + (\tan^{-1}x)(x^2+1) + C'$$

$[\because C' = C\cdot(x^2+1)]$ **(1)**

11. Given differential equation is

$$x\frac{dy}{dx} - y = x\tan\left(\frac{y}{x}\right) \Rightarrow \frac{dy}{dx} = \frac{y + x\tan\left(\frac{y}{x}\right)}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right) \quad ...(i)$$

which is a homogeneous differential equation as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i),

we get **(1)**

$$v + x\frac{dv}{dx} = v + \tan v \Rightarrow x\frac{dv}{dx} = \tan v$$

$$\Rightarrow \frac{dv}{\tan v} = \frac{dx}{x}$$

$$\Rightarrow \cot v\, dv = \frac{dx}{x} \quad \left[\because \frac{1}{\tan v} = \cot v\right] \text{ (1)}$$

On integrating both sides, we get

$$\int \cot v\, dv = \int \frac{dx}{x}$$

$$\Rightarrow \log|\sin v| = \log|x| + C \quad \text{(1)}$$

$$\left[\because \int \cot v\, dv = \log|\sin v|\right]$$

$$\Rightarrow \log|\sin v| - \log|x| = C$$

$$\Rightarrow \log\left|\frac{\sin v}{x}\right| = C$$

$$[\because \log m - \log n = \log (m/n)]$$

$$\therefore \log\left|\frac{1}{x}\sin\frac{y}{x}\right| = C \quad \left[\text{put } v = \frac{y}{x}\right]$$

$$\Rightarrow \frac{1}{x}\sin\frac{y}{x} = e^C$$

$$\Rightarrow \sin\frac{y}{x} = Ax \quad [\because e^C = A]$$

$$\Rightarrow y = x\sin^{-1}(Ax).$$

which is the required solution. **(1)**

12. Given, $\frac{dy}{dx} = -\frac{x}{1+\sin x} - \frac{y\cos x}{1+\sin x}$

or $$\frac{dy}{dx} + \frac{y\cos x}{1+\sin x} = -\frac{x}{1+\sin x} \quad ...(i)$$

which is in the linear form, $\frac{dy}{dx} + Py = Q$, where

$$P = \frac{\cos x}{1+\sin x},\ Q = -\frac{x}{1+\sin x} \quad (1\tfrac{1}{2})$$

Now, $\text{IF} = e^{\int \frac{\cos x}{1+\sin x}dx} = e^{\log(1+\sin x)} = 1+\sin x$ **(1)**

and the general solution is

$$y(1+\sin x) = \int -x\, dx + C$$

$$[\because y\cdot(\text{IF}) = \int Q\cdot(\text{IF})\, dx + C] \quad (1/2)$$

$$\Rightarrow y(1+\sin x) = -\frac{x^2}{2} + C \quad (1)$$

13. Given differential equation is

$$e^x\tan y\, dx + (2-e^x)\sec^2 y\, dy = 0$$

which can be rewritten as

$$e^x\tan y\, dx = (e^x - 2)\sec^2 y\, dy$$

$$\Rightarrow \frac{\sec^2 y}{\tan y}dy = \frac{e^x}{e^x - 2}dx$$

$$\Rightarrow \int\frac{\sec^2 y}{\tan y}dy = \int\frac{e^x}{e^x-2}dx$$

$$\Rightarrow \log|\tan y| = \log|e^x - 2| + C \quad (1)$$

$$\left[\because \int\frac{f'(x)}{f(x)}dx = \log|f(x)| + C\right]$$

$$\Rightarrow \log|\tan y| - \log|e^x - 2| = C$$

$$\Rightarrow \log\left|\frac{\tan y}{e^x - 2}\right| = C$$

$$\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right] \quad (1)$$

$$\Rightarrow \frac{\tan y}{e^x - 2} = e^C \quad [\because \log m = n \Rightarrow m = e^n]$$

$$\Rightarrow \tan y = e^C(e^x - 2) \quad (1)$$

Now, it is given that $y = \frac{\pi}{4}$ when $x = 0$

$$\therefore \tan\frac{\pi}{4} = e^C(e^0 - 2) \Rightarrow 1 = e^C(1-2) \Rightarrow e^C = -1$$

Thus, the particular solution of the given differential equation is $\tan y = 2 - e^x$. **(1)**

14. Given differential equation is

$$\frac{dy}{dx} + 2y\tan x = \sin x$$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$.

Here, $P = 2\tan x$ and $Q = \sin x$

$$\therefore \text{IF} = e^{\int P\,dx} = e^{2\int\tan x\, dx} = e^{2\log|\sec x|} \quad (1)$$

$$= e^{\log\sec^2 x} \quad [\because m\log n = \log n^m]$$

$$= \sec^2 x \quad [\because e^{\log x} = x]$$

The general solution is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\, dx + C \quad \ldots\text{(i)(1)}$$

$$\Rightarrow \quad y \sec^2 x = \int (\sin x \cdot \sec^2 x)\, dx + C$$

$$\Rightarrow \quad y \sec^2 x = \int \sin x \cdot \frac{1}{\cos^2 x}\, dx + C$$

$$\Rightarrow \quad y \sec^2 x = \int \tan x \sec x\, dx + C$$

$$\Rightarrow \quad y \sec^2 x = \sec x + C \quad \ldots\text{(ii)}$$

Also, given that $y = 0$, when $x = \frac{\pi}{3}$.

On putting $y = 0$ and $x = \frac{\pi}{3}$ in Eq. (ii), we get

$$0 \times \sec^2 \frac{\pi}{3} = \sec \frac{\pi}{3} + C$$

$$\Rightarrow \quad 0 = 2 + C \Rightarrow C = -2 \quad \textbf{(1)}$$

On putting the value of C in Eq. (ii), we get

$$y \sec^2 x = \sec x - 2$$

$$\therefore \quad y = \cos x - 2\cos^2 x$$

which is the required particular solution of the given differential equation. **(1)**

15. Given, differential equation is

$$(x^2 - y^2)\, dx + 2xy\, dy = 0,$$

which can be re-written as

$$(x^2 - y^2)\, dx = -2xy\, dy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2 - y^2}{-2xy} = \frac{y^2 - x^2}{2xy} = \frac{\left(\frac{y}{x}\right)^2 - 1}{2\left(\frac{y}{x}\right)} \quad \textbf{(1)}$$

$\because$ In RHS, degree of numerator and denominator is same.

$\therefore$ It is a homogeneous differential equation and can be written as

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Now, put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ **(1)**

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v} = \frac{-v^2 - 1}{2v}$$

$$\Rightarrow \quad \frac{2v}{v^2 + 1}\, dv = -\frac{dx}{x} \quad \textbf{(1/2)}$$

$$\Rightarrow \int \frac{2v}{v^2 + 1}\, dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \log|v^2 + 1| = -\log|x| + \log C$$

$$\left[\because \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + C\right]$$

$$\Rightarrow \log\left|\frac{y^2}{x^2} + 1\right| = -\log|x| + \log C \quad \left[\because v = \frac{y}{x}\right] \textbf{(1)}$$

$$\Rightarrow \log\left|\frac{y^2 + x^2}{x^2} \cdot x\right| = \log C$$

$$\Rightarrow \quad \frac{y^2 + x^2}{x} = C \Rightarrow y^2 + x^2 = Cx,$$

which is the required solution. **(1/2)**

16. First, divide the given differential equation by $(x^2 + 1)$ to convert it into the form of linear differential equation and then solve it.

Given differential equation is

$$(x^2 + 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 + 1} \quad \textbf{(1)}$$

On dividing both sides by $(x^2 + 1)$, we get

$$\frac{dy}{dx} + \frac{2x}{x^2 + 1}y = \frac{1}{(x^2 + 1)^2}$$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, here $P = \frac{2x}{x^2 + 1}$ and $Q = \frac{1}{(x^2 + 1)^2}$

Now, integrating factor, $\text{IF} = e^{\int P dx} = e^{\int \frac{2x}{x^2 + 1} dx}$

$$= e^{\log|x^2 + 1|} = x^2 + 1 \quad \textbf{(1)}$$

$$\left[\begin{aligned}&\text{put } x^2 + 1 = t \Rightarrow 2x\, dx = dt, \text{ then}\\ &\int \frac{2x}{x^2 + 1}\, dx = \int \frac{1}{t}\, dt = \log|t| = \log|x^2 + 1|\end{aligned}\right]$$

So, the required general solution is

$$y \times \text{IF} = \int (Q \times \text{IF})\, dx + C$$

$$\Rightarrow \quad y(x^2 + 1) = \int \frac{1}{(x^2 + 1)^2} \times (x^2 + 1)\, dx + C \; \textbf{(1)}$$

$$\Rightarrow \quad y(x^2 + 1) = \int \frac{1}{x^2 + 1}\, dx + C$$

$$\Rightarrow \quad y(x^2 + 1) = \tan^{-1} x + C \quad \ldots\text{(i)}$$

when $x = 1$, then $y = 0$

$$\therefore 0 = \tan^{-1} 1 + C \Rightarrow C = \frac{-\pi}{4}$$

Now, $y(x^2 + 1) = \tan^{-1} x - \frac{\pi}{4}$ **[from Eq. (i)]**

which is the required differential equation. **(1)**

17. Do same as Q. No. 15.

18. Given, differential equation can be rewritten as

$$\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} \quad \dots \text{(i)}$$

This is a homogeneous differential equation, so, put $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then, Eq. (i) becomes

$$v + x\frac{dv}{dx} = \frac{x^3 - 3x(vx)^2}{(vx)^3 - 3x^2(vx)}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{1-3v^2}{v^3 - 3v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1-3v^2}{v^3 - 3v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 - 3v^2 - v^4 + 3v^2}{v^3 - 3v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 - v^4}{v^3 - 3v} \quad \text{(1)}$$

$$\Rightarrow \quad \left(\frac{v^3 - 3v}{1 - v^4}\right) dv = \frac{dx}{x}$$

On integrating both sides, we get

$$\int \left(\frac{v^3 - 3v}{1 - v^4}\right) dv = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{v^3}{1-v^4}\, dv - 3\int \frac{v}{1-v^4}\, dv = \log x + \log C \quad \dots \text{(ii)}$$

$$\Rightarrow -\frac{1}{4}\log(1 - v^4) - \frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right| = \log x + \log C \quad \text{(1)}$$

$$\Rightarrow \quad -\frac{1}{4}\log\left[(1 - v^4)\left(\frac{1+v^2}{1-v^2}\right)^3\right] = \log(Cx)$$

$$\Rightarrow -\frac{1}{4}\log\left[(1 - v^2)(1 + v^2) \times \frac{(1+v^2)^3}{(1-v^2)^3}\right] = \log(Cx)$$

$$\Rightarrow \quad \log\left[\frac{(1+v^2)^4}{(1-v^2)^2}\right]^{-1/4} = \log Cx$$

$$\Rightarrow \quad \frac{(1+v^2)^4}{(1-v^2)^2} = (Cx)^{-4} \quad \text{(1)}$$

$$\Rightarrow \quad \frac{(1 + y^2/x^2)^4}{(1 - y^2/x^2)^2} = \frac{1}{C^4x^4} \quad [\because y = v - x]$$

$$\Rightarrow \quad \frac{(x^2 + y^2)^4}{x^4(x^2 - y^2)^2} = \frac{1}{C^4x^4}$$

$$\Rightarrow \quad (x^2 - y^2) = C^2(x^2 + y^2)^2$$

[taking square root]

$$\Rightarrow \quad (x^2 - y^2) = C_1\,(x^2 + y^2)^2,$$

where $C_1 = C^2$ **Hence proved.** (1)

19. Given differential equation is

$$x\frac{dy}{dx} + y = x\cos x + \sin x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = \cos x + \frac{\sin x}{x}$$

[dividing both sides by x]

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, here $P = \frac{1}{x}$ and $Q = \cos x + \frac{\sin x}{x}$

$$\therefore \quad \text{IF} = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log|x|} = x$$

The general solution is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\, dx + C \quad \text{(1)}$$

$$\Rightarrow \quad yx = \int \left(\cos x + \frac{\sin x}{x}\right) x\, dx + C$$

$$\Rightarrow \quad yx = \int (x\cos x + \sin x)\, dx + C$$

$$\Rightarrow \quad xy = \int \underset{\text{I}}{x}\, \underset{\text{II}}{\cos x}\, dx + \int \sin x\, dx + C$$

$$\Rightarrow \quad xy = x\int \cos x\, dx - \int\left[\frac{d}{dx}(x)\int \cos x\, dx\right] dx + \int \sin x\, dx + C$$

[using integration by parts]

$$\Rightarrow \quad xy = x\sin x - \int 1 \cdot \sin x\, dx - \cos x + C$$

$$\Rightarrow \quad xy = x\sin x + \cos x - \cos x + C$$

$$\Rightarrow \quad xy = x\sin x + C$$

$$\Rightarrow \quad y = \sin x + C \cdot \frac{1}{x} \quad \dots \text{(i)} \quad \text{(1)}$$

Also, given that at $x = \frac{\pi}{2}; y = 1$

On putting $x = \frac{\pi}{2}$ and $y = 1$ in Eq. (i), we get

$$1 = 1 + C \cdot \frac{2}{\pi} \Rightarrow C = 0 \quad \text{(1)}$$

On putting the value of C in Eq. (i), we get

$$y = \sin x$$

which is the required solution of given differential equation. (1)

20. Given, $(\tan^{-1} x - y)\, dx = (1 + x^2)\, dy$

$$\Rightarrow \frac{dy}{dx} = \frac{\tan^{-1} x - y}{(1 + x^2)} \Rightarrow \frac{dy}{dx} = \frac{\tan^{-1} x}{1+x^2} - \frac{1}{1+x^2} y$$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{1+x^2} y = \frac{\tan^{-1} x}{1+x^2} \quad \text{...(i)} \quad (1)$$

which is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, here $P = \frac{1}{1+x^2}$ and $Q = \frac{\tan^{-1} x}{1+x^2}$

Now, $\text{IF} = e^{\int P dx} = e^{\int \frac{1}{1+x^2} dx} = e^{\tan^{-1} x}$ (1)

$\therefore$ The general solution is given by

$$y \cdot \text{IF} = \int Q \cdot \text{IF}\, dx + C$$

$$\Rightarrow y \cdot e^{\tan^{-1} x} = \int \frac{\tan^{-1} x}{1+x^2} \cdot e^{\tan^{-1} x} dx + C$$

Put $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

$$\therefore ye^{\tan^{-1} x} = \int t \cdot e^t dt + C = t \cdot e^t - \int 1 \cdot e^t\, dt + C \quad (1)$$

[using integration by parts]

$$\Rightarrow ye^{\tan^{-1} x} = t \cdot e^t - e^t + C$$

$$\Rightarrow ye^{\tan^{-1} x} = \tan^{-1} x \cdot e^{\tan^{-1} x} - e^{\tan^{-1} x} + C$$

$[\because t = e^{\tan^{-1} x}]$

$$\Rightarrow ye^{\tan^{-1} x} = (\tan^{-1} x - 1)e^{\tan^{-1} x} + C \quad (1)$$

21. We have, $ydx - (x + 2y^2)dy = 0$

$$\Rightarrow y\frac{dx}{dy} = x + 2y^2 \Rightarrow \frac{dx}{dy} - \frac{1}{y}x = 2y \quad (1)$$

which is a linear differential equation of the form $\frac{dx}{dy} + Px = Q$, here $P = \frac{-1}{y}$ and $Q = 2y$.

$$\therefore \text{IF} = e^{\int Pdy} = e^{\int -\frac{1}{y} dy} = e^{-\log y} = \frac{1}{y} \quad (1)$$

Hence, required general solution of the differential equation is

$$x \cdot \text{IF} = \int (Q \cdot \text{IF})\, dy + C \Rightarrow x \times \frac{1}{y} = \int 2y \times \frac{1}{y} dy + C \quad (1)$$

$$\Rightarrow \frac{x}{y} = 2y + C \Rightarrow x = 2y^2 + Cy \quad (1)$$

22. We have, $\frac{dy}{dx} - y = \sin x$, which is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, here $P = -1$ and $Q = \sin x$

$$\therefore \text{IF} = e^{\int P dx} = e^{\int (-1) dx} = e^{-x}$$

Now, the general solution of given differential equation is given by

$$y \cdot (\text{IF}) = \int (\text{IF}) \cdot Q dx + C$$

$$\Rightarrow y \cdot e^{-x} = \int e^{-x} \sin x\, dx + C \quad \text{...(i)}$$

Let $I = \int \underset{\text{II}}{e^{-x}} \underset{\text{I}}{\sin x} \cdot dx$ ...(ii) (1)

By using the method of integration by parts, we get

$$I = \sin x \frac{e^{-x}}{(-1)} - \int \cos x \frac{e^{-x}}{(-1)} dx$$

$$= -\sin x\, e^{-x} + \int \underset{\text{II}}{e^{-x}} \underset{\text{I}}{\cos x}\, dx \quad (1)$$

Again, by using integration by parts, we get

$$I = -\sin x\, e^{-x} + \cos x \frac{e^{-x}}{(-1)} - \int (-\sin x) \frac{e^{-x}}{(-1)} dx$$

$$= -\sin x e^{-x} - \cos x e^{-x} - \int e^{-x} \sin x dx$$

$$= -\sin x e^{-x} - \cos x e^{-x} - I \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow 2I = -e^{-x}(\sin x + \cos x)$$

$$\Rightarrow I = -\frac{e^{-x}}{2}(\sin x + \cos x) \quad (1)$$

Then, from Eq. (i), we get

$$y \cdot e^{-x} = -\frac{e^{-x}}{2}(\sin x + \cos x) + C$$

$$\Rightarrow y = -\frac{1}{2}(\sin x + \cos x) + Ce^x \quad (1)$$

23. Given differential equation is

$(1 - y^2)(1 + \log |x|)\, dx + 2xy\, dy = 0$.

On separating the variables, we get

$$\frac{(1 + \log |x|)}{x} dx + \frac{2y}{1 - y^2} dy = 0$$

[dividing both sides by $x(1 - y^2)$] (1/2)

On integrating, we get

$$\int \left(\frac{1}{x} + \frac{\log |x|}{x}\right) dx + \int \frac{2y}{1 - y^2} dy = 0$$

$$\Rightarrow \log |x| + \frac{(\log |x|)^2}{2} - \log |1 - y^2| = \log C \quad \text{...(i)} \quad (1)$$

Also, given $y = 0$ and $x = 1$

$\therefore\ \log 1 + \dfrac{(\log 1)^2}{2} - \log|1-0| = \log C$ **(1)**

$\Rightarrow\ 0+0-0 = \log C \Rightarrow \log C = 0$

On putting $\log C = 0$ in Eq. (i), we get

$$\log|x| + \frac{(\log|x|)^2}{2} - \log|1-y^2| = 0$$ **(1½)**

4. Given differential equation is

$$(1+y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$$

It can be rewritten as

$$(1+y^2)\frac{dx}{dy} + x - e^{\tan^{-1} y} = 0$$

or $$\frac{dx}{dy} + \frac{1}{(1+y^2)}x = \frac{e^{\tan^{-1} y}}{1+y^2}$$

[dividing both sides by $(1+y^2)$]

It is a linear differential equation of the form

$$\frac{dx}{dy} + Px = Q.$$

here, $P = \dfrac{1}{1+y^2}$ and $Q = \dfrac{e^{\tan^{-1} y}}{1+y^2}$ **(1)**

Now, integrating factor, $\text{IF} = e^{\int P\,dy}$

$$= e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$$ **(1/2)**

$\therefore$ The general solution of linear differential equation is given by

$$x \times \text{IF} = \int (Q \times \text{IF})\,dy + C$$

$$\Rightarrow x \times e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1+y^2} \times e^{\tan^{-1} y}\,dy + C$$ **(1/2)**

$$\Rightarrow\ xe^{\tan^{-1} y} = \int \frac{e^{2\tan^{-1} y}}{1+y^2}\,dy + C \quad \ldots(i)$$

On putting $\tan^{-1} y = t \Rightarrow \dfrac{1}{1+y^2}dy = dt$ in Eq. (i), we get

$$xe^{\tan^{-1} y} = \int e^{2t}dt + C$$ **(1)**

$$\Rightarrow\ xe^{\tan^{-1} y} = \frac{e^{2t}}{2} + C$$

$$\Rightarrow\ xe^{\tan^{-1} y} = \frac{e^{2\tan^{-1} y}}{2} + C \quad [\text{put } t = \tan^{-1} y]$$ **(1)**

25. Do same as Q. No. 12.

The general solution is

$$y(1+\sin x) = -\frac{x^2}{2} + C \quad \ldots(i)$$

Since, $y = 1$, when $x = 0$

$\therefore\ 1(1+\sin 0) = -\dfrac{0}{2} + C \Rightarrow C = 1 + 0 = 1$

On putting $C = 1$ in Eq. (i), we get

$$y(1+\sin x) = -\frac{x^2}{2} + 1$$

Hence, particular solution of the given differential equation is $y(1+\sin x) = -\dfrac{x^2}{2} + 1$.

26.

> First, replace x by λx and y by λy in $F(x, y)$ of given differential equation to check that it is homogeneous. If it is homogeneous, then put $x = vy$ and $\dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$ and then solve it.

Given differential equation is

$$2ye^{x/y}dx + (y - 2xe^{x/y})dy = 0.$$

It can be written as

$$\frac{dx}{dy} = \frac{2xe^{x/y} - y}{2ye^{x/y}} \quad \ldots(i)$$

Let $$F(x, y) = \frac{\left(2xe^{\frac{x}{y}} - y\right)}{\left(2ye^{\frac{x}{y}}\right)}$$

On replacing x by λx and y by λy both sides, we get

$$F(\lambda x, \lambda y) = \frac{\left(2\lambda x\, e^{\frac{\lambda x}{\lambda y}} - \lambda y\right)}{\left(2\lambda y\, e^{\frac{\lambda x}{\lambda y}}\right)}$$

$$\Rightarrow\ F(\lambda x, \lambda y) = \frac{\lambda(2xe^{x/y} - y)}{\lambda(2ye^{x/y})} = \lambda^0\,[F(x, y)]$$

Thus, $F(x, y)$ is a homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation. **(1)**

To solve it, put $x = vy$

$\Rightarrow\ \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$ in Eq.(i), we get

$$v + y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v}$$

$$\Rightarrow\ y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v} - v = \frac{2ve^v - 1 - 2ve^v}{2e^v}$$

$$\Rightarrow\ 2e^v dv = \frac{-dy}{y}$$ **(1)**

On integrating both sides, we get

$$\int 2e^v\, dv = -\int \frac{dy}{y}$$

$$\Rightarrow \quad 2e^v = -\log|y| + C$$

$$\Rightarrow \quad 2e^{x/y} + \log|y| = C \qquad \left[\text{put } v = \frac{x}{y}\right] \quad \ldots\text{(ii)} \quad \textbf{(1)}$$

Also, given that $x = 0$, when $y = 1$.

On substituting $x=0$ and $y=1$ in Eq. (ii), we get

$$2e^0 + \log|1| = C \Rightarrow C = 2$$

On substituting the value of C in Eq. (ii), we get

$$2e^{x/y} + \log|y| = 2$$

which is the required particular solution of the given differential equation. **(1)**

27. We have, $y + x\dfrac{dy}{dx} = x - y\dfrac{dy}{dx}$

$$\Rightarrow \quad x\frac{dy}{dx} + y\frac{dy}{dx} = x - y$$

$$\frac{dy}{dx} = \frac{x-y}{x+y} \quad \ldots\text{(i)}$$

This is a homogeneous differential equation.

On putting $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ in Eq. (i),

we get

$$v + x\frac{dv}{dx} = \frac{x - vx}{x + vx} \quad \textbf{(1)}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v} - v = \frac{1-v-v-v^2}{1+v} = \frac{1-2v-v^2}{1+v}$$

$$\Rightarrow \quad \frac{1+v}{v^2+2v-1}\,dv = -\frac{1}{x}dx$$

On integrating both sides, we get

$$\int \frac{1+v}{v^2+2v-1}\,dv = -\int \frac{1}{x}\,dx \quad \textbf{(1)}$$

$$\Rightarrow \frac{1}{2}\log|v^2+2v-1| = -\log|x| + \log C$$

$$\left[\because \frac{d}{dv}(v^2+2v-1) = 2v+2 = 2(v+1)\right]$$

$$\Rightarrow \frac{1}{2}\log|v^2+2v-1| + \log|x| = \log C$$

$$\Rightarrow \log|v^2+2v-1| + 2\log|x| = 2\log C$$

$$\Rightarrow \quad \log\left|\frac{y^2}{x^2} + \frac{2y}{x} - 1\right| + \log x^2 = \log C^2 \quad \textbf{(1)}$$

$$[\text{put } v = y/x \text{ and } n\log m = \log m^n]$$

$$\Rightarrow \log\left(\frac{y^2}{x^2} + \frac{2y}{x} - 1\right)x^2 = \log C^2$$

$$[\because \log m + \log n = \log mn]$$

$$\Rightarrow \quad y^2 + 2xy - x^2 = C^2$$

$$\therefore \quad y^2 + 2xy - x^2 = C_1 \text{ where, } C_1 = C^2 \quad \textbf{(1)}$$

28. We have, $y^2dx + (x^2 - xy + y^2)dy = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y^2}{x^2 - xy + y^2} \quad \ldots\text{(i)}$$

This is homogeneous differential equation.

Now, on putting $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ in Eq. (i),

we get

$$v + x\frac{dv}{dx} = \frac{-v^2x^2}{x^2 - vx^2 + v^2x^2} \quad \textbf{(1)}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{-v^2}{1 - v + v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v^2}{1-v+v^2} - v \Rightarrow x\frac{dv}{dx} = \frac{-v-v^3}{1-v+v^2}$$

$$\therefore \quad \frac{1-v+v^2}{v(1+v^2)}dv = -\frac{1}{x}dx \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{1+v^2}{v(1+v^2)}dv - \int \frac{v}{v(1+v^2)}dv = -\int \frac{1}{x}\,dx$$

$$\Rightarrow \quad \int \frac{1}{v}dv - \int \frac{1}{1+v^2}dv = -\int \frac{1}{x}dx$$

$$\Rightarrow \quad \log|v| - \tan^{-1}v = -\log|x| + \log C \quad \textbf{(1)}$$

$$\Rightarrow \quad \log\left|\frac{vx}{C}\right| = \tan^{-1}v \Rightarrow \left|\frac{vx}{C}\right| = e^{\tan^{-1}v}$$

$$\Rightarrow \quad \left|\frac{y}{C}\right| = e^{\tan^{-1}(y/x)} \quad [\because vx = y]$$

$\therefore |y| = C\,e^{\tan^{-1}(y/x)}$, which is the required solution. **(1)**

29. We have, $(\cot^{-1}y + x)dy = (1 + y^2)\,dx$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\cot^{-1}y + x}{1+y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} + \left(-\frac{1}{1+y^2}\right)x = \frac{\cot^{-1}y}{1+y^2} \quad \textbf{(1/2)}$$

This is a linear differential equation of the form $\dfrac{dx}{dy} + Px = Q$, here $P = \dfrac{-1}{1+y^2}$ and $Q = \dfrac{\cot^{-1}y}{1+y^2}$.

$$\text{IF} = e^{-\int \frac{1}{1+y^2} dy} = e^{\cot^{-1} y}$$

∴

Now, the solution of linear differential equation is given by $x \cdot \text{IF} = \int (Q \times \text{IF})\, dy + C$

∴ $$x e^{\cot^{-1} y} = \int \frac{\cot^{-1} y}{(1+y^2)} e^{\cot^{-1} y} dy + C \quad \ldots\text{(i)} \textbf{(1½)}$$

On putting $\cot^{-1} y = t \Rightarrow \frac{1}{1+y^2} dy = -dt$ in Eq. (i),

we get

$$x e^{\cot^{-1} y} = -\int t e^t\, dt + C \qquad \textbf{(1)}$$

$$= -e^t (t-1) + C$$

⇒ $$x e^{\cot^{-1} y} = e^{\cot^{-1} y}(1 - \cot^{-1} y) + C$$

$[\because t = \cot^{-1} y]$

which is the required solution. **(1)**

30. Given differential equation is

$$x\frac{dy}{dx} + y - x + xy \cot x = 0,\ x \neq 0$$

Above equation can be written as

$$x\frac{dy}{dx} + y(1 + x\cot x) = x$$

On dividing both sides by x, we get

$$\frac{dy}{dx} + y\left(\frac{1 + x\cot x}{x}\right) = 1$$

⇒ $$\frac{dy}{dx} + y\left(\frac{1}{x} + \cot x\right) = 1 \qquad \textbf{(1)}$$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

here, $P = \frac{1}{x} + \cot x$ and $Q = 1$.

∴ $$\text{IF} = e^{\int P dx} = e^{\int\left(\frac{1}{x} + \cot x\right) dx} = e^{\log|x| + \log \sin x}$$

$$\left[\because \int \frac{1}{x} dx = \log|x| \text{ and } \int \cot x\, dx = \log|\sin x|\right]$$

$$= e^{\log|x \sin x|} \quad [\because \log m + \log n = \log mn]$$

⇒ $\text{IF} = x \sin x$ **(1)**

The solution of given linear differential equation is

$$y \times \text{IF} = \int (Q \times \text{IF})\, dx + C$$

∴ $$y \times x \sin x = \int 1 \times x \sin x\, dx + C$$

⇒ $$y \cdot x \sin x = \int \underset{\text{I}}{x} \underset{\text{II}}{\sin x}\, dx + C$$

⇒ $$y \cdot x \sin x = x \int \sin x\, dx - \int\left(\frac{d}{dx}(x) \cdot \int \sin x\, dx\right) dx + C$$

[using integration by parts]

$$\Rightarrow y\, x \sin x = -x\cos x - \int 1(-\cos x)\, dx + C \qquad \textbf{(1)}$$

$$\Rightarrow y\, x \sin x = -x\cos x + \int \cos x\, dx + C$$

$$\Rightarrow y x \sin x = -x\cos x + \sin x + C$$

On dividing both sides by $x \sin x$, we get

$$y = \frac{-x\cos x + \sin x + C}{x \sin x}$$

∴ $$y = -\cot x + \frac{1}{x} + \frac{C}{x \sin x}$$

which is the required solution. **(1)**

31. Given differential equation is

$$x^2 dy + (xy + y^2)\, dx = 0$$

⇒ $$x^2 dy = -(xy + y^2)\, dx$$

⇒ $$\frac{dy}{dx} = \frac{-(xy + y^2)}{x^2} = -\left(\frac{y}{x} + \frac{y^2}{x^2}\right) \quad \ldots\text{(i)}$$

which is a homogeneous differential equation as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq (i),

we get

$$v + x\frac{dv}{dx} = -(v + v^2) \Rightarrow x\frac{dv}{dx} = -v - v^2 - v$$

$$\Rightarrow x\frac{dv}{dx} = -v^2 - 2v \Rightarrow \frac{dv}{v^2 + 2v} = \frac{dx}{-x} \qquad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{dv}{v^2 + 2v} = -\int \frac{dx}{x} \Rightarrow \int \frac{dv}{v^2 + 2v + 1 - 1} = -\int \frac{dx}{x}$$

⇒ $$\int \frac{dv}{(v+1)^2 - (1)^2} = -\int \frac{dx}{x}$$

⇒ $$\frac{1}{2}\log\left|\frac{v+1-1}{v+1+1}\right| = -\log|x| + C \qquad \textbf{(1)}$$

$$\left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right]$$

⇒ $$\frac{1}{2}\log\left|\frac{v}{v+2}\right| = -\log|x| + C$$

⇒ $$\frac{1}{2}\log\left|\frac{\frac{y}{x}}{\frac{y}{x} + 2}\right| = -\log|x| + C \quad \left[\text{put } v = \frac{y}{x}\right]$$

$$\Rightarrow \quad \frac{1}{2}\log\left|\frac{y}{y+2x}\right| = -\log|x| + C \qquad \ldots(ii)$$

Also, given that at $x = 1, y = 1$.

On putting $x = y = 1$ in Eq. (ii), we get

$$\frac{1}{2}\log\left|\frac{1}{1+2}\right| = -\log 1 + C$$

$$\Rightarrow \quad \frac{1}{2}\log\left|\frac{1}{3}\right| = -\log 1 + C$$

$$\Rightarrow \quad C = \frac{1}{2}\log\frac{1}{3} \qquad [\because \log 1 = 0] \text{ (1)}$$

On putting the value of C in Eq. (ii), we get

$$\frac{1}{2}\log\left|\frac{y}{y+2x}\right| = -\log|x| + \frac{1}{2}\log\frac{1}{3}$$

$$\Rightarrow \quad \log\left|\frac{y}{y+2x}\right| = -2\log|x| + \log\frac{1}{3}$$

$$\Rightarrow \quad \log\left|\frac{y}{y+2x}\right| = \log|x|^{-2} + \log\frac{1}{3}$$

$$[\because n\log|m| = \log|m|^n]$$

$$\Rightarrow \quad \log\left|\frac{y}{y+2x}\right| = \log\frac{1}{x^2} + \log\frac{1}{3}$$

$$\Rightarrow \quad \log\left|\frac{y}{y+2x}\right| = \log\frac{1}{3x^2}$$

$$[\because \log m + \log n = \log mn]$$

$$\Rightarrow \quad \frac{y}{y+2x} = \frac{1}{3x^2} \Rightarrow y\cdot 3x^2 = y + 2x$$

$$\Rightarrow \quad y(3x^2 - 1) = 2x$$

$$\therefore \quad y = \frac{2x}{3x^2 - 1}$$

which is the required particular solution. **(1)**

32. Given differential equation is

$$\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x.$$

$$\Rightarrow \quad \frac{1}{1+y}\,dy = -\frac{\cos x}{2+\sin x}\,dx$$

On integrating both sides, we get

$$\int\frac{1}{1+y}\,dy = -\int\frac{\cos x}{2+\sin x}\,dx$$

$$\Rightarrow \quad \log|1+y| = -\log|2+\sin x| + \log C \qquad \text{(1)}$$

$$\left[\begin{array}{l}\text{put } 2+\sin x = t \Rightarrow \cos x\,dx = dt,\\ \text{then } \int\frac{\cos x}{2+\sin x}dx = \int\frac{dt}{t} = \log|t| + C\\ \qquad\qquad = \log|2+\sin x| + C\end{array}\right]$$

$$\Rightarrow \quad \log|1+y| + \log|2+\sin x| = \log C$$

$$\Rightarrow \quad \log(|1+y||2+\sin x|) = \log C$$

$$[\because \log m + \log n = \log mn]$$

$$\Rightarrow \quad (1+y)(2+\sin x) = C \qquad \ldots(i)$$

Also, given that at $x = 0, y = 1$.

On putting $x = 0$ and $y = 1$ in Eq. (i), we get

$$(1+1)(2+\sin 0) = C \Rightarrow C = 4 \qquad \text{(1)}$$

On putting $C = 4$ in Eq. (i), we get

$$(1+y)(2+\sin x) = 4$$

$$\Rightarrow \quad 1 + y = \frac{4}{2+\sin x}$$

$$\Rightarrow \quad y = \frac{4}{2+\sin x} - 1$$

$$\Rightarrow \quad y = \frac{4-2-\sin x}{2+\sin x} \Rightarrow y = \frac{2-\sin x}{2+\sin x} \qquad \text{(1)}$$

Now, at $x = \frac{\pi}{2}$, $y\left(\frac{\pi}{2}\right) = \dfrac{2-\sin\frac{\pi}{2}}{2+\sin\frac{\pi}{2}}$

$$\therefore \quad y\left(\frac{\pi}{2}\right) = \frac{1}{3} \qquad \left[\because \sin\frac{\pi}{2} = 1\right] \text{(1)}$$

33. Given differential equation is

$$\frac{dy}{dx} = \frac{x(2\log|x| + 1)}{\sin y + y\cos y}$$

On separating the variables, we get

$$(\sin y + y\cos y)\,dy = x(2\log|x| + 1)\,dx$$

$$\Rightarrow \sin y\,dy + y\cos y\,dy = 2x\log|x|\,dx + x\,dx \qquad \text{(1)}$$

On integrating both sides, we get

$$\int\sin y\,dy + \int \underset{\text{I}}{y}\,\underset{\text{II}}{\cos y}\,dy = 2\int \underset{\text{II}}{x}\,\underset{\text{I}}{\log|x|}\,dx + \int x\,dx$$

$$\Rightarrow \; -\cos y + \left[y\int\cos y\,dy - \int\left\{\frac{d}{dy}(y)\int\cos y\,dy\right\}dy\right]$$

$$= 2\left[\log|x|\int x\,dx - \int\left\{\frac{d}{dx}(\log|x|)\int x\,dx\right\}dx\right] + \frac{x^2}{2}$$

[by using integration by parts] **(1)**

$$\Rightarrow \quad -\cos y + y\sin y - \int\sin y\,dy$$

$$= 2\left[\frac{x^2}{2}\log|x| - \int\left\{\frac{1}{x}\cdot\frac{x^2}{2}\right\}dx\right] + \frac{x^2}{2}$$

$$\Rightarrow \quad -\cos y + y\sin y + \cos y$$

$$= x^2\log|x| - \int x\,dx + \frac{x^2}{2}$$

$$\Rightarrow \quad y\sin y = x^2\log|x| - \frac{x^2}{2} + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = x^2\log|x| + C \qquad \ldots(i) \text{ (1)}$$

Also, given that $y = \frac{\pi}{2}$, when $x = 1$.

On putting $y = \frac{\pi}{2}$ and $x = 1$ in Eq. (i), we get

$$\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = (1)^2 \log (1) + C$$

$$\Rightarrow \quad C = \frac{\pi}{2} \quad \left[\because \sin\frac{\pi}{2} = 1, \log 1 = 0\right]$$

On substituting the value of C in Eq. (i), we get

$$y \sin y = x^2 \log |x| + \frac{\pi}{2}$$

which is the required particular solution. **(1)**

34. Do same as Q. No. 16.

$$\left[\textbf{Ans. } y(x^2 - 1) = \log\left|\frac{x-1}{x+1}\right| + C\right]$$

35. Given differential equation is

$$e^x \sqrt{1-y^2}\, dx + \frac{y}{x}\, dy = 0$$

$$\Rightarrow \quad e^x \sqrt{1-y^2}\, dx = \frac{-y}{x}\, dy$$

On separating the variables, we get

$$\frac{-y}{\sqrt{1-y^2}}\, dy = x\, e^x dx \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{-y}{\sqrt{1-y^2}}\, dy = \int x\, e^x\, dx$$

On putting $1 - y^2 = t \Rightarrow -y\, dy = \frac{dt}{2}$ in LHS, we get

$$\int \frac{1}{2\sqrt{t}}\, dt = \int \underset{\text{I}}{x}\, \underset{\text{II}}{e^x}\, dx$$

$$\Rightarrow \quad \frac{1}{2}[2\sqrt{t}] = x \int e^x dx - \int \left[\frac{d}{dx}(x) \int e^x dx\right] dx$$

[using integration by parts]

$$\Rightarrow \sqrt{1-y^2} = x\, e^x - \int e^x\, dx \quad [\text{put } t = 1 - y^2] \textbf{(1)}$$

$$\Rightarrow \sqrt{1-y^2} = x\, e^x - e^x + C \quad \text{...(i)}$$

Also, given that $y = 1$, when $x = 0$

On putting $y = 1$ and $x = 0$ in Eq. (i), we get

$$\sqrt{1-1} = 0 - e^0 + C$$

$$\Rightarrow \quad C = 1 \quad [\because e^0 = 1] \textbf{(1)}$$

On substituting the value of C in Eq. (i), we get

$$\sqrt{1-y^2} = x\, e^x - e^x + 1$$

which is the required particular solution of given differential equation. **(1)**

36. First, separate the variables, then integrate by using integration by parts.

Given differential equation is

$$\operatorname{cosec} x \log |y| \frac{dy}{dx} + x^2 y^2 = 0 \quad \text{...(i)}$$

It can be rewritten as

$$\operatorname{cosec} x \log |y| \frac{dy}{dx} = -x^2 y^2$$

On separating the variables, we get

$$\frac{\log |y|}{y^2}\, dy = \frac{-x^2}{\operatorname{cosec} x}\, dx$$

On integrating both sides, we get

$$\int \frac{\log |y|}{y^2}\, dy = -\int \frac{x^2}{\operatorname{cosec} x}\, dx$$

$$\Rightarrow \quad I_1 = -I_2 \quad \text{...(ii)} \textbf{(1)}$$

where, $I_1 = \int \frac{\log|y|}{y^2}\, dy$

and $I_2 = \int \frac{x^2}{\operatorname{cosec} x}\, dx = \int x^2 \sin x\, dx$

Consider, $I_1 = \int \frac{\log|y|}{y^2} dy$

Put $\log y = t \Rightarrow y = e^t$, then $\frac{dy}{y} = dt$

$$\therefore I_1 = \int \underset{\text{I}}{t}\, \underset{\text{II}}{e^{-t}}\, dt = t \int e^{-t}\, dt - \int \left[\frac{d}{dt}(t) \int e^{-t}\, dt\right] dt$$

[using integration by parts]

$$= -t\, e^{-t} - \int (-e^{-t})\, dt$$

$$= -t\, e^{-t} + \int e^{-t} dt = -t\, e^{-t} - e^{-t} + C_1$$

$$= -\frac{\log |y|}{y} - \frac{1}{y} + C_1 \quad \text{...(iii)} \textbf{(1)}$$

$$\left[\because t = \log |y| \text{ and } e^{-t} = \frac{1}{y}\right]$$

and $I_2 = \int \underset{\text{I}}{x^2}\, \underset{\text{II}}{\sin x}\, dx$

$$= x^2 \int \sin x\, dx - \int \left[\frac{d}{dx}(x^2) \int \sin x\, dx\right] dx$$

[using integration by parts]

$$= x^2(-\cos x) - \int [2x(-\cos x)]\, dx$$

$$= -x^2 \cos x + 2\int \underset{\text{I}}{x}\, \underset{\text{II}}{\cos x}\, dx$$

$$= -x^2 \cos x + 2\left[x \int \cos x\, dx - \int \left\{\frac{d}{dx}(x) \int \cos x\, dx\right\} dx\right.$$

$$= -x^2 \cos x + 2[x \sin x - \int \sin x \, dx]$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + C_2 \quad \text{...(iv)(1)}$$

On putting the values of I_1 and I_2 from Eqs.(iii) and (iv) in Eq. (ii), we get

$$-\frac{\log|y|}{y} - \frac{1}{y} + C_1 = -x^2 \cos x + 2x \sin x + 2 \cos x + C_2$$

$$\Rightarrow \quad \frac{(1 + \log|y|)}{y} = x^2 \cos x - 2x \sin x - 2 \cos x - C_2 - C_1$$

$$\Rightarrow \quad \frac{(1 + \log|y|)}{y} = x^2 \cos x - 2x \sin x - 2 \cos x + C$$

where, $C = -C_2 - C_1$

which is the required solution of given differential equation. **(1)**

37. Given differential equation is

$$x \cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x \quad \text{...(i)}$$

which is a homogeneous differential equation as

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i),

we get

$$x \cos v \left[v + x\frac{dv}{dx}\right] = vx \cos v + x \quad \text{(1)}$$

$$vx \cos v + x^2 \cos v \frac{dv}{dx} = vx \cos v + x$$

$$\Rightarrow \quad x^2 \cos v \frac{dv}{dx} = x$$

$$\Rightarrow \quad \cos v \, dv = \frac{dx}{x} \quad \text{(1)}$$

On integrating both sides, we get

$$\int \cos v \, dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \sin v = \log|x| + C \quad \text{(1)}$$

$$\Rightarrow \quad \sin\left(\frac{y}{x}\right) = \log|x| + C \quad \left[\text{put } v = \frac{y}{x}\right]$$

which is the required solution of given differential equation. **(1)**

38. Given differential equation is

$$x\frac{dy}{dx} - y + x \operatorname{cosec}\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0$$

[dividing both sides by x]

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right) \quad \text{...(i)}$$

which is a homogeneous differential equation as

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

On putting $y = vx$,

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \text{ in Eq. (i), we get}$$

$$v + x\frac{dv}{dx} = \frac{vx}{x} - \operatorname{cosec}\left(\frac{vx}{x}\right)$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v - \operatorname{cosec} v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\operatorname{cosec} v \Rightarrow \frac{dv}{\operatorname{cosec} v} = \frac{-dx}{x} \quad \text{(1)}$$

On integrating both sides, we get

$$\int \frac{dv}{\operatorname{cosec} v} = \int -\frac{dx}{x}$$

$$\Rightarrow \quad \int \sin v \, dv = \int -\frac{dx}{x}$$

$$\Rightarrow \quad -\cos v = -\log|x| + C$$

$$\Rightarrow \quad -\cos\frac{y}{x} = -\log|x| + C \quad \left[\text{put } v = \frac{y}{x}\right]$$

$$\Rightarrow \quad \cos\frac{y}{x} = (\log|x| - C) \quad \text{...(ii) (1½)}$$

[multiply both sides by -1]

Also, given that $x = 1$ and $y = 0$.

On putting above values in Eq. (ii), we get

$$\Rightarrow \quad \cos 0 = \log|1| - C$$

$$\Rightarrow \quad 1 = 0 - C \Rightarrow C = -1$$

$$\therefore \quad \cos\frac{y}{x} = \log|x| + 1 \quad \text{[from Eq. (ii)]}$$

which is required particular solution of given differential equation. **(1½)**

39. Given differential equation is

$$\frac{dy}{dx} = 1 + x + y + xy$$

$$\Rightarrow \quad \frac{dy}{dx} = 1(1 + x) + y(1 + x)$$

$$\Rightarrow \quad \frac{dy}{dx} = (1 + x)(1 + y) \quad \text{...(i) (1)}$$

On separating variables, we get

$$\frac{1}{(1 + y)}dy = (1 + x)\,dx \quad \text{...(ii)}$$

On integrating both sides of Eq. (ii), we get

$$\int \frac{1}{1 + y}\,dy = \int (1 + x)\,dx$$

$\Rightarrow \quad \log|1+y| = x + \dfrac{x^2}{2} + C \qquad \ldots(\text{iii})$ **(1)**

Also, given that $y = 0$, when $x = 1$.

On substituting $x = 1$, $y = 0$ in Eq. (iii), we get

$\log|1+0| = 1 + \dfrac{1}{2} + C \Rightarrow C = -\dfrac{3}{2}$ $[\because \log 1 = 0]$ **(1)**

Now, on substituting the value of C in Eq. (iii), we get

$$\log|1+y| = x + \frac{x^2}{2} - \frac{3}{2}$$

which is the required particular solution of given differential equation. **(1)**

40. Solve as Q. No. 24.

Hint Given differential equation is a linear differential equation of the form $\dfrac{dy}{dx} + Py = Q$ and its solution is given by $y \cdot (\text{IF}) = \int Q \cdot (\text{IF}) + C$, where $\text{IF} = e^{\int P\,dx}$.

$$\left[\textbf{Ans. } y\,e^{\tan^{-1}x} = \frac{e^{2\tan^{-1}x}}{2} + C\right]$$

41. Given differential equation is

$$\log\left(\frac{dy}{dx}\right) = 3x + 4y$$

$\Rightarrow \quad \dfrac{dy}{dx} = e^{3x+4y}$ $[\because \log m = n \Rightarrow e^n = m]$

$\Rightarrow \quad \dfrac{dy}{dx} = e^{3x}\,e^{4y}$ **(1)**

On separating the variables, we get

$$\frac{1}{e^{4y}}\,dy = e^{3x}\,dx$$

On integrating both sides, we get

$\int e^{-4y}\,dy = \int e^{3x}\,dx \Rightarrow \dfrac{e^{-4y}}{-4} = \dfrac{e^{3x}}{3} + C \qquad \ldots(\text{i})$ **(1)**

Also, given that $y = 0$, when $x = 0$.

On putting $y = 0$ and $x = 0$ in Eq. (i), we get

$\dfrac{e^{-4(0)}}{-4} = \dfrac{e^{3(0)}}{3} + C \Rightarrow -\dfrac{1}{4} = \dfrac{1}{3} + C$ $[\because e^{-0} = e^0 = 1]$

$\Rightarrow \quad C = -\dfrac{1}{4} - \dfrac{1}{3}$

$\therefore \quad C = -\dfrac{7}{12}$ **(1)**

On substituting the value of C in Eq. (i), we get

$$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} - \frac{7}{12} \Rightarrow 4e^{3x} + 3e^{-4y} - 7 = 0$$

which is the required particular solution of given differential equation. **(1)**

42. Given differential equation is

$$x(1+y^2)\,dx - y(1+x^2)\,dy = 0$$

$\Rightarrow \quad x(1+y^2)\,dx = y(1+x^2)\,dy$

On separating the variables, we get

$\dfrac{y}{(1+y^2)}\,dy = \dfrac{x}{(1+x^2)}\,dx$ **(1)**

On integrating both sides, we get

$$\int \frac{y}{1+y^2}\,dy = \int \frac{x}{(1+x^2)}\,dx$$

$\Rightarrow \quad \dfrac{1}{2}\log|1+y^2| = \dfrac{1}{2}\log|1+x^2| + C \qquad \ldots(\text{i})$

$\left[\begin{array}{l}\text{put } 1+y^2 = u \Rightarrow 2y\,dy = du,\\ \text{then } \int \dfrac{y}{1+y^2}\,dy = \int \dfrac{1}{2u}\,du = \dfrac{1}{2}\log|u|\\ \text{and put } 1+x^2 = v \Rightarrow 2x\,dx = dv,\\ \text{then } \int \dfrac{x}{1+x^2}\,dx = \dfrac{1}{2}\int \dfrac{1}{v}\,dv = \dfrac{1}{2}\log|v|\end{array}\right]$

Also, given that $y = 1$, when $x = 0$. **(1)**

On substituting the values of x and y in Eq. (i), we get

$$\frac{1}{2}\log|1+(1)^2| = \frac{1}{2}\log|1+(0)^2| + C$$

$\Rightarrow \quad \dfrac{1}{2}\log 2 = C$ $[\because \log 1 = 0]$

On putting $C = \dfrac{1}{2}\log 2$ in Eq. (i), we get

$$\frac{1}{2}\log|1+y^2| = \frac{1}{2}\log|1+x^2| + \frac{1}{2}\log 2$$

$\Rightarrow \quad \log|1+y^2| = \log|1+x^2| + \log 2$ **(1)**

$\Rightarrow \quad \log|1+y^2| - \log|1+x^2| = \log 2$

$\Rightarrow \quad \log\left|\dfrac{1+y^2}{1+x^2}\right| = \log 2$

$\left[\because \log m - \log n = \log\dfrac{m}{n}\right]$

$\Rightarrow \quad \dfrac{1+y^2}{1+x^2} = 2$

$\Rightarrow \quad 1+y^2 = 2 + 2x^2$

$\Rightarrow \quad y^2 - 2x^2 - 1 = 0$

which is the required particular solution of given differential equation. **(1)**

43. Given differential equation is

$$(x \log|x|)\frac{dy}{dx} + y = \frac{2}{x}\log|x|$$

On dividing both sides by $x \log x$, we get

$$\frac{dy}{dx} + \frac{y}{x\log|x|} = \frac{2}{x^2}\frac{\log|x|}{\log|x|} = \frac{2}{x^2}$$

which is a linear differential equation of first order and is of the form $\frac{dy}{dx} + Py = Q$.

here, $P = \frac{1}{x\log|x|}$ and $Q = \frac{2}{x^2}$ **(1)**

$\therefore \quad \text{IF} = e^{\int P\,dx} = e^{\int \frac{1}{x\log|x|}dx} = e^{\log|\log x|}$

$\left[\because I = \int \frac{1}{x\log|x|}dx, \text{ put } \log|x| = t \Rightarrow \frac{1}{x}dx = dt \quad \therefore I = \int \frac{1}{t}dt = \log|t| = \log|\log x|\right]$

$\Rightarrow \quad \text{IF} = \log|x| \qquad [\because e^{\log x} = x]$ **(1)**

Now, solution of above equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\,dx + C$$

$$y\log|x| = \int \frac{2}{x^2}\log|x|\,dx$$

$$\Rightarrow \quad y\log|x| = 2\left[\log|x|\int\frac{1}{x^2}dx - \int\left(\frac{d}{dx}(\log|x|)\cdot\int\frac{1}{x^2}dx\right)dx\right]$$

[by using integration by parts]

$$\Rightarrow \quad y\log|x| = 2\left[\log|x|\cdot\left(-\frac{1}{x}\right) - \int\frac{1}{x}\cdot\left(-\frac{1}{x}\right)dx\right] \quad \textbf{(1)}$$

$$\Rightarrow \quad y\log|x| = 2\left[-\frac{1}{x}\log|x| + \int\frac{1}{x^2}dx\right]$$

$$\therefore \quad y\log|x| = -\frac{2}{x}\log|x| - \frac{2}{x} + C,$$

which is the required solution. **(1)**

44. Given differential equation is

$$\frac{dy}{dx} + y\cot x = 2\cos x$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \cot x$ and $Q = 2\cos x$

$\therefore \quad \text{IF} = e^{\int P\,dx} = e^{\int \cot x\,dx} = e^{\log|\sin x|} \Rightarrow \text{IF} = \sin x$ **(1)**

The general solution is given by

$$y \times \text{IF} = \int (\text{IF} \times Q)\,dx + C$$

$$\Rightarrow \quad y\sin x = \int 2\sin x\cos x\,dx + C$$

$$\Rightarrow \quad y\sin x = \int \sin 2x\,dx + C$$

$$\Rightarrow \quad y\sin x = -\frac{\cos 2x}{2} + C \quad \text{...(i)} \ \textbf{(1)}$$

Also, given that $y = 0$, when $x = \frac{\pi}{2}$.

On putting $x = \frac{\pi}{2}$ and $y = 0$ in Eq. (i), we get

$$0\sin\frac{\pi}{2} = -\frac{\cos\left(2\frac{\pi}{2}\right)}{2} + C$$

$$\Rightarrow \quad C - \frac{\cos\pi}{2} = 0$$

$$\Rightarrow \quad C + \frac{1}{2} = 0 \qquad [\because \cos\pi = -1]$$

$$\therefore \quad C = -\frac{1}{2} \quad \textbf{(1)}$$

On putting the value of C in Eq. (i), we get

$$y\sin x = -\frac{\cos 2x}{2} - \frac{1}{2}$$

$\therefore \quad 2y\sin x + \cos 2x + 1 = 0$

which is the required solution. **(1)**

45. Given differential equation is

$$(x^2 - yx^2)\,dy + (y^2 + x^2y^2)\,dx = 0$$

$$\Rightarrow \quad x^2(1 - y)\,dy + y^2(1 + x^2)\,dx = 0$$

$$\Rightarrow \quad -x^2(1 - y)\,dy = y^2(1 + x^2)\,dx$$

$$\Rightarrow \quad x^2(y - 1)\,dy = y^2(1 + x^2)\,dx$$

$$\Rightarrow \quad \frac{y-1}{y^2}dy = \frac{1+x^2}{x^2}dx \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int\frac{y-1}{y^2}dy = \int\frac{1+x^2}{x^2}dx$$

$$\Rightarrow \quad \int\frac{1}{y}dy - \int\frac{1}{y^2}dy = \int\frac{1}{x^2}dx + \int 1\,dx$$

$$\Rightarrow \quad \log|y| + \frac{1}{y} = \frac{-1}{x} + x + C \quad \text{...(i)} \ \textbf{(1)}$$

Also, given that $y = 1$, when $x = 1$

On putting $y = 1$ and $x = 1$ in Eq. (i), we get

$$\log|1| + 1 = -1 + 1 + C$$

$$\Rightarrow \quad C = 1 \quad \textbf{(1)}$$

On putting the value of C in Eq.(i), we get

$$\log|y| + \frac{1}{y} = \frac{-1}{x} + x + 1$$

which is the required solution. **(1)**

46. First, convert the given differential equation in homogeneous and then put $y = vx$.

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Further, separate the variables and integrate it, then substitute the value of v and get the required result.

Given differential equation is rewritten as

$$\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)\cdot y - \left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)\cdot x\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left[x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right]\cdot y}{\left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)\cdot x} \quad \text{...(i)}$$

which is a homogeneous differential equation.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i), we get **(1)**

$$v + x\frac{dv}{dx} = \frac{(x\cos v + vx\sin v)\cdot vx}{(vx\sin v - x\cos v)\cdot x}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v} \quad \textbf{(1)}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \frac{v\sin v - \cos v}{v\cos v}dv = 2\frac{dx}{x}$$

On integrating both sides, we get

$$\int\frac{v\sin v - \cos v}{v\cos v}dv = 2\int\frac{dx}{x}$$

$$\Rightarrow \int\left(\frac{v\sin v}{v\cos v} - \frac{\cos v}{v\cos v}\right)dv = 2\int\frac{dx}{x}$$

$$\Rightarrow \int\left(\tan v - \frac{1}{v}\right)dv = 2\int\frac{dx}{x}$$

$$\Rightarrow \log|\sec v| - \log|v| = 2\log|x| + C \quad \textbf{(1)}$$

$$\left[\because \int\tan v\,dv = \log|\sec v| \text{ and } \int\frac{1}{x}dx = \log|x|\right]$$

$$\Rightarrow \log|\sec v| - \log|v| - 2\log|x| = C$$

$$\Rightarrow \log|\sec v| - \log|v| - \log|x|^2 = C$$

$$[\because n\log m = \log m^n]$$

$$\Rightarrow \log|\sec v| - \log\left|vx^2\right| = C$$

$$[\because \log m + \log n = \log mn]$$

$$\Rightarrow \log\left|\frac{\sec v}{vx^2}\right| = C$$

$$\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$\Rightarrow \log\left|\frac{\sec\frac{y}{x}}{\frac{y}{x}\cdot x^2}\right| = C \quad \left[\text{put } v = \frac{y}{x}\right]$$

$$\therefore \log\left|\frac{\sec\frac{y}{x}}{xy}\right| = C \Rightarrow \frac{\sec\frac{y}{x}}{xy} = e^C$$

$$\Rightarrow \sec\frac{y}{x} = Axy \quad [\because e^C = A]$$

which is the required solution. **(1)**

47. We have, $\frac{dy}{dx} - y = \cos x$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, here $P = -1$ and $Q = \cos x$

$$\therefore \text{IF} = e^{\int P dx} = e^{\int(-1)dx} = e^{-x}$$

The general solution is given by

$$y \times \text{IF} = \int(\text{IF} \times Q)\,dx + C$$

$$\Rightarrow y\cdot e^{-x} = \int e^{-x}\cos x\,dx + C \quad \text{...(i)}$$

Now, $\int \underset{\text{I}}{e^{-x}}\underset{\text{II}}{\cos x}\,dx = e^{-x}\sin x + \int \underset{\text{I}}{e^{-x}}\underset{\text{II}}{\sin x}\,dx$

[integrating by parts]

$$= e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\cos x\,dx$$

$$\Rightarrow 2\int e^{-x}\cos x\,dx = e^{-x}(\sin x - \cos x)$$

$$\Rightarrow \int e^{-x}\cos x\,dx = \frac{1}{2}e^{-x}(\sin x - \cos x) \quad \textbf{(2)}$$

On substituting this value in Eq. (i), we get

$$y\cdot e^{-x} = \frac{1}{2}e^{-x}(\sin x - \cos x) + C \quad \text{...(ii)}$$

On putting $x = 0$, $y = 1$ in Eq. (ii), we get

$$1\cdot e^{-0} = \frac{1}{2}e^{-0}(\sin 0 - \cos 0) + C$$

$$\Rightarrow 1 = \frac{1}{2}(-1) + C \Rightarrow C = \frac{3}{2} \quad \textbf{(1)}$$

On putting $C = \frac{3}{2}$ in Eq. (ii), we get

$$y\cdot e^{-x} = \frac{1}{2}e^{-x}(\sin x - \cos x) + \frac{3}{2}$$

$$\Rightarrow y = \frac{1}{2}(\sin x - \cos x) + \frac{3}{2}e^x$$

which is the required particular solution. **(1)**

48. We have, $x\frac{dy}{dx} + 2y = x^2, (x \neq 0)$

$$\Rightarrow \quad \frac{dy}{dx} + \left(\frac{2}{x}\right) \cdot y = x \quad \text{...(i)} \ \textbf{(1/2)}$$

This is linear differential equation of the form $\frac{dy}{dx} + Py = Q$, here $P = \frac{2}{x}$ and $Q = x$.

$$\therefore \text{IF} = e^{\int P dx} = e^{\int (2/x) dx} = e^{2\log x} = e^{\log x^2} = x^2 \quad \textbf{(1)}$$

The general solution is given by

$$y \cdot \text{IF} = \int (\text{IF} \times Q)\, dx + C$$

$$\Rightarrow \quad y \cdot x^2 = \int x^2 \times x\, dx + C$$

$$\Rightarrow \quad y \cdot x^2 = \int x^3\, dx + C$$

$$\therefore \quad y \cdot x^2 = \frac{x^4}{4} + C \quad \text{...(ii)} \ \textbf{(1½)}$$

On putting $x = 2$, $y = 1$ in Eq. (ii), we get

$$1 \cdot 2^2 = \frac{2^4}{4} + C \Rightarrow 4 = 4 + C \Rightarrow C = 0$$

$$\therefore \quad y \cdot x^2 = \frac{x^4}{4} \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad y = \frac{x^2}{4}$$

which is the required particular solution. **(1)**

49. We have, $\frac{dy}{dx} + y \cot x = 2x + x^2 \cot x, (x \neq 0)$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$.

Here, $P = \cot x$ and $Q = 2x + x^2 \cot x$.

$$\therefore \quad \text{IF} = e^{\int P dx} = e^{\int \cot x\, dx} = e^{\log|\sin x|} = \sin x \quad \textbf{(1)}$$

The general solution is given by

$$y \cdot \text{IF} = \int (\text{IF} \times Q) dx + C$$

$$\Rightarrow y \cdot \sin x = \int (2x + x^2 \cdot \cot x) \sin x\, dx + C$$

$$= 2\int x \sin x\, dx + \int x^2 \cos x\, dx + C$$

$$= 2\int x \sin x\, dx + x^2 \sin x - \int 2x \sin x\, dx + C$$

$$\Rightarrow \quad y \cdot \sin x = x^2 \sin x + C \quad \text{...(i)} \ \textbf{(1)}$$

On putting $x = \frac{\pi}{2}$ and $y = 0$ in Eq. (i), we get

$$0 \cdot \sin\frac{\pi}{2} = \left(\frac{\pi}{2}\right)^2 \cdot \sin\frac{\pi}{2} + C \Rightarrow C = -\frac{\pi^2}{4} \quad \textbf{(1)}$$

On putting $C = \frac{-\pi^2}{4}$ in Eq. (i), we get

$$y \cdot \sin x = x^2 \sin x - \frac{\pi^2}{4}$$

$$\therefore \quad y = x^2 - \frac{\pi^2}{4} \operatorname{cosec} x$$

[dividing both sides by $\sin x$] **(1)**

50. Given differential equation is

$$\frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x$$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, here $P = \cot x$ and $Q = 4x \operatorname{cosec} x$ **(1)**

$$\therefore \quad \text{IF} = e^{\int P dx} = e^{\int \cot x\, dx}$$

$$= e^{\log|\sin x|} = \sin x \quad [\because e^{\log|x|} = x] \quad \textbf{(1)}$$

The solution of linear differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\, dx + C$$

$$\Rightarrow \quad y \times \sin x = \int 4x \operatorname{cosec} x \cdot \sin x\, dx + C$$

$$\Rightarrow \quad y \sin x = \int 4x \cdot \frac{1}{\sin x} \cdot \sin x\, dx + C$$

$$\Rightarrow \quad y \sin x = \int 4x\, dx + C$$

$$\Rightarrow \quad y \sin x = 2x^2 + C \quad \text{...(i)} \ \textbf{(1)}$$

Also, given that $y = 0$, when $x = \frac{\pi}{2}$.

On putting $y = 0$ and $x = \frac{\pi}{2}$ in Eq. (i), we get

$$0 = 2 \times \frac{\pi^2}{4} + C \Rightarrow C = \frac{-\pi^2}{2}$$

On putting $C = -\frac{\pi^2}{2}$ in Eq. (i), we get

$$y \sin x = 2x^2 - \frac{\pi^2}{2}$$

$$\therefore \quad y = 2x^2 \operatorname{cosec} x - \frac{\pi^2}{2} \operatorname{cosec} x$$

[dividing both sides by $\sin x$]

which is the required solution. **(1)**

51. We have, $xy\frac{dy}{dx} = (x + 2)(y + 2)$

On separating the variables, we get

$$\frac{y\,dy}{y + 2} = \frac{x + 2}{x} dx \quad \textbf{(1)}$$

$\Rightarrow \left(\frac{y+2-2}{y+2}\right)dy = \left(1+\frac{2}{x}\right)dx$

$\Rightarrow \left(1-\frac{2}{y+2}\right)dy = \left(1+\frac{2}{x}\right)dx$

On integrating both sides, we get

$\int\left(1-\frac{2}{y+2}\right)dy = \int\left(1+\frac{2}{x}\right)dx$ **(1)**

$\Rightarrow y - 2\log|y+2| = x + 2\log|x| + C$ ...(i)

Given that $y=-1$, when $x=1$

On putting $x=1$ and $y=-1$ in Eq. (i), we get

$-1-2\log(1) = 1 + 2\log|1| + C$

$\Rightarrow -1 = 1 + C \Rightarrow C = -2$ **(1)**

On putting $C=-2$ in Eq. (i), we get

$y - 2\log|y+2| = x + 2\log|x| - 2$

which is required particular solution. **(1)**

52. Given differential equation is

$2x^2\frac{dy}{dx} - 2xy + y^2 = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x} - \frac{y^2}{2x^2}$ ...(i)

which is a homogeneous differential equation as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ **(1)**

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i),

we get

$\Rightarrow v + x\frac{dv}{dx} = v - \frac{v^2}{2} \Rightarrow x\frac{dv}{dx} = \frac{-v^2}{2} \Rightarrow \frac{2dv}{v^2} = -\frac{1}{x}dx$ **(1)**

On integrating both sides, we get

$2\int v^{-2}dv = -\log|x| + C$

$\Rightarrow \frac{2v^{-1}}{-1} = -\log|x| + C$ **(1)**

$\Rightarrow \frac{-2}{v} = -\log|x| + C$

$\Rightarrow \frac{-2x}{y} = -\log|x| + C$ $\left[\text{put } v = \frac{y}{x}\right]$

$\Rightarrow -2x = y(-\log|x| + C)$

$\therefore y = \frac{-2x}{-\log|x| + C}$

which is the required solution. **(1)**

53. Given differential equation is

$\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$

$\Rightarrow \frac{dy}{dx} = 1(1+x^2) + y^2(1+x^2)$ **(1)**

$\Rightarrow \frac{dy}{dx} = (1+x^2)(1+y^2)$

On separating the variables, we get **(1)**

$\Rightarrow \frac{dy}{1+y^2} = (1+x^2)dx$

On integrating both sides, we get

$\int\frac{dy}{1+y^2} = \int(1+x^2)dx$

$\Rightarrow \tan^{-1}y = x + \frac{x^3}{3} + C$ ...(i)

Also, given that $y=1$, when $x=0$.

On putting $x=0$ and $y=1$ in Eq. (i), we get

$\tan^{-1}1 = C$

$\Rightarrow \tan^{-1}(\tan\pi/4) = C$ $\left[\because \tan\frac{\pi}{4} = 1\right]$ **(1)**

$\Rightarrow C = \pi/4$

On putting the value of C in Eq. (i), we get

$\tan^{-1}y = x + \frac{x^3}{3} + \frac{\pi}{4}$

$\therefore y = \tan\left(x + \frac{x^3}{3} + \frac{\pi}{4}\right)$

which is the required solution. **(1)**

54. Given differential equation is

$\frac{dy}{dx} + y\sec x = \tan x$

which is a linear differential equation of first order and is of the form

$\frac{dy}{dx} + Py = Q$ ...(i)

Here, $P = \sec x$ and $Q = \tan x$ **(1)**

$\therefore \text{IF} = e^{\int P dx} = e^{\int \sec x\, dx} = e^{\log|\sec x + \tan x|}$

$[\because \int \sec x\, dx = \log|\sec x + \tan x|]$

$\Rightarrow \text{IF} = \sec x + \tan x$ **(1)**

The general solution is $y \times \text{IF} = \int(Q\times\text{IF})\,dx + C$

$y(\sec x + \tan x) = \int \tan x\cdot(\sec x + \tan x)\,dx$

$\Rightarrow y(\sec x + \tan x) = \int \sec x\tan x\,dx + \int \tan^2 x\,dx$

$\Rightarrow y(\sec x + \tan x) = \sec x + \int(\sec^2 x - 1)\,dx$ **(1)**

$\Rightarrow y(\sec x + \tan x) = \sec x + \tan x - x + C$

$[\because \int \sec^2 x\,dx = \tan x]$

On dividing both sides by $(\sec x + \tan x)$, we get

$y = 1 - \frac{x}{\sec x + \tan x} + \frac{C}{\sec x + \tan x}$. **(1)**

55. Given differential equation is

$$x(x^2-1)\frac{dy}{dx}=1 \Rightarrow \frac{dy}{dx}=\frac{1}{x(x^2-1)}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x(x-1)(x+1)}$$

$[\because a^2-b^2=(a-b)(a+b)]$

$$\Rightarrow \quad dy=\frac{dx}{x(x-1)(x+1)}$$

On integrating both sides, we get

$$\int dy=\int\frac{dx}{x(x-1)(x+1)}$$

$$\Rightarrow \quad y=I+K \quad \text{...(i)}$$

where, $I=\int\frac{dx}{x(x-1)(x+1)}$ **(1)**

By using partial fraction method,

let $\frac{1}{x(x-1)(x+1)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$

$\Rightarrow 1=A(x-1)(x+1)+Bx(x+1)+Cx(x-1)$

$\Rightarrow 1=A(x^2-1)+B(x^2+x)+C(x^2-x)$

On comparing the coefficients of x^2, x and constant terms from both sides, we get

$$A+B+C=0$$

$$B-C=0$$

and $$-A=1$$

$$\Rightarrow \quad A=-1$$

On solving above equations, we get

$$A=-1, B=\frac{1}{2} \text{ and } C=\frac{1}{2},$$

then $\frac{1}{x(x-1)(x+1)}=\frac{-1}{x}+\frac{1/2}{x-1}+\frac{1/2}{x+1}$ **(1)**

On integrating both sides w.r.t. x, we get

$$I=\int\frac{1}{x(x-1)(x+1)}dx=\int\frac{-1}{x}dx+\frac{1}{2}\int\frac{dx}{x-1}+\frac{1}{2}\int\frac{dx}{x+1}$$

$$\Rightarrow \quad I=-\log|x|+\frac{1}{2}\log|x-1|+\frac{1}{2}\log|x+1|$$

On putting the value of I in Eq. (i), we get

$$y=-\log|x|+\frac{1}{2}\log|x-1|+\frac{1}{2}\log|x+1|+K \quad \text{...(ii)}$$

Also, given that $y=0$, when $x=2$.

On putting $y=0$ and $x=2$ in Eq. (ii), we get

$$0=-\log 2+\frac{1}{2}\log 1+\frac{1}{2}\log 3+K$$

$$\Rightarrow \quad K=\log 2-\frac{1}{2}\log 1-\frac{1}{2}\log 3$$

$$\Rightarrow \quad K=\log 2-\log\sqrt{3} \quad [\because \log 1=0]$$

$$\Rightarrow \quad K=\log\frac{2}{\sqrt{3}}$$ **(1)**

On putting the value of K in Eq. (ii), we get

$$y=-\log|x|+\frac{1}{2}\log|x-1|+\frac{1}{2}\log|x+1|+\log\frac{2}{\sqrt{3}}$$

which is the required solution. **(1)**

56. Given differential equation is

$$(1+x^2)\,dy+2xy\,dx=\cot x\,dx \quad [\because x\neq 0]$$

$$\Rightarrow \quad (1+x^2)\,dy=(\cot x-2xy)\,dx$$

On dividing both sides by $(1+x^2)$, we get

$$dy=\frac{\cot x-2xy}{1+x^2}dx$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{\cot x}{1+x^2}$$ **(1)**

which is a linear differential equation of 1st order and is of the form

$$\frac{dy}{dx}+Py=Q$$

Here, $P=\frac{2x}{1+x^2}$ and $Q=\frac{\cot x}{1+x^2}$

$$\therefore \quad \text{IF}=e^{\int P dx}=e^{\int\frac{2x}{1+x^2}dx}=e^{\log|1+x^2|}=1+x^2$$ **(1)**

$$\left[\begin{array}{l}\text{put } 1+x^2=t \Rightarrow 2x\,dx=dt, \text{ then} \\ \int\frac{2x}{1+x^2}dx=\int\frac{dt}{t}=\log|t|=\log|1+x^2|\end{array}\right]$$

The solution of linear differential equation is given by

$$y\times\text{IF}=\int(Q\times\text{IF})\,dx+C$$

$$\therefore \quad y(1+x^2)=\int\frac{\cot x}{1+x^2}\times(1+x^2)\,dx+C$$

$$\Rightarrow \quad y(1+x^2)=\int\cot x\,dx+C$$ **(1)**

$$\Rightarrow \quad y(1+x^2)=\log|\sin x|+C$$

$[\because \int\cot x\,dx=\log|\sin x|]$

On dividing both sides by $(1+x^2)$, we get

$$y=\frac{\log|\sin x|}{1+x^2}+\frac{C}{1+x^2}$$

which is the required solution. **(1)**

57. We have, $x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} + \sin\left(\frac{y}{x}\right) = 0$$

[dividing both sides by x]...(i)

This is a homogeneous differential equation as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ **(1)**

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i), we get

$$v + x\frac{dv}{dx} - v + \sin v = 0 \Rightarrow x\frac{dv}{dx} + \sin v = 0$$

$$\Rightarrow \quad \operatorname{cosec} v\, dv + \frac{dx}{x} = 0 \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int \operatorname{cosec} v\, dv + \int \frac{dx}{x} = \log C$$

$$\Rightarrow \log|\operatorname{cosec} v - \cot v| + \log x = \log C$$

$$\Rightarrow \quad x(\operatorname{cosec} v - \cot v) = C$$

$$\Rightarrow \quad x\left[\operatorname{cosec}\left(\frac{y}{x}\right) - \cot\left(\frac{y}{x}\right)\right] = C$$

$$\left[\because v = \frac{y}{x}\right] \text{...(ii)} \quad \textbf{(1)}$$

On putting $x = 2$ and $y = \pi$ in Eq. (ii) we get

$$2\left[\operatorname{cosec}\left(\frac{\pi}{2}\right) - \cot\left(\frac{\pi}{2}\right)\right] = C \Rightarrow C = 2$$

On putting $C = 2$ in Eq. (ii), we get

$$\therefore x\left[\operatorname{cosec}\left(\frac{y}{x}\right) - \cot\left(\frac{y}{x}\right)\right] = 2$$

which is the required particular solution. **(1)**

58. Given differential equation is

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - \sin^2\left(\frac{y}{x}\right) \text{...(i)} \quad \textbf{(1)}$$

which is a homogeneous differential equation as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i), we get

$$v + x\frac{dv}{dx} = v - \sin^2 v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \quad \operatorname{cosec}^2 v\, dv = -\frac{dx}{x} \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int \operatorname{cosec}^2 v\, dv + \int \frac{dx}{x} = 0$$

$$\Rightarrow \quad -\cot v + \log|x| = C$$

$$\Rightarrow \quad -\cot\left(\frac{y}{x}\right) + \log|x| = C \quad \left[\because v = \frac{y}{x}\right]$$

$\Rightarrow y = x \cdot \cot^{-1}(\log x - C)$; which is the required solution. **(2)**

59. Given differential equation is

$(1 + y^2)(1 + \log|x|)\,dx + x\,dy = 0$

On separating the variables, we get

$$\frac{1 + \log|x|}{x}\,dx = \frac{-dy}{1 + y^2} \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{1 + \log|x|}{x}\,dx = -\int \frac{dy}{1 + y^2}$$

$$\Rightarrow \quad \int \frac{1}{x}\,dx + \int \frac{\log|x|}{x}\,dx = -\int \frac{dy}{1 + y^2}$$

$$\Rightarrow \quad \log|x| + I_1 + K = -\tan^{-1} y \quad \text{...(i)} \textbf{(1)}$$

where, $I_1 = \int \frac{\log|x|}{x}\,dx$

Put $\log|x| = t \Rightarrow \frac{1}{x}dx = dt$

$$\therefore \quad I_1 = \int t\,dt$$

$$= \frac{t^2}{2} + C_1 = \frac{(\log|x|)^2}{2} + C_1 \quad \textbf{(1)}$$

On putting the value of I_1 in Eq.(i), we get

$$\log|x| + \frac{(\log|x|)^2}{2} + C = -\tan^{-1} y$$

[where $C = C_1 + K$]

$$\Rightarrow \quad \tan^{-1} y = -\left[\log|x| + \frac{(\log|x|)^2}{2} + C\right]$$

$$\therefore \quad y = \tan\left[-\log|x| - \frac{(\log|x|)^2}{2} - C\right]$$

which is the required solution. **(1)**

60. Do same as Q. No. 13.

$$\left[\textbf{Ans. } y = \tan^{-1}\left(\frac{e^x - 1}{C}\right)\right]$$

61. First, transform the given differential equation in the form of $\frac{dy}{dx} = F(x, y)$. Now, replace $x = \lambda x$ and $y = \lambda y$ and verify whether $F(\lambda x, \lambda y) = \lambda^n F(x, y), n \in Z$. If above equation is satisfied, then given equation is said to be homogeneous equation. Then, we use the substitution $y = vx$ and solve the differential equation by using variable separable method.

Given differential equation is

$$y\,dx + x \log\left|\frac{y}{x}\right| dy - 2x\,dy = 0$$

$$\Rightarrow \quad y\,dx = \left[2x - x\log\left|\frac{y}{x}\right|\right]dy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{2x - x\log\left|\frac{y}{x}\right|} \quad \ldots(i)$$

Now, let $F(x, y) = \dfrac{y}{2x - x\log\left|\frac{y}{x}\right|}$

On replace x by λx and y by λy both sides, we get

$$F(\lambda x, \lambda y) = \frac{\lambda y}{2\lambda x - \lambda x \log\left|\frac{\lambda y}{\lambda x}\right|}$$

$$= \frac{\lambda y}{\lambda\left[2x - x\log\left|\frac{y}{x}\right|\right]}$$

$$\Rightarrow \quad F(\lambda x, \lambda y) = \lambda^0 \frac{y}{2x - x\log\left|\frac{y}{x}\right|} = \lambda^0 F(x, y)$$

So, the given differential equation is homogeneous. (1)

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{vx}{2x - x\log\left|\frac{vx}{x}\right|} = \frac{v}{2 - \log|v|}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{2 - \log|v|} - v = \frac{v - 2v + v\log|v|}{2 - \log|v|}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v + v\log|v|}{2 - \log|v|}$$

$$\Rightarrow \quad \frac{2 - \log|v|}{v\log|v| - v}\,dv = \frac{dx}{x} \quad (1)$$

On integrating both sides, we get

$$\int \frac{2 - \log|v|}{v(\log|v| - 1)}\,dv = \int \frac{dx}{x}$$

On putting $\log|v| = t \Rightarrow \frac{1}{v}dv = dt$

Then, $$\int \frac{2 - t}{t - 1}\,dt = \log|x| + C$$

$$\Rightarrow \quad \int\left(\frac{1}{t - 1} - 1\right)dt = \log|x| + C \quad (1)$$

$$\Rightarrow \quad \log|t - 1| - t = \log|x| + C$$

$$\Rightarrow \quad \log|\log v - 1| - \log v = \log|x| + C \quad [\text{put } t = \log|v|]$$

$$\Rightarrow \quad \log\left|\frac{\log v - 1}{v}\right| = \log|x| + C$$

$$\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$\Rightarrow \log\left|\frac{\log v - 1}{v}\right| - \log|x| = C \Rightarrow \log\left|\frac{\log v - 1}{vx}\right| = C$$

$$\therefore \quad \log\left|\frac{\log\frac{y}{x} - 1}{y}\right| = C \quad \left[\because y = vx \Rightarrow v = \frac{y}{x}\right]$$

which is the required solution. (1)

62. Given differential equation is

$$(y + 3x^2)\frac{dx}{dy} = x \Rightarrow \frac{dy}{dx} = \frac{y}{x} + 3x$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = 3x \quad (1)$$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$.

Here, $P = \frac{-1}{x}$ and $Q = 3x$ (1)

$$\therefore \quad \text{IF} = e^{\int P dx} = e^{\int -\frac{1}{x}dx} = e^{-\log|x|} = e^{\log x^{-1}} = x^{-1}$$

$$\Rightarrow \quad \text{IF} = x^{-1} = \frac{1}{x}$$

The solution of linear differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\,dx + C$$

$$\Rightarrow \quad y \times \frac{1}{x} = \int\left(3x \times \frac{1}{x}\right)dx \quad (1)$$

$$\Rightarrow \quad \frac{y}{x} = \int 3dx \Rightarrow \frac{y}{x} = 3x + C$$

$$\therefore \quad y = 3x^2 + Cx$$

which is the required solution. (1)

63. Do same as Q. No. 62. [Ans. $y = 2x^2 + Cx$]

64. Do same as Q.No. 62. $\left[\textbf{Ans. } y = \frac{x^3}{4} + \frac{C}{x}\right]$

65. Given differential equation is

$$(1 + e^{2x})\, dy + (1 + y^2)e^x\, dx = 0$$

Above equation may be written as

$$\frac{dy}{1 + y^2} = \frac{-e^x}{1 + e^{2x}} dx$$

On integrating both sides, we get

$$\int \frac{dy}{1 + y^2} = -\int \frac{e^x}{1 + e^{2x}} dx \quad \textbf{(1)}$$

On putting $e^x = t \Rightarrow e^x\, dx = dt$ in RHS, we get

$$\tan^{-1} y = -\int \frac{1}{1 + t^2} dt$$

$$\Rightarrow \quad \tan^{-1} y = -\tan^{-1} t + C$$

$$\Rightarrow \quad \tan^{-1} y = -\tan^{-1}(e^x) + C \quad \ldots(i)$$

[put $t = e^x$] **(1)**

Also, given that $y = 1$, when $x = 0$.

On putting above values in Eq. (i), we get

$$\tan^{-1} 1 = -\tan^{-1}(e^0) + C$$

$$\Rightarrow \quad \tan^{-1} = -\tan^{-1} 1 + C \quad [\because e^0 = 1]$$

$$\Rightarrow \quad 2\tan^{-1} 1 = C$$

$$\Rightarrow \quad 2\tan^{-1}\left(\tan\frac{\pi}{4}\right) = C$$

$$\Rightarrow \quad C = 2 \times \frac{\pi}{4} = \frac{\pi}{2} \quad \textbf{(1)}$$

On putting $C = \frac{\pi}{2}$ in Eq. (i), we get

$$\tan^{-1} y = -\tan^{-1} e^x + \frac{\pi}{2}$$

$$\Rightarrow \quad y = \tan\left[\frac{\pi}{2} - \tan^{-1}(e^x)\right] = \cot\,[\tan^{-1}(e^x)]$$

$$= \cot\left[\cot^{-1}\left(\frac{1}{e^x}\right)\right] \quad \left[\because \tan^{-1} x = \cot^{-1}\frac{1}{x}\right]$$

$$\therefore \quad y = \frac{1}{e^x}$$

which is the required solution. **(1)**

66. Given differential equation is

$$xy \log\left|\frac{y}{x}\right| dx + \left[y^2 - x^2 \log\left|\frac{y}{x}\right|\right] dy = 0$$

$$xy \log\left|\frac{y}{x}\right| dx = \left[x^2 \log\left|\frac{y}{x}\right| - y^2\right] dy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{xy \log\left|\frac{y}{x}\right|}{x^2 \log\left|\frac{y}{x}\right| - y^2} = \frac{\frac{y}{x}\log\left|\frac{y}{x}\right|}{\log\left|\frac{y}{x}\right| - \frac{y^2}{x^2}} \quad \ldots\text{(i)}\textbf{(1)}$$

which is a homogeneous differential equation as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{v \log|v|}{\log|v| - v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v \log|v|}{\log|v| - v^2} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v \log|v| - v\log|v| + v^3}{\log|v| - v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^3}{\log|v| - v^2}$$

$$\Rightarrow \quad \frac{\log|v| - v^2}{v^3} dv = \frac{dx}{x} \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{\log|v| - v^2}{v^3} dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{\log|v|}{v^3} dv - \int \frac{1}{v} dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \int \underset{\text{II}}{|v|^{-3}} \underset{\text{I}}{\log|v|} dv - \log|v| = \log|x| + C_1$$

Using integration by parts, we get

$$\log|v| \int v^{-3} dv - \int\left[\frac{d}{dv}(\log|v|) \cdot \int v^{-3} dv\right] dv$$

$$= \log|v| + \log|x| + C_1$$

$$\Rightarrow \quad \frac{v^{-2}}{-2}\log|v| - \int \frac{1}{v}\frac{v^{-2}}{(-2)} dv = \log|v| + \log|x| + C_1$$

$$\Rightarrow \quad \frac{-1}{2v^2}\log|v| + \frac{1}{2}\int v^{-3} dv = \log|v| + \log|x| + C_1$$

$$\Rightarrow \quad \frac{-1}{2v^2}\log|v| + \frac{1}{2}\cdot\frac{v^{-2}}{(-2)} = \log|v| + \log|x| + C_1$$

$$\Rightarrow \quad \frac{-1}{2v^2}\log|v| - \frac{1}{4v^2} = \log|vx| + C_1 \quad \textbf{(1)}$$

$[\because \log m + \log n = \log mn]$

$$\Rightarrow \quad \frac{-1}{2}\cdot\frac{x^2}{y^2}\log\left|\frac{y}{x}\right| - \frac{1}{4}\cdot\frac{x^2}{y^2} = \log\left|\frac{y}{x}\cdot x\right| + C_1$$

$\left[\text{put } v = \frac{y}{x}\right]$

$$\Rightarrow \quad \frac{-x^2}{2y^2}\log\left|\frac{y}{x}\right| - \frac{x^2}{4y^2} = \log|y| + C_1$$

$$\Rightarrow \quad \frac{-x^2}{y^2}\left[\frac{\log\left|\frac{y}{x}\right|}{2} + \frac{1}{4}\right] = \log|y| + C_1$$

$$\Rightarrow \quad \frac{x^2}{4y^2}\left[2\log\left|\frac{y}{x}\right| + 1\right] + \log|y| = -C_1$$

$$\therefore \quad x^2\left[2\log\left|\frac{y}{x}\right| + 1\right] + 4y^2\log|y| = 4y^2C,$$

where $\quad C = -C_1$ **(1)**

which is the required solution.

67. Given differential equation is

$$\frac{dy}{dx} = y\tan x$$

It can be written as $\frac{dy}{y} = \tan x\, dx$ **(1)**

On integrating both sides, we get

$$\int\frac{dy}{y} = \int\tan x\, dx$$

$$\Rightarrow \quad \log|y| = \log|\sec x| + C \quad \text{...(i)} \; \textbf{(1)}$$

$$\left[\because \int\frac{1}{y}\,dy = \log|y| \text{ and } \int\tan x\, dx = \log|\sec x|\right]$$

Also, given that $y = 1$, when $x = 0$.

On putting $x = 0$ and $y = 1$ in Eq. (i), we get

$$\log 1 = \log(\sec 0) + C$$

$$\Rightarrow \quad 0 = \log 1 + C \quad [\because \sec 0 = 1] \; \textbf{(1)}$$

$$\Rightarrow \quad C = 0 \quad [\because \log 1 = 0]$$

On putting $C = 0$ in Eq. (i), we get

$$\log|y| = \log|\sec x|$$

$$\therefore \quad y = \sec x$$

which is the required solution. **(1)**

68. Given differential equation is

$$(x^2+1)\frac{dy}{dx} + 2xy = \sqrt{x^2+4}$$

On dividing both sides by (x^2+1), we get

$$\frac{dy}{dx} + \frac{2xy}{x^2+1} = \frac{\sqrt{x^2+4}}{x^2+1} \quad \textbf{(1)}$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $\quad P = \frac{2x}{x^2+1}$ and $Q = \frac{\sqrt{x^2+4}}{x^2+1}$

$$\therefore \quad \text{IF} = e^{\int P dx} = e^{\int\frac{2x}{x^2+1}dx} = e^{\log|x^2+1|}$$

$$\left[\text{put } x^2+1 = t \Rightarrow 2x\,dx = dt\right.$$

$$\left.\therefore \int\frac{2x}{x^2+1}dx = \int\frac{dt}{t} = \log|t| = \log|x^2+1|\right]$$

$$\Rightarrow \quad \text{IF} = x^2+1 \quad [\because e^{\log x} = x] \; \textbf{(1)}$$

The solution of this equation is given by

$$y \times \text{IF} = \int Q \times \text{IF}\, dx + C$$

$$\therefore \quad y(x^2+1) = \int\frac{\sqrt{x^2+4}}{x^2+1}(x^2+1)dx \quad \textbf{(1)}$$

$$\Rightarrow y(x^2+1) = \int\sqrt{x^2+4}\,dx$$

$$\Rightarrow y(x^2+1) = \int\sqrt{x^2+(2)^2}\,dx$$

$$\Rightarrow y(x^2+1) = \frac{x}{2}\sqrt{x^2+4} + \frac{4}{2}\log|x+\sqrt{x^2+4}| + C$$

$$\left[\because \int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + C\right]$$

$$\therefore \quad y(x^2+1) = \frac{x}{2}\sqrt{x^2+4} + 2\log|x+\sqrt{x^2+4}| + C$$

which is the required solution. **(1)**

69. Given differential equation is

$$(x^3+x^2+x+1)\frac{dy}{dx} = 2x^2+x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x^2+x}{x^3+x^2+x+1}$$

It is a variable separable type differential equation.

$$\therefore \quad dy = \frac{2x^2+x}{x^3+x^2+x+1}dx$$

On integrating both sides, we get

$$\int dy = \int\frac{2x^2+x}{x^3+x^2+x+1}dx$$

$$\Rightarrow \quad y = \int\frac{2x^2+x}{x^2(x+1)+1(x+1)}dx + C$$

$$\Rightarrow \quad y = \int\frac{2x^2+x}{(x+1)(x^2+1)}dx \quad \text{...(i)} \; \textbf{(1)}$$

Using partial fractions method,

$$\text{let } \frac{2x^2+x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \quad \text{...(ii)}$$

$$\Rightarrow \frac{2x^2+x}{(x+1)(x^2+1)} = \frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)}$$

$$\Rightarrow 2x^2+x = A(x^2+1)+(Bx+C)(x+1)$$

$$\Rightarrow 2x^2+x = A(x^2+1)+B(x^2+x)+C(x+1)$$

On comparing the coefficients of x^2, x and constant terms from both sides, we get

$$A+B=2;\ B+C=1$$

and $A+C=0 \Rightarrow A=-C$

On solving above equations, we get

$$A=\frac{1}{2}, B=\frac{3}{2} \text{ and } C=\frac{-1}{2} \quad \text{(1)}$$

On substituting the values of A, B and C in Eq. (ii), we get

$$\frac{2x^2+x}{(x+1)(x^2+1)} = \frac{\frac{1}{2}}{x+1} + \frac{\frac{3}{2}x-\frac{1}{2}}{x^2+1}$$

On integrating both sides, we get

$$\int \frac{2x^2+x}{(x+1)(x^2+1)}dx = \frac{1}{2}\int\frac{dx}{x+1} + \frac{3}{2}\int\frac{x}{x^2+1}dx - \frac{1}{2}\int\frac{dx}{x^2+1}$$

$$\Rightarrow y = \frac{1}{2}\log|x+1| + I_1 - \frac{1}{2}\tan^{-1}x + C_2 \quad \text{...(iii)}$$

[from Eq. (i)] (1)

where, $I_1 = \frac{3}{2}\int\frac{x}{x^2+1}dx$

Put $x^2+1=t \Rightarrow 2xdx = dt \Rightarrow xdx = \frac{dt}{2}$

$$\therefore \quad I_1 = \frac{3}{4}\int\frac{dt}{t} = \frac{3}{4}\log|t|+C_1$$

$$= \frac{3}{4}\log|x^2+1|+C_1$$

On putting the value of I_1 in Eq. (iii), we get

$$y = \frac{1}{2}\log|x+1| + \frac{3}{4}\log|x^2+1| - \frac{1}{2}\tan^{-1}x + C$$

[where, $C = C_1 + C_2$]

which is the required solution. (1)

70. Do same as Q.No. 27.

The general solution is

$$\log|x| + C = -\frac{1}{2}\log\left(\frac{y^2}{x^2}+\frac{y}{x}+1\right) + \sqrt{3}\tan^{-1}\left[\left(\frac{2y}{x}+1\right)/\sqrt{3}\right] \quad \text{...(i)}$$

On putting $x=1$ and $y=0$ in Eq. (i), we get

$$0+C = \frac{-1}{2}\log(0+0+1) + \sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \quad C = \frac{\pi}{2\sqrt{3}}$$

$$\therefore \quad \log x + \frac{\pi}{2\sqrt{3}} = -\frac{1}{2}[\log(y^2+xy+x^2) - \log x^2] + \sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right)$$

$$\Rightarrow \quad \frac{\pi}{2\sqrt{3}} = -\frac{1}{2}\log(x^2+xy+y^2) + \sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right)$$

which is the required particular solution.

71. First, consider the function of differential equation as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$. Put $y = vx$ and convert the given differential equation in v and x. Further, integrate it and substitute $v = \frac{y}{x}$ to get the required solution.

Given differential equation is

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2} = \frac{\frac{y}{x}}{1+\frac{y^2}{x^2}} \quad \text{...(i)}$$

which is a homogeneous differential equation as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$. (1)

Now, put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ from Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{v}{1+v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{1+v^2} - v \quad \text{(1)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v-v-v^3}{1+v^2} \Rightarrow x\frac{dv}{dx} = -\frac{v^3}{1+v^2}$$

$$\Rightarrow \quad \frac{1+v^2}{v^3}dv = -\frac{dx}{x} \quad \text{(1)}$$

On integrating both sides, we get

$$\int\left(\frac{1}{v^3}+\frac{1}{v}\right)dv = -\int\frac{dx}{x}$$

$$\Rightarrow \quad -\frac{1}{2v^2} + \log|v| = -\log|x| + C \quad \text{(1)}$$

$$\Rightarrow \quad -\frac{x^2}{2y^2} + \log\left|\frac{y}{x}\right| = -\log|x| + C \quad \left[\text{put } v = \frac{y}{x}\right]$$

$$\Rightarrow -\frac{x^2}{2y^2} + \log|y| - \log|x| = -\log|x| + C$$

$$\left[\because \log\frac{m}{n} = \log m - \log n\right]$$

$$\Rightarrow \quad -\frac{x^2}{2y^2} + \log|y| = C \quad \text{...(ii)(1)}$$

Also, it is given that $y = 1$, when $x = 0$.

From Eq. (ii), we have

$$0 + \log|1| = C \Rightarrow C = 0$$

On putting $C = 0$ in Eq. (ii), we get

$$-\frac{x^2}{2y^2} + \log|y| = 0 \Rightarrow \log|y| = \frac{x^2}{2y^2}$$

$$\therefore \quad y = e^{\frac{x^2}{2y^2}}$$

which is the required solution. **(1)**

72. Do same as Q. No. 58.

The solution of differential equation is

$$-\cot\left(\frac{y}{x}\right) + \log|x| = C \quad \text{...(i)}$$

Also, given that $y = \frac{\pi}{4}$, when $x = 1$.

On putting $x = 1$ and $y = \frac{\pi}{4}$ in Eq. (i), we get

$$-\cot\left(\frac{\pi}{4}\right) + \log 1 = C \Rightarrow C = -1 \quad \left[\because \cot\frac{\pi}{4} = 1\right]$$

On putting this value of C in Eq. (i), we get

$$-\cot\left(\frac{y}{x}\right) + \log|x| = -1$$

$$\therefore \quad 1 + \log|x| - \cot\left(\frac{y}{x}\right) = 0$$

which is the required particular solution of given differential equation.

73. We have, $\frac{dy}{dx} - 3y\cot x = \sin 2x$ ...(i)

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, here $P = -3\cot x$ and $Q = \sin 2x$. **(1)**

$$\therefore \quad \text{IF} = e^{\int P dx} = e^{-3\int \cot x\, dx}$$

$$\Rightarrow \quad \text{IF} = e^{-3\log|\sin x|} = e^{\log|\sin x|^{-3}} = |\sin x|^{-3} \quad \textbf{(1)}$$

$\therefore$ The general solution of differential equation is given by

$$y \times \text{IF} = \int (\text{IF} \times Q)\, dx + C$$

$$\Rightarrow \quad y\cdot(\sin x)^{-3} = \int (\sin x)^{-3}(\sin 2x)\, dx + C$$

$$= \int \frac{2\sin x \cos x}{\sin^3 x} dx + C$$

$$\therefore \quad y\cdot(\sin x)^{-3} = \int \frac{2\cos x}{\sin^2 x} dx + C \quad \text{...(i) (1)}$$

On putting $\sin x = t \Rightarrow \cos x\, dx = dt$ in Eq. (i), we get

$$y\cdot(\sin x)^{-3} = 2\int \frac{dt}{t^2} + C = 2 \times \frac{t^{-1}}{-1} + C$$

$$\Rightarrow \quad y(\sin x)^{-3} = -\frac{2}{t} + C$$

$$\Rightarrow \quad y(\sin x)^{-3} = \frac{-2}{\sin x} + C \quad [\text{put } t = \sin x]$$

$$\Rightarrow \quad y = -2\sin^2 x + C\sin^3 x \quad \text{...(ii) (1½)}$$

On putting $x = \frac{\pi}{2}$ and $y = 2$ in Eq. (ii), we get

$$2 = -2\sin^2\frac{\pi}{2} + C\sin^3\frac{\pi}{2} \Rightarrow 2 = -2\cdot 1 + C\cdot 1$$

$$\Rightarrow \quad C = 4$$

$\therefore y = -2\sin^2 x + 4\sin^3 x$, which is required particular solution. **(1½)**

74. Do same as Q. No. 29.

The solution of differential equation is

$$x e^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + C$$

Also, it is given that $x = 1$, when $y = 0$.

Therefore, we have $1\cdot e^0 = e^0(0 - 1) + C$

$$\Rightarrow \quad 1 = -1 + C \quad \Rightarrow \quad C = 2$$

Hence, the required solution is

$$x e^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + 2$$

75. First, consider $\frac{dy}{dx}$ as equal to $F(x, y)$. Then, replace x by λx and y by λy on both sides, we get

$$F(x, y) = \lambda^0 F(x, y).$$

Put $y = vx$ and convert the given equation in terms of v and x, then separate the variables and integrate it. Further put $v = \frac{y}{x}$ and simplify it to get the required result.

Given differential equation is

$$\frac{dy}{dx} = \frac{y^2}{xy - x^2} \quad \text{...(i)}$$

Let $$F(x, y) = \frac{y^2}{xy - x^2}$$

Now, on replacing x by λx and y by λy, we get

$$F(\lambda x, \lambda y) = \frac{\lambda^2 y^2}{\lambda^2 (xy - x^2)} = \lambda^0 \frac{y^2}{xy - x^2} = \lambda^0 F(x, y)$$

Thus, the given differential equation is a homogeneous differential equation. **(1)**

Now, to solve it, put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

From Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{v^2 x^2}{vx^2 - x^2} = \frac{v^2}{v-1} \quad \textbf{(1)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2}{v-1} - v = \frac{v^2 - v^2 + v}{v-1}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{v-1} \Rightarrow \frac{v-1}{v}dv = \frac{dx}{x} \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int\left(1 - \frac{1}{v}\right)dv = \int\frac{dx}{x}$$

$$\Rightarrow \quad v - \log|v| = \log|x| + C \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{y}{x} - \log\left|\frac{y}{x}\right| = \log|x| + C \quad \left[\text{put } v = \frac{y}{x}\right] \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{y}{x} - \log|y| + \log|x| = \log|x| + C$$

$$\left[\because \log\left(\frac{m}{n}\right) = \log m - \log n\right]$$

$$\therefore \quad \frac{y}{x} - \log|y| = C$$

which is the required solution. **(1)**

76. Given differential equation is

$$\sqrt{1 + x^2 + y^2 + x^2y^2} + xy\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \sqrt{1(1 + x^2) + y^2(1 + x^2)} = -xy\frac{dy}{dx}$$

$$\Rightarrow \quad \sqrt{(1 + x^2)(1 + y^2)} = -xy\frac{dy}{dx}$$

$$\Rightarrow \quad \sqrt{1 + x^2}\cdot\sqrt{1 + y^2} = -xy\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{y}{\sqrt{1 + y^2}}dy = -\frac{\sqrt{1 + x^2}}{x}dx \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int\frac{y}{\sqrt{1 + y^2}}dy = -\int\frac{\sqrt{1 + x^2}}{x^2}\cdot x\,dx \quad \textbf{(1)}$$

On putting $1 + y^2 = t$ and $1 + x^2 = u^2$

$$\Rightarrow \quad 2y\,dy = dt \text{ and } 2x\,dx = 2u\,du$$

$$\Rightarrow \quad y\,dy = \frac{dt}{2} \text{ and } x\,dx = u\,du \quad \textbf{(1)}$$

$$\therefore \quad \frac{1}{2}\int\frac{dt}{\sqrt{t}} = -\int\frac{u}{u^2 - 1}\cdot u\,du$$

$$\Rightarrow \quad \frac{1}{2}\int t^{-1/2}dt = -\int\frac{u^2}{u^2 - 1}du$$

$$\Rightarrow \quad \frac{1}{2}\frac{t^{1/2}}{1/2} = -\int\frac{(u^2 - 1 + 1)}{u^2 - 1}du \quad \textbf{(1)}$$

$$\Rightarrow \quad t^{1/2} = -\int\frac{u^2 - 1}{u^2 - 1}du - \int\frac{1}{u^2 - 1}du$$

$$\Rightarrow \quad \sqrt{1 + y^2} = -\int du - \int\frac{1}{u^2 - (1)^2}du$$

[put $1 + y^2 = t$]

$$\Rightarrow \quad \sqrt{1 + y^2} = -u - \frac{1}{2}\log\left|\frac{u-1}{u+1}\right| + C \quad \textbf{(1)}$$

$$\left[\because \int\frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

$$\therefore \quad \sqrt{1 + y^2} = -\sqrt{1 + x^2} - \frac{1}{2}\log\left|\frac{\sqrt{1 + x^2} - 1}{\sqrt{1 + x^2} + 1}\right| + C$$

which is the required solution. **(1)**

77. Given differential equation is

$$\left[y - x\cos\left(\frac{y}{x}\right)\right]dy + \left[y\cos\left(\frac{y}{x}\right) - 2x\sin\left(\frac{y}{x}\right)\right]dx = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)}{y - x\cos\left(\frac{y}{x}\right)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\sin\left(\frac{y}{x}\right) - \frac{y}{x}\cos\left(\frac{y}{x}\right)}{\frac{y}{x} - \cos\left(\frac{y}{x}\right)} \quad \text{...(i)}$$

[divide numerator and denominator by x]

which is a homogeneous differential equation as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dy}{dx}$ in Eq. (i), we get **(1)**

$$v + x\frac{dv}{dx} = \frac{2\sin v - v\cos v}{v - \cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2\sin v - v\cos v}{v - \cos v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2\sin v - v\cos v - v^2 + v\cos v}{v - \cos v} \quad \textbf{(1)}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2\sin v - v^2}{v - \cos v} \Rightarrow \left(\frac{v - \cos v}{v^2 - 2\sin v}\right)dv = -\frac{dx}{x}$$

On integrating both sides, we get

$$\int\left(\frac{v - \cos v}{v^2 - 2\sin v}\right)dv = -\int\frac{dx}{x} \quad (1)$$

$$\Rightarrow \frac{1}{2}\log\left|v^2 - 2\sin v\right| = -\log|x| + \log C_1$$

$$\left[\because \frac{d}{dv}(v^2 - 2\sin v) = 2v - 2\cos v = 2(v - \cos v)\right]$$

$$\Rightarrow \log\sqrt{v^2 - 2\sin v} = \log\left|\frac{C_1}{x}\right| \quad (1)$$

$$\Rightarrow \sqrt{v^2 - 2\sin v} = \frac{C_1}{x}$$

$$\Rightarrow \sqrt{\frac{y^2}{x^2} - 2\sin\left(\frac{y}{x}\right)} = \frac{C_1}{x} \quad \left[\text{put } v = \frac{y}{x}\right]$$

$$\Rightarrow \sqrt{y^2 - 2x^2\sin\left(\frac{y}{x}\right)} = C_1 \quad (1)$$

$$\Rightarrow y^2 - 2x^2\sin\left(\frac{y}{x}\right) = C_1^2 \quad [\text{squaring both sides}]$$

$$\therefore y^2 - 2x^2\sin\left(\frac{y}{x}\right) = C, \text{ where } C = C_1^2 \quad (1)$$

78. Given differential equation is

$$(3xy + y^2)dx + (x^2 + xy)dy = 0$$

It can be rewritten as

$$\frac{dy}{dx} = -\frac{3xy + y^2}{x^2 + xy} \Rightarrow \frac{dy}{dx} = -\frac{3\frac{y}{x} + \frac{y^2}{x^2}}{1 + \frac{y}{x}} \quad \ldots(i)$$

which is a homogeneous differential equation as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq.(i), we get

$$v + x\frac{dv}{dx} = -\frac{3v + v^2}{1 + v}$$

$$\Rightarrow x\frac{dv}{dx} = -\left(\frac{3v + v^2 + v + v^2}{1 + v}\right) \quad (1)$$

$$\Rightarrow x\frac{dv}{dx} = -\left(\frac{2v^2 + 4v}{1 + v}\right)$$

$$\Rightarrow \frac{(1 + v)dv}{2(v^2 + 2v)} = -\frac{dx}{x}$$

On integrating both sides, we get

$$\int\frac{1 + v}{2(v^2 + 2v)}dv = -\int\frac{dx}{x} \quad \ldots(ii)\ (1)$$

Again, put $v^2 + 2v = z \Rightarrow (2v + 2)dv = dz$

$$\Rightarrow (1 + v)dv = \frac{dz}{2}$$

Then, Eq. (ii) becomes,

$$\int\frac{1}{2}\times\frac{dz}{2z} = -\int\frac{dx}{x} \quad (1)$$

$$\Rightarrow \frac{1}{4}\log|z| = -\log|x| + \log|C|$$

$$\Rightarrow \frac{1}{4}[\log|z| + 4\log|x|] = \log|C|$$

$$\Rightarrow \log|zx^4| = 4\log|C|$$

$$\Rightarrow zx^4 = C^4 \Rightarrow zx^4 = C_1,$$

where $C_1 = C^4$

$$\Rightarrow x^4(v^2 + 2v) = C_1 \quad [\text{put } z = v^2 + 2v]$$

$$\Rightarrow x^4\left(\frac{y^2}{x^2} + \frac{2y}{x}\right) = C_1 \quad \left[\text{put } v = \frac{y}{x}\right] \ldots(iii)\ (1)$$

Also, given that $y = 1$ for $x = 1$.

On putting $x = 1$ and $y = 1$ in Eq. (iii), we get

$$1\left(\frac{1}{1} + \frac{2}{1}\right) = C_1 \Rightarrow C_1 = 3 \quad (1)$$

So, on putting $C_1 = 3$ in Eq. (iii), we get

$$x^4\left(\frac{y^2}{x^2} + \frac{2y}{x}\right) = 3$$

$$\therefore y^2x^2 + 2yx^3 = 3 \quad (1)$$

which is the required particular solution.

79. We have, $(x^2 + xy)dy = (x^2 + y^2)dx$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2 + y^2}{x^2 + xy}\right) \quad \ldots(i)$$

This is a homogeneous differential equation.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v\cdot 1 + x\frac{dv}{dx}$ in Eq. (i), we get

$$v + x\frac{dv}{dx} = \left(\frac{x^2 + v^2x^2}{x^2 + x\cdot xv}\right) \quad (1)$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 + v^2}{1 + v} - v = \frac{1 + v^2 - v - v^2}{1 + v} = \frac{1 - v}{1 + v}$$

$$\therefore \left(\frac{1 + v}{1 - v}\right)dv = \frac{1}{x}dx \quad (1)$$

On integrating both sides, we get

$$\int\left(\frac{1 + v}{1 - v}\right)dv = \int\frac{1}{x}dx$$

$$\Rightarrow \int\left[-1 + \frac{2}{1 - v}\right]dv = \log|x| + \log C$$

$$\Rightarrow \quad -v - 2\log(1-v) = \log|x| + \log C \quad \textbf{(1)}$$
$$\Rightarrow \quad -v = 2\log(1-v) + \log|x| + \log C$$
$$\Rightarrow \quad -v = \log(1-v)^2 + \log\{C|x|\}$$
$$[\because \log m + \log n = \log mn]$$
$$\Rightarrow \quad -v = \log\{C|x|(1-v)^2\} \quad \textbf{(1)}$$
$$\Rightarrow \quad C|x|(1-v)^2 = e^{-v}$$
$$\Rightarrow \quad C|x|\left(1-\frac{y}{x}\right)^2 = e^{-y/x}\left[\because v = \frac{y}{x}\right] \quad \ldots(\text{ii})$$

On putting $x = 1$ and $y = 0$ in Eq. (ii), we get

$$C \cdot 1(1-0) = e^0 \Rightarrow C = 1 \quad \textbf{(1)}$$

Thus, the required solution is

$$|x|\left(1-\frac{y}{x}\right)^2 = e^{-y/x} \Rightarrow (x-y)^2 = |x|e^{-y/x} \quad \textbf{(1)}$$

which is the required particular solutions.

80. Given differential equation is

$$x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) = y\sin\left(\frac{y}{x}\right) - x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - \frac{1}{\sin\frac{y}{x}} \quad \ldots(\text{i})$$

$$\left[\text{dividing both sides by } x\sin\left(\frac{y}{x}\right)\right]$$

Let $\quad F(x,y) = \frac{y}{x} - \frac{1}{\sin\frac{y}{x}}$

On replacing x by λx and y by λy both sides, we get

$$F(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} - \frac{1}{\sin\frac{\lambda y}{\lambda x}} = \lambda^0\left(\frac{y}{x} - \frac{1}{\sin\frac{y}{x}}\right) = \lambda^0 F(x,y)$$

So, given differential equation is homogeneous. **(2)**

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i),

we get **(1)**

$$v + x\frac{dv}{dx} = v - \frac{1}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{1}{\sin v} \Rightarrow \sin v\, dv = -\frac{dx}{x}$$

On integrating both sides, we get

$$\int \sin v\, dv = -\int\frac{dx}{x}$$

$$\Rightarrow \quad -\cos v = -\log|x| + C$$

$$\Rightarrow \quad -\cos\frac{y}{x} = -\log|x| + C \left[\text{put } v = \frac{y}{x}\right] \ldots(\text{ii}) \quad \textbf{(1½)}$$

Also, given that $x = 1$, when $y = \frac{\pi}{2}$.

On putting $x = 1$ and $y = \frac{\pi}{2}$ in Eq. (ii), we get

$$-\cos\left(\frac{\pi}{2}\right) = -\log|1| + C \Rightarrow -0 = -0 + C \Rightarrow C = 0$$

On putting the value of C in Eq. (ii), we get

$$\cos\frac{y}{x} = \log|x|$$

which is the required solution. **(1½)**

81. Given differential equation is

$$\frac{dx}{dy} + x\cot y = 2y + y^2\cot y, \ (y \neq 0)$$

which is a linear differential equation of the form $\frac{dx}{dy} + Px = Q$, here $P = \cot y$ and $Q = 2y + y^2\cot y$ **(1)**

$$\therefore \quad \text{IF} = e^{\int P dy} = e^{\int \cot y\, dy} = e^{\log|\sin y|} = \sin y$$

The solution of the differential equation is given by

$$x \times \text{IF} = \int Q \times \text{IF}\, dy + C \quad \textbf{(1)}$$

$$\therefore \quad x\sin y = \int (2y + y^2\cot y)\sin y\, dy + C$$
$$= 2\int y\sin y\, dy + \int \underset{\text{I}}{y^2}\,\underset{\text{II}}{\cos y}\, dy + C$$
$$= 2\int y\sin y\, dy + y^2\int\cos y\, dy$$
$$-\int\left[\left(\frac{d}{dy}y^2\right)\int\cos y\, dy\right]dy + C$$

[using integration by parts]

$$= 2\int y\sin y\, dy + y^2\sin y - \int 2y\sin y\, dy$$
$$= 2\int y\sin y\, dy + y^2\sin y - 2\int y\sin y\, dy + C$$
$$\Rightarrow \quad x\sin y = y^2\sin y + C \quad \ldots(\text{i}) \quad \textbf{(2)}$$

Also, given that $x = 0$, when $y = \frac{\pi}{2}$.

On putting $x = 0$ and $y = \frac{\pi}{2}$ in Eq. (i), we get

$$0 = \left(\frac{\pi}{2}\right)^2\sin\frac{\pi}{2} + C \Rightarrow C = -\frac{\pi^2}{4}\left[\because \sin\frac{\pi}{2} = 1\right] \quad \textbf{(1)}$$

On putting the value of C in Eq. (i), we get

$$x\sin y = y^2\sin y - \frac{\pi^2}{4} \Rightarrow x = y^2 - \frac{\pi^2}{4}\cdot\text{cosec } y$$

which is required particular solution of given differential equation. **(1)**

82. Do same as Q. No. 74.

[**Ans.** $x = \tan^{-1}y - 1 + e^{-\tan^{-1}y}$]

Objective Questions

(For Complete Chapter)

1 Mark Questions

1. Order of the equation $\left(1+5\dfrac{dy}{dx}\right)^{3/2}=10\dfrac{d^3y}{dx^3}$ is

(a) 2 (b) 3 (c) 1 (d) 0

2. The order and degree of the differential equation $y=x\dfrac{dy}{dx}+\dfrac{2}{dy/dx}$, are

(a) 1, 2 (b) 1, 3 (c) 2, 1 (d) 1, 1

3. The degree of the differential equation $x=1+\left(\dfrac{dy}{dx}\right)+\dfrac{1}{2!}\left(\dfrac{dy}{dx}\right)^2+\dfrac{1}{3!}\left(\dfrac{dy}{dx}\right)^3+\cdots$, is

(a) 3 (b) 2
(c) 1 (d) not defined

4. The family of curves $y=e^{a\sin x}$, where a is an arbitrary constant is represented by the differential equation

(a) $\log y=\tan x\dfrac{dy}{dx}$ (b) $y\log y=\tan x\dfrac{dy}{dx}$
(c) $y\log y=\sin x\dfrac{dy}{dx}$ (d) $\log y=\cos x\dfrac{dy}{dx}$

5. $y=2e^{2x}-e^{-x}$ is a solution of the differential equation

(a) $y_2+y_1+2y=0$ (b) $y_2-y_1+2y=0$
(c) $y_2+y_1=0$ (d) $y_2-y_1-2y=0$

6. The order of the differential equation whose solution is $y=a\cos x+b\sin x+ce^{-x}$, is

(a) 3 (b) 1 (c) 2 (d) 4

7. Solution of $e^{dy/dx}=x$, when $x=1$ and $y=0$ is

(a) $y=x(\log x-1)+4$
(b) $y=x(\log x-1)+3$
(c) $y=x(\log x+1)+1$
(d) $y=x(\log x-1)+1$

8. The general solution of the differential equation $\dfrac{dy}{dx}=e^y(e^x+e^{-x}+2x)$ is

(a) $e^{-y}=e^x-e^{-x}+x^2+C$
(b) $e^{-y}=e^{-x}-e^x-x^2+C$
(c) $e^{-y}=-e^{-x}-e^x-x^2+C$
(d) $e^y=e^{-x}+e^x+x^2+C$

9. Solution of the differential equation $xdy-ydx=0$ represents a

(a) parabola (b) circle
(c) hyperbola (d) straight line

10. The solution of $\dfrac{dy}{dx}=\dfrac{ax+g}{by+f}$ represents a circle, when

(a) $a=b$ (b) $a=-b$
(c) $a=-2b$ (d) $a=2b$

11. An integrating factor of the differential equation $x\dfrac{dy}{dx}+y\log x=xe^x x^{-\frac{1}{2}\log x}$ $(x>0)$ is

(a) $x^{\log x}$ (b) $(\sqrt{x})^{\log x}$
(c) $(\sqrt{e})^{(\log x)^2}$ (d) e^{x^2}

12. Integrating factor (IF) of the differential equation $\dfrac{dy}{dx}-\dfrac{3x^2y}{1+x^3}=\dfrac{\sin^2(x)}{1+x}$ is

(a) e^{1+x^3} (b) $\log(1+x^3)$
(c) $1+x^3$ (d) $\dfrac{1}{1+x^3}$

Solutions

1. (b) Given, $\left(1+5\dfrac{dy}{dx}\right)^{3/2}=10\dfrac{d^3y}{dx^3}$

On squaring both sides, we get

$$\left(1+5\frac{dy}{dx}\right)^3=100\left(\frac{d^3y}{dx^3}\right)^2$$

$$\Rightarrow 1+125\left(\frac{dy}{dx}\right)^3+15\frac{dy}{dx}\left(1+5\frac{dy}{dx}\right)=100\left(\frac{d^3y}{dx^3}\right)^2$$

Clearly, the order of highest derivative occuring in the differential equation is 3. Hence, the order of given differential equation is 3.

2. (a) Given differential equation is

$$y=x\frac{dy}{dx}+\frac{2}{dy/dx}\Rightarrow y\frac{dy}{dx}=x\left(\frac{dy}{dx}\right)^2+2$$

Here, order = 1 and degree = 2

3. (c) Given differential equation is

$$x = 1 + \frac{dy}{dx} + \frac{1}{2!}\left(\frac{dy}{dx}\right)^2 + \frac{1}{3!}\left(\frac{dy}{dx}\right)^3 + \cdots$$

$$\Rightarrow \quad x = e^{\frac{dy}{dx}} \Rightarrow \frac{dy}{dx} = \log_e x$$

Hence, degree of differential equation is 1.

4. (b) Given curve is $y = e^{a \sin x}$

On taking log both sides, we get

$$\sin x = \frac{\log y}{a}$$

$$\therefore \frac{dy}{dx} = e^{a \sin x} \cdot a \cos x \Rightarrow \frac{dy}{dx} = y \cos x \cdot \frac{\log y}{\sin x}$$

$$\Rightarrow y \log y = \tan x \frac{dy}{dx}$$

5. (d) Given, $y = 2e^{2x} - e^{-x}$

$$\Rightarrow y_1 = 4e^{2x} + e^{-x} \Rightarrow y_2 = 8e^{2x} - e^{-x}$$

$$\Rightarrow y_2 = 4e^{2x} + e^{-x} + 4e^{2x} - 2e^{-x}$$

$$\Rightarrow y_2 = y_1 + 2(2e^{2x} - e^{-x})$$

$$\Rightarrow y_2 = y_1 + 2y \Rightarrow y_2 - y_1 - 2y = 0$$

6. (a) In given equation, there are three parameters. So, its differential equation is third order differential equation.

7. (d) Given, $e^{dy/dx} = x$

On taking log both sides, we get

$$\frac{dy}{dx} = \log x \Rightarrow dy = \log x \cdot dx$$

On integrating both sides, we get

$$\int dy = \int \log x \cdot dx$$

$$\Rightarrow y = x \log x - \int \frac{x}{x} + C$$

$$= x \log x - x + C \quad \ldots(i)$$

Also, it is given that at $x = 1$, $y = 0$

$$\therefore 0 = (1) \log 1 - 1 + C \Rightarrow C = 1$$

On substituting it in Eq. (i), we get

$$y = x \log x - x + 1$$

$$\Rightarrow y = x(\log x - 1) + 1$$

8. (b) Given, $\frac{dy}{dx} = e^y (e^x + e^{-x} + 2x)$

$$\Rightarrow \frac{dy}{e^y} = dx(e^x + e^{-x} + 2x)$$

On integrating both sides, we get

$$\int \frac{dy}{e^y} = \int dx(e^x + e^{-x} + 2x)$$

$$\Rightarrow e^{-y} = e^x - e^{-x} + x^2 + C$$

$$\Rightarrow e^{-y} = e^{-x} - e^x - x^2 + C$$

9. (d) Given differential equation is

$$xdy = ydx \Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\Rightarrow \log_e y = \log_e x + \log_e C$$

$$\Rightarrow y = Cx$$

which is a straight line.

10. (b) We have, $\frac{dy}{dx} = \frac{ax + g}{by + f}$

$$\Rightarrow (by + f)\, dy = (ax + g)\, dx$$

On integrating both sides, we get

$$\frac{by^2}{2} + fy = \frac{ax^2}{2} + gx + C$$

$$\Rightarrow ax^2 - by^2 + 2gx - 2fy + C = 0$$

which represents a circle, if $a = -b$.

11. (c) Here, $\frac{dy}{dx} + y\frac{1}{x}\log x = e^x x^{-(1/2)\log x}$

$$\therefore \text{IF} = e^{\int \frac{1}{x}\log x\, dx} = e^{\frac{(\log x)^2}{2}} = (\sqrt{e})^{(\log x)^2}$$

12. (d) Given, $\frac{dy}{dx} - \frac{3x^2 y}{1 + x^3} = \frac{\sin^2 x}{1 + x}$

From the given equation, $P = -\frac{3x^2}{1 + x^3}$, $Q = \frac{\sin^2 x}{1 + x}$

$$\text{IF} = e^{\int P\, dx} = e^{\int \frac{-3x^2}{1 + x^3} dx}$$

Put $1 + x^3 = t \Rightarrow 3x^3 dx = dt$

$$\therefore \text{IF} = e^{\int -\frac{1}{t} dt} = e^{-\log t} = e^{-\log}(1 + x^3)$$

$$= e^{\log}(1 + x^3)^{-1}$$

Hence, $\text{IF} = (1 + x^3)^{-1} = \frac{1}{1 + x^3}$