DCAM classes

Dynamic Classes for Academic Mastery

inverse trigonometry:cbse(solutions)

Solutions

1. We have, $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$

$= \tan^{-1}(\sqrt{3}) - \{\pi - \cot^{-1}(\sqrt{3})\}$

$[\because \cot^{-1}(-x) = \pi - \cot^{-1} x;\ x \in R]$

$= \tan^{-1}\sqrt{3} - \pi + \cot^{-1}\sqrt{3}$

$= (\tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3}) - \pi$

$= \frac{\pi}{2} - \pi = -\frac{\pi}{2}$ $\left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2};\ x \in R\right]$ **(1)**

which is the required principal value.

2. We have, $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$

$= \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \sec^{-1}\left(\sec\frac{2\pi}{3}\right)$

$\left[\because \tan\frac{\pi}{3} = \sqrt{3} \text{ and } \sec\frac{2\pi}{3} = -2\right]$

$= \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$

$\left[\begin{array}{l}\because \tan^{-1}(\tan\theta) = \theta; \forall \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{and} \\ \sec^{-1}(\sec\theta) = \theta; \forall \theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}\end{array}\right]$ **(1)**

which is the required principal value.

3. Given, $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1} x\right) = 1$

$\Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1} x = \sin^{-1}(1)$

$[\because \sin\theta = x \Rightarrow \theta = \sin^{-1} x]$

$\Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1} x = \sin^{-1}\left(\sin\frac{\pi}{2}\right)$ $\left[\because \sin\frac{\pi}{2} = 1\right]$

$\Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1} x = \frac{\pi}{2}$ **(1/2)**

$\Rightarrow \sin^{-1}\frac{1}{5} = \frac{\pi}{2} - \cos^{-1} x$

$\Rightarrow \sin^{-1}\frac{1}{5} = \sin^{-1} x$

$\left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2};\ x \in [-1, 1]\right]$

$\therefore x = \frac{1}{5}$ **(1/2)**

4. Given, $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}, xy < 1$

We know that,

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right), xy < 1$$

$\therefore \quad \tan^{-1}\left(\frac{x+y}{1-xy}\right)=\frac{\pi}{4} \Rightarrow \frac{x+y}{1-xy}=\tan\frac{\pi}{4}$ **(1/2)**

$\Rightarrow \quad \frac{x+y}{1-xy}=1 \quad \left[\because \tan\frac{\pi}{4}=1\right]$

$\Rightarrow \quad x+y=1-xy$

$\therefore \quad x+y+xy=1$ **(1/2)**

5. We have, $\cos^{-1}\left(\frac{-1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$

$=\left[\pi-\cos^{-1}\left(\frac{1}{2}\right)\right]+2\sin^{-1}\left(\frac{1}{2}\right)$

$[\because \cos^{-1}(-x)=\pi-\cos^{-1}x;\ \forall x\in[-1,1]]$

$=\left[\pi-\cos^{-1}\left(\cos\frac{\pi}{3}\right)\right]+2\sin^{-1}\left(\sin\frac{\pi}{6}\right)$ **(1/2)**

$\left[\because \cos\frac{\pi}{3}=\frac{1}{2} \text{ and } \sin\frac{\pi}{6}=\frac{1}{2}\right]$

$=\left[\pi-\frac{\pi}{3}\right]+2\times\frac{\pi}{6}$

$\left[\because \cos^{-1}(\cos\theta)=\theta;\ \forall\theta\in[0,\pi] \text{ and } \sin^{-1}(\sin\theta)=\theta;\ \forall\theta\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]\right]$

$=\frac{2\pi}{3}+\frac{\pi}{3}=\frac{2\pi+\pi}{3}=\pi$ **(1/2)**

6. First, we check the given angle lies in the principal value branch. If it is so, then use the property $\cos^{-1}(\cos\theta)=\theta,\ \forall\theta\in[0,180°]$. Otherwise reduce the angle such that, it lies in principal value branch.

We know that, principal value branch of $\cos^{-1}x$ is $[0, 180°]$.

Since, $680°\notin[0,180°]$, so write $680°$ as

$2\times360°-40°$

Now, $\cos^{-1}[\cos(680)°]=\cos^{-1}[\cos(2\times360°-40°)]$

$=\cos^{-1}(\cos40°)\quad[\because\cos(4\pi-\theta)=\cos\theta]$ **(1/2)**

Since, $40°\in[0,180°]$

$\therefore \cos^{-1}[\cos(680°)]=40°$

$[\because \cos^{-1}(\cos\theta)=\theta;\ \forall\theta\in[0,180°]]$

which is the required principal value. **(1/2)**

7. We have, $\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]$

$=\tan^{-1}\left[-\sin\left(\frac{\pi}{2}\right)\right]\quad\left[\because \sin^{-1}(-x)=-\sin^{-1}x,\ x\in(-1,1)\right]$

$=\tan^{-1}(-1)\quad\left[\because \sin\left(\frac{\pi}{2}\right)=1\right]$

$=\tan^{-1}\left(-\tan\frac{\pi}{4}\right)\quad\left[\because \tan\frac{\pi}{4}=1\right]$

$=\tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right]=\frac{-\pi}{4}$

$\left[\because \tan^{-1}(\tan\theta)=\theta;\forall\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$ **(1)**

which is the required principal value.

8. First, use $\cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$, then put $\cot^{-1}\sqrt{3}=\frac{\pi}{6}$ and simplify it.

We have, $\cot\left(\frac{\pi}{2}-2\cot^{-1}\sqrt{3}\right)$

$=\tan(2\cot^{-1}\sqrt{3})\quad\left[\because \cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta\right]$ **(1/2)**

$=\tan\left(2\times\frac{\pi}{6}\right)\quad\left[\because \cot^{-1}\sqrt{3}=\cot^{-1}\left(\cot\frac{\pi}{6}\right)=\frac{\pi}{6}\right]$

$=\tan\left(\frac{\pi}{3}\right)=\sqrt{3}$ **(1/2)**

9. We have, $\cos^{-1}\frac{\sqrt{3}}{2}+\cos^{-1}\left(-\frac{1}{2}\right)$

$=\cos^{-1}\frac{\sqrt{3}}{2}+\left[\pi-\cos^{-1}\left(\frac{1}{2}\right)\right]$ **(1/2)**

$[\because \cos^{-1}(-x)=\pi-\cos^{-1}x,\ \forall x\in[-1,1]]$

$=\cos^{-1}\left(\cos\frac{\pi}{6}\right)+\left[\pi-\cos^{-1}\left(\cos\frac{\pi}{3}\right)\right]$

$=\frac{\pi}{6}+\pi-\frac{\pi}{3}=\frac{\pi+6\pi-2\pi}{6}=\frac{5\pi}{6}$ **(1/2)**

$[\because \cos^{-1}(\cos\theta)=\theta;\ \forall\theta\in[0,\pi]]$

which is the required principal value.

10. We have, $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)$

$=\tan^{-1}\left(\tan\frac{\pi}{4}\right)+\cos^{-1}\left(-\cos\frac{\pi}{3}\right)$

$\left[\because \tan\frac{\pi}{4}=1 \text{ and } \cos\frac{\pi}{3}=\frac{1}{2}\right]$

$=\frac{\pi}{4}+\cos^{-1}\left[\cos\left(\pi-\frac{\pi}{3}\right)\right]$ **(1/2)**

[since, principal value branch of $\cos^{-1}x$ is $[0,\pi]$, so we write $-\cos\theta=\cos(\pi-\theta)$]

$=\frac{\pi}{4}+\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$

$=\frac{\pi}{4}+\frac{2\pi}{3}=\frac{3\pi+8\pi}{12}=\frac{11\pi}{12}$

$[\because \cos^{-1}(\cos\theta)=\theta;\ \forall\theta\in[0,\pi]]$ **(1/2)**

which is the required principal value.

Alternate Method

We have,

$$\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) = \tan^{-1}(1) + \pi - \cos^{-1}\left(\frac{1}{2}\right)$$

$[\because \cos^{-1}(-x) = \pi - \cos^{-1} x; \forall x \in [-1,1]]$ **(1/2)**

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) + \pi - \cos^{-1}\left(\cos\frac{\pi}{3}\right)$$

$$= \frac{\pi}{4} + \pi - \frac{\pi}{3} = \frac{3\pi + 12\pi - 4\pi}{12} = \frac{11\pi}{12}$$ **(1/2)**

which is the required principal value.

11. We have,

$$\tan\left(2\tan^{-1}\frac{1}{5}\right) = \tan\left[\tan^{-1}\left(\frac{2\times\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2}\right)\right]$$ **(1/2)**

$$\left[\because 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right); -1 < x < 1\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{2\times 5}{24}\right)\right] = \tan\left[\tan^{-1}\left(\frac{5}{12}\right)\right] = \frac{5}{12}$$

$[\because \tan(\tan^{-1} x) = x; \forall x \in R]$ **(1/2)**

12. We have, $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$$= \tan^{-1}\left[2\sin\left\{2\cos^{-1}\left(\cos\frac{\pi}{6}\right)\right\}\right] \quad \left[\because \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}\right]$$

$$= \tan^{-1}\left[2\sin\left\{2\times\frac{\pi}{6}\right\}\right]$$ **(1/2)**

$[\because \cos^{-1}(\cos\theta) = \theta; \forall \theta \in [0, \pi]]$

$$= \tan^{-1}\left(2\sin\frac{\pi}{3}\right) = \tan^{-1}\left(2\cdot\frac{\sqrt{3}}{2}\right) \quad \left[\because \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\right]$$

$$= \tan^{-1}(\sqrt{3}) = \tan^{-1}\left(\tan\frac{\pi}{3}\right) = \frac{\pi}{3}$$

$$\left[\because \tan\frac{\pi}{3} = \sqrt{3} \text{ and } \tan^{-1}(\tan\theta) = \theta, \forall \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$$ **(1/2)**

13. Do same as Q. No. 5. $\left[\textbf{Ans. } \frac{2\pi}{3}\right]$

14. Do same as Q. No. 5. $\left[\textbf{Ans. } \frac{2\pi}{3}\right]$

15. We have,

$$\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right] = \sin\left[\frac{\pi}{3} + \sin^{-1}\left(\frac{1}{2}\right)\right]$$

$[\because \sin^{-1}(-x) = -\sin^{-1} x; \forall x \in [-1,1]]$

$$= \sin\left[\frac{\pi}{3} + \sin^{-1}\left(\sin\frac{\pi}{6}\right)\right] \quad \left[\because \sin\frac{\pi}{6} = \frac{1}{2}\right]$$

$$= \sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right] = \sin\frac{\pi}{2} = 1$$

$$\left[\because \sin^{-1}(\sin\theta) = \theta; \forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right]$$ **(1)**

16.

> First, we check the given angle lies in the principal value branch. If it is so, then use the property $\tan^{-1}(\tan\theta) = \theta, \forall \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Otherwise reduce the angle such that it lies in principal value branch.

We know that, principal value branch of $\tan^{-1} x$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Since, $\frac{3\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so write as $\frac{3\pi}{4} = \pi - \frac{\pi}{4}$

Now, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left[\tan\left(\pi - \frac{\pi}{4}\right)\right]$ **(1/2)**

$$= \tan^{-1}\left(-\tan\frac{\pi}{4}\right) \quad [\because \tan(\pi - \theta) = -\tan\theta]$$

$$= \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] [\because -\tan\theta = \tan(-\theta)]$$

$\because -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan^{-1}(\tan\theta) = \theta; \forall \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\therefore \quad \tan^{-1}\left(\tan\frac{3\pi}{4}\right) = -\frac{\pi}{4}$$ **(1/2)**

17. We know that, the principal value branch of $\cos^{-1} x$ is $[0, \pi]$.

Since, $\frac{7\pi}{6} \notin [0, \pi]$, so write as $\frac{7\pi}{6} = 2\pi - \frac{5\pi}{6}$

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{5\pi}{6}\right)\right]$ **(1/2)**

$$= \cos^{-1}\left(\cos\frac{5\pi}{6}\right) \quad [\because \cos(2\pi - \theta) = \cos\theta]$$

$\because \frac{5\pi}{6} \in [0, \pi]$ and $\cos^{-1}(\cos\theta) = \theta; \forall \theta \in [0, \pi]$

$$\therefore \quad \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \frac{5\pi}{6}$$ **(1/2)**

18. We know that, the principal value branch of $\cos^{-1} x$ is $[0, \pi]$ and for $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Since, $\frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so write $\frac{2\pi}{3} = \left(\pi - \frac{\pi}{3}\right)$

Now, $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

$=\frac{2\pi}{3}+\sin^{-1}\left[\sin\left(\pi-\frac{\pi}{3}\right)\right]$ **(1/2)**

$[\because \cos^{-1}(\cos\theta)=\theta;\forall\,\theta\in[0,\pi]]$

$=\frac{2\pi}{3}+\sin^{-1}\left(\sin\frac{\pi}{3}\right)$ $[\because \sin(\pi-\theta)=\sin\theta]$

$=\frac{2\pi}{3}+\frac{\pi}{3}$ $\left[\because \sin^{-1}(\sin\theta)=\theta;\forall\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right]$

$=\frac{3\pi}{3}=\pi$ **(1/2)**

which is the required principal value.

19. We have,

$\tan^{-1}(-1)=\tan^{-1}\left(-\tan\frac{\pi}{4}\right)$ $\left[\because \tan\frac{\pi}{4}=1\right]$

$=\tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right]$ $[\because -\tan\theta=\tan(-\theta)]$

$=-\frac{\pi}{4}$ $\left[\because \tan^{-1}(\tan\theta)=\theta;\forall\theta\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)\right]$

which is the required principal value. **(1)**

Alternate Method

We have,

$\tan^{-1}(-1)=-\tan^{-1}(1)$

$[\because \tan^{-1}(-x)=-\tan^{-1}x;\,x\in R]$

$=-\tan^{-1}\left(\tan\frac{\pi}{4}\right)$ $\left[\because \tan\frac{\pi}{4}=1\right]$

$=\frac{-\pi}{4}$ $\left[\because \tan^{-1}(\tan\theta)=\theta;\forall\,\theta\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)\right]$

which is the required principal value. **(1)**

20. We have,

$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\sin^{-1}\left(-\sin\frac{\pi}{3}\right)$

$\left[\because \sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}\right]$

$=\sin^{-1}\left[\sin\left(-\frac{\pi}{3}\right)\right]$ $[\because -\sin\theta=\sin(-\theta)]$

$=-\frac{\pi}{3}$ $\left[\because \sin^{-1}(\sin\theta)=\theta;\forall\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right]$

Hence, $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{3}$ **(1)**

Alternate Method

We have, $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)=-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$[\because \sin^{-1}(-x)=-\sin^{-1}x;\,x\in[-1,1]]$

$=-\sin^{-1}\left(\sin\frac{\pi}{3}\right)$ $\left[\because \sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}\right]$

$=-\frac{\pi}{3}$ $\left[\because \sin^{-1}(\sin\theta)=\theta;\forall\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right]$

which is the required principal value. **(1)**

21. Do same as Q.No. 20. $\left[\textbf{Ans.}-\frac{\pi}{6}\right]$

22. We have, $\sec^{-1}(-2)=\pi-\sec^{-1}(2)$

$[\because \sec^{-1}(-x)=\pi-\sec^{-1}(x);\,|x|\ge 1]$

$=\pi-\sec^{-1}\left(\sec\frac{\pi}{3}\right)=\pi-\frac{\pi}{3}$

$\left[\because \sec\frac{\pi}{3}=2 \text{ and } \sec^{-1}(\sec\theta)=\theta;\forall\theta\in[0,\pi]-\left\{\frac{\pi}{2}\right\}\right]$

$=\frac{2\pi}{3}$

which is the required principal value. **(1)**

23. The domain of the function $\sin^{-1}x$ is $[-1,1]$. **(1)**

24. Do same as Q.No. 17. $\left[\textbf{Ans.}\ \frac{\pi}{6}\right]$

25. Given, $\tan^{-1}\sqrt{3}+\cot^{-1}x=\frac{\pi}{2}$

$\Rightarrow \tan^{-1}\sqrt{3}=\frac{\pi}{2}-\cot^{-1}x$

$\Rightarrow \tan^{-1}\sqrt{3}=\tan^{-1}x$

$\left[\because \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2};\,x\in R\right]$

$\therefore\quad x=\sqrt{3}$ **(1)**

26. Consider, $\text{RHS}=\sin^{-1}(3x-4x^3)$ ...(i)

Let $x=\sin\theta$,

then $\theta=\sin^{-1}x$

Now, from Eq. (i), we get

$\text{RHS}=\sin^{-1}(3\sin\theta-4\sin^3\theta)$

$=\sin^{-1}(\sin 3\theta)$ $[\because \sin 3A=3\sin A-4\sin^3 A]$

$=3\theta$ **(1)**

$\left[\begin{array}{l}\because \sin^{-1}(\sin\theta)=\theta,\ \forall\theta\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]\\ \text{and here } -\frac{1}{2}\le x\le\frac{1}{2}\Rightarrow -\frac{\pi}{6}\le\sin^{-1}x\le\frac{\pi}{6}\end{array}\right]$

$=3\sin^{-1}x$ $[\because \theta=\sin^{-1}x]$

$=\text{LHS}$ **Hence proved.** **(1)**

27. Consider, RHS $= \cos^{-1}(4x^3 - 3x)$ ...(i)

Let $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

Now, from Eq. (i), we get

RHS $= \cos^{-1}(4\cos^3\theta - 3\cos\theta)$

$= \cos^{-1}(\cos 3\theta)$ $[\because \cos 3A = 4\cos^3 A - 3\cos A]$

$= 3\theta$ $\left[\begin{array}{l}\because \cos^{-1}(\cos\theta) = \theta \forall \theta \in [0, \pi] \\ \text{here, } \frac{1}{2} \le x \le 1 \Rightarrow 0 \le \cos^{-1} x \le \frac{\pi}{3}\end{array}\right]$ **(1)**

$= 3\cos^{-1} x =$ LHS **Hence proved.** **(1)**

28.

> First, use the formula
> $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$; $xy < 1$, then
> simplify it and get the values of x. Further, verify the given equation by obtained values of x.

Given, $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$ ...(i)

$\Rightarrow \tan^{-1}\left[\frac{(x+1)+(x-1)}{1-(x+1)(x-1)}\right] = \tan^{-1}\frac{8}{31}$ **(1)**

$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right); xy < 1\right]$

$\Rightarrow \tan^{-1}\left(\frac{2x}{1-(x^2-1)}\right) = \tan^{-1}\frac{8}{31}$

$\Rightarrow \frac{2x}{2-x^2} = \frac{8}{31}$

$\Rightarrow 62x = 16 - 8x^2$ **(1)**

$\Rightarrow 8x^2 + 62x - 16 = 0$

$\Rightarrow 4x^2 + 31x - 8 = 0$ [dividing by 2]

$\Rightarrow 4x^2 + 32x - x - 8 = 0$

$\Rightarrow 4x(x+8) - 1(x+8) = 0$

$\Rightarrow (x+8)(4x-1) = 0$

$\therefore x = -8$ or $x = 1/4$ **(1)**

But $x = -8$ gives LHS $= \tan^{-1}(-7) + \tan^{-1}(-9)$

$= -\tan^{-1}(7) - \tan^{-1}(9)$,

which is negative, while RHS is positive.

So, $x = -8$ is not possible.

Hence, $x = \frac{1}{4}$ is the only solution of the given equation. **(1)**

29. Let $\cos^{-1}\frac{4}{5} = x$, then

$\Rightarrow \cos x = \frac{4}{5}$

$\Rightarrow \sin x = \sqrt{1-\cos^2 x} = \sqrt{1-\left(\frac{4}{5}\right)^2}$

$\Rightarrow \sin x = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

$\Rightarrow \tan x = \frac{3}{4}$

$\Rightarrow x = \tan^{-1}\left(\frac{3}{4}\right) = \cos^{-1}\left(\frac{4}{5}\right)$ **(1)**

Now,

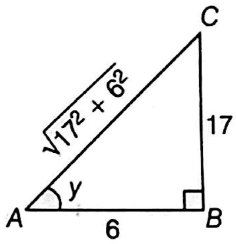
$\sin\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right) = \sin\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right)$

$= \sin\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right)\right)$

$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1\right]$

$= \sin\left(\tan^{-1}\left(\frac{\frac{9+8}{12}}{\frac{12-6}{12}}\right)\right) = \sin\left(\tan^{-1}\left(\frac{17}{6}\right)\right)$ **(1)**

Now, let $\tan^{-1}\left(\frac{17}{6}\right) = y$



$\Rightarrow \tan y = \frac{17}{6}$

$\Rightarrow \sin y = \frac{17}{\sqrt{325}} = \frac{17}{5\sqrt{13}}$

$\Rightarrow y = \sin^{-1}\left(\frac{17}{5\sqrt{13}}\right)$ **(1)**

$= \tan^{-1}\left(\frac{17}{6}\right)$

Hence,

$\sin\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right) = \sin\left(\sin^{-1}\frac{17}{5\sqrt{13}}\right) = \frac{17}{5\sqrt{13}}$

$[\because \sin(\sin^{-1} x) = x, x \in [-1, 1]]$ **(1)**

30. Given, $\tan^{-1} 3x + \tan^{-1} 2x = \frac{\pi}{4}$ ...(i)

$\Rightarrow \tan^{-1}\left(\frac{3x+2x}{1-3x\times 2x}\right) = \frac{\pi}{4}$ **(1)**

$\left[\because \tan^{-1} A + \tan^{-1} B = \tan^{-1}\left(\frac{A+B}{1-AB}\right); AB < 1\right]$

$$\Rightarrow \tan^{-1}\left(\frac{5x}{1-6x^2}\right)=\frac{\pi}{4} \Rightarrow \frac{5x}{1-6x^2}=\tan\frac{\pi}{4}$$

$[\because \tan^{-1}x=\theta \Rightarrow x=\tan\theta]$

$$\Rightarrow \frac{5x}{1-6x^2}=1$$

$$\Rightarrow 5x=1-6x^2 \quad \textbf{(1)}$$

$$\Rightarrow 6x^2+5x-1=0$$

$$\Rightarrow 6x^2+6x-x-1=0$$

$$\Rightarrow 6x(x+1)-1(x+1)=0$$

$$\Rightarrow (6x-1)(x+1)=0$$

$$\Rightarrow 6x-1=0 \text{ or } x+1=0$$

$$\therefore x=1/6 \text{ or } x=-1 \quad \textbf{(1)}$$

But $x=-1$ does not satisfy the Eq. (i), as LHS becomes negative. So, $x=\frac{1}{6}$ is the only solution of the given equation. **(1)**

31. $\tan^{-1}4x+\tan^{-1}6x=\frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left(\frac{4x+6x}{1-4x\cdot 6x}\right)=\frac{\pi}{4}$$

$$\left[\because \tan^{-1}A+\tan^{-1}B=\tan^{-1}\left(\frac{A+B}{1-AB}\right)\right]$$

$$\Rightarrow \frac{4x+6x}{1-4x\cdot 6x}=\tan\frac{\pi}{4} \quad \textbf{(1)}$$

$$\Rightarrow \frac{10x}{1-24x^2}=1$$

$$\Rightarrow 10x=1-24x^2$$

$$\Rightarrow 24x^2+10x-1=0 \quad \textbf{(1)}$$

$$\Rightarrow 24x^2+12x-2x-1=0$$

$$\Rightarrow 12x(2x+1)-1(2x+1)=0$$

$$\Rightarrow (2x+1)(12x-1)=0$$

$$\Rightarrow 2x+1=0$$

$$x=-\frac{1}{2}$$

or

$$12x-1=0 \quad \textbf{(1)}$$

$$x=\frac{1}{12}$$

But $x=-\frac{1}{2}$ does not satisfy the given equation.

Hence, the required solution is $x=\frac{1}{12}$ **(1)**

32. Given,

$$\tan^{-1}\left[\frac{x-3}{x-4}\right]+\tan^{-1}\left[\frac{x+3}{x+4}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-3}{x-4}+\frac{x+3}{x+4}}{1-\left(\frac{x-3}{x-4}\right)\left(\frac{x+3}{x+4}\right)}\right]=\frac{\pi}{4} \quad \textbf{(1)}$$

$$\left[\because \tan^{-1}a+\tan^{-1}b=\tan^{-1}\left(\frac{a+b}{1-ab}\right); ab<1\right]$$

$$\Rightarrow \left[\frac{\frac{(x-3)(x+4)+(x+3)(x-4)}{(x-4)(x+4)}}{\frac{(x-4)(x+4)-(x-3)(x+3)}{(x-4)(x+4)}}\right]=\tan\left(\frac{\pi}{4}\right) \quad \textbf{(1)}$$

$$\Rightarrow \frac{x^2+4x-3x-12+x^2-4x+3x-12}{x^2-16-x^2+9}=1$$

$$\left[\because \tan\left(\frac{\pi}{4}\right)=1\right]$$

$$\Rightarrow \frac{2x^2-24}{-7}=1 \quad \textbf{(1)}$$

$$\Rightarrow 2x^2-24=-7$$

$$\Rightarrow 2x^2=-7+24$$

$$\Rightarrow 2x^2=17 \Rightarrow x^2=\frac{17}{2}$$

$$\therefore x=\pm\sqrt{\frac{17}{2}} \quad \textbf{(1)}$$

33. To Prove

$$\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right)=\frac{2b}{a}$$

$$\text{LHS}=\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right) \quad \textbf{(1)}$$

Put $\cos^{-1}\frac{a}{b}=\theta \Rightarrow \cos\theta=\frac{a}{b}$

$$\text{LHS}=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$=\frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\theta}{2}}+\frac{\tan\frac{\pi}{4}-\tan\frac{\theta}{2}}{1+\tan\frac{\pi}{4}\tan\frac{\theta}{2}} \quad \textbf{(1)}$$

$$=\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}+\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$$

$$=\frac{\left(1+\tan\frac{\theta}{2}\right)^2+\left(1-\tan\frac{\theta}{2}\right)^2}{\left(1-\tan\frac{\theta}{2}\right)\left(1+\tan\frac{\theta}{2}\right)}$$

$$= 2\left(\frac{\frac{1+\tan^2\frac{\theta}{2}}{2}}{1-\tan^2\frac{\theta}{2}}\right) = \frac{2}{\cos\theta} = \frac{2}{a/b} = \frac{2b}{a}$$ **(1)**

$$\left[\because \cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right]$$

= RHS **Hence proved. (1)**

34. We have, $\cos(\tan^{-1} x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$ ...(i)

Let $\tan^{-1} x = \theta$ and $\cot^{-1}\frac{3}{4} = \phi$; $\forall \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\phi \in (0, \pi)$

$\Rightarrow \tan\theta = x$ and $\cot\phi = \frac{3}{4}$

$\Rightarrow \sec\theta = \sqrt{1+\tan^2\theta}$ and $\operatorname{cosec}\phi = \sqrt{1+\cot^2\phi}$

[taking positive square root as $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\phi \in (\theta, \pi)$] **(1)**

$\Rightarrow \sec\theta = \sqrt{1+x^2}$

and $\operatorname{cosec}\phi = \sqrt{1+\left(\frac{3}{4}\right)^2} = \sqrt{\frac{16+9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$

$\Rightarrow \frac{1}{\cos\theta} = \sqrt{1+x^2}$ and $\frac{1}{\sin\phi} = \frac{5}{4}$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{1+x^2}}$ and $\sin\phi = \frac{4}{5}$

$\Rightarrow \theta = \cos^{-1}\frac{1}{\sqrt{1+x^2}}$ and $\phi = \sin^{-1}\frac{4}{5}$ **(1)**

$\Rightarrow \tan^{-1} x = \cos^{-1}\frac{1}{\sqrt{1+x^2}}$ and $\cot^{-1}\frac{3}{4} = \sin^{-1}\frac{4}{5}$

On substituting these values in Eq. (i), we get

$$\cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right) = \sin\left(\sin^{-1}\frac{4}{5}\right)$$

$\Rightarrow \frac{1}{\sqrt{1+x^2}} = \frac{4}{5}$ [$\because \cos(\cos^{-1} x) = x; \forall x \in [-1,1]$ and $\sin(\sin^{-1} x) = x; \forall x \in [-1,1]$] **(1)**

On squaring both sides, we get

$16(x^2+1) = 25 \Rightarrow 16x^2 = 9 \Rightarrow x^2 = \frac{9}{16}$

$\Rightarrow x = \pm\frac{3}{4}$ [taking square root both sides]

But $x = \frac{-3}{4}$ does not satisfy the given equation.

Hence, the required solution is $x = \frac{3}{4}$. **(1)**

35. To prove,

$$\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$$

$$\text{LHS} = \left(\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7}\right) + \left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right)$$ **(1)**

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right); xy < 1\right]$$

$$= \tan^{-1}\left(\frac{\frac{7+5}{35}}{\frac{35-1}{35}}\right) + \tan^{-1}\left(\frac{\frac{8+3}{24}}{\frac{24-1}{24}}\right)$$

$$= \tan^{-1}\left(\frac{12}{34}\right) + \tan^{-1}\left(\frac{11}{23}\right)$$

$$= \tan^{-1}\left(\frac{6}{17}\right) + \tan^{-1}\left(\frac{11}{23}\right)$$ **(1)**

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right) = \tan^{-1}\left(\frac{\frac{138+187}{391}}{\frac{391-66}{391}}\right)$$ **(1)**

$$= \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1}(1) = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$$

= RHS $\left[\because \tan^{-1}(\tan\theta) = \theta; \forall\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\right]$

Hence proved. (1)

36. Given equation is

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$

$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}\left(\frac{2}{\sin x}\right)$

$\left[\because 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right); -1 < x < 1\right]$ **(1)**

$\Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x}$ [$\because 1-\cos^2 x = \sin^2 x$] ...(i)

$\Rightarrow \sin x\cos x - \sin^2 x = 0$

$\Rightarrow \sin x(\cos x - \sin x) = 0$ **(1)**

$\Rightarrow \sin x = 0$ or $\cos x = \sin x$

$\Rightarrow \sin x = \sin 0$

or $\cot x = 1 = \cot \pi/4$

$\therefore x = 0$ or $\frac{\pi}{4}$ **(1)**

But here at $x = 0$, the given equation does not exist.

Hence, $x = \frac{\pi}{4}$ is the only solution. **(1)**

37. Given,
$\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1) = \tan^{-1}3x$

$\Rightarrow \tan^{-1}(x-1)+\tan^{-1}(x+1) = \tan^{-1}3x - \tan^{-1}x$

$\Rightarrow \tan^{-1}\left(\frac{x-1+x+1}{1-(x-1)(x+1)}\right) = \tan^{-1}\left(\frac{3x-x}{1+3x\times x}\right)$ **(1)**

$\left[\because \tan^{-1}x+\tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy<1 \text{ and } \tan^{-1}x-\tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right), xy>-1\right]$

$\Rightarrow \tan^{-1}\left(\frac{2x}{1-(x^2-1)}\right) = \tan^{-1}\left(\frac{2x}{1+3x^2}\right)$

$\Rightarrow \frac{2x}{2-x^2} = \frac{2x}{1+3x^2}$ **(1)**

$\Rightarrow 2x(1+3x^2) = 2x(2-x^2)$

$\Rightarrow 2x[1+3x^2-(2-x^2)] = 0$

$\Rightarrow x(4x^2-1) = 0 \Rightarrow x = 0 \text{ or } 4x^2-1 = 0$

$\therefore x = 0 \text{ or } x = \pm\frac{1}{2}$ **(2)**

38. To prove, $\tan^{-1}\left(\frac{6x-8x^3}{1-12x^2}\right) - \tan^{-1}\left(\frac{4x}{1-4x^2}\right) = \tan^{-1}2x; |2x| < \frac{1}{\sqrt{3}}$

We consider,

$\text{LHS} = \tan^{-1}\left(\frac{6x-8x^3}{1-12x^2}\right) - \tan^{-1}\left(\frac{4x}{1-4x^2}\right)$

$= \tan^{-1}\left[\frac{\left(\frac{6x-8x^3}{1-12x^2}\right)-\left(\frac{4x}{1-4x^2}\right)}{1+\left(\frac{6x-8x^3}{1-12x^2}\right)\left(\frac{4x}{1-4x^2}\right)}\right]$

$\left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right); xy>-1\right]$ **(1)**

$= \tan^{-1}\left[\frac{\frac{(6x-8x^3)(1-4x^2)-4x(1-12x^2)}{(1-12x^2)(1-4x^2)}}{\frac{(1-12x^2)(1-4x^2)+(6x-8x^3)(4x)}{(1-12x^2)(1-4x^2)}}\right]$ **(1)**

$= \tan^{-1}\left(\frac{6x-24x^3-8x^3+32x^5-4x+48x^3}{1-4x^2-12x^2+48x^4+24x^2-32x^4}\right)$ **(1)**

$= \tan^{-1}\left(\frac{2x+16x^3+32x^5}{16x^4+8x^2+1}\right)$

$= \tan^{-1}\left[\frac{2x(16x^4+8x^2+1)}{(16x^4+8x^2+1)}\right] = \tan^{-1}2x = \text{RHS}$ **(1)**

Hence proved.

39. To prove, $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2};$ $0<x<\frac{\pi}{2}$

$\text{LHS} = \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

$= \cot^{-1}\left[\frac{\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}+\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}-\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}\right]$ **(1/2)**

$\left[\because 1 = \sin^2\frac{x}{2}+\cos^2\frac{x}{2} \text{ and } \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right]$

$= \cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right]$

$\left[\because 0<x<\frac{\pi}{2} \Rightarrow 0<\frac{x}{2}<\frac{\pi}{4} \text{ so, } \cos\frac{x}{2}>\sin\frac{x}{2} \text{ or } \cos\frac{x}{2}-\sin\frac{x}{2}>0\right]$ **(1½)**

$= \cot^{-1}\left(\frac{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|+\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|-\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}\right)$

$\left[\because \sqrt{x^2} = |x|\right]$

$= \cot^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right)$ **(1)**

$= \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right) = \cot^{-1}\left(\cot\frac{x}{2}\right)$

$= \frac{x}{2}$ $\left[\because 0<\frac{x}{2}<\frac{\pi}{4} \text{ and } \cot^{-1}(\cot\theta) = \theta; \forall\theta\in(0,\pi)\right]$ **(1)**

Hence, LHS = RHS **Hence proved.**

Alternate Method

$\text{LHS} = \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

$$= \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} \times \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right)$$

[by rationalising denominator] **(1)**

$$= \cot^{-1}\left[\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})^2}{(\sqrt{1+\sin x})^2-(\sqrt{1-\sin x})^2}\right]$$

$[\because (a-b)(a+b) = a^2 - b^2]$

$$= \cot^{-1}\left(\frac{1+\sin x+1-\sin x+2\sqrt{1-\sin^2 x}}{1+\sin x-1+\sin x}\right)$$

$[\because (a+b)^2 = a^2+b^2+2ab]$

$$= \cot^{-1}\left(\frac{2+2\cos x}{2\sin x}\right)$$

$\left[\because \sqrt{1-\sin^2 x} = |\cos x| = \cos x;\ \text{as } 0<x<\frac{\pi}{2}\right]$ **(1)**

$$= \cot^{-1}\left(\frac{1+\cos x}{\sin x}\right) = \cot^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$ **(1)**

$\left[\because 1+\cos\theta = 2\cos^2\frac{\theta}{2} \text{ and } \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]$

$$= \cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$= \frac{x}{2} = \text{RHS}$$

$\left[\because 0<\frac{x}{2}<\frac{\pi}{4} \text{ and } \cot^{-1}(\cot\theta) = \theta; \forall\theta\in(0,\pi)\right]$ **(1)**

Hence proved.

40. Do same as Q.No. 32. **Ans.** $\sqrt{\frac{7}{2}}, -\sqrt{\frac{7}{2}}$

41. Do same as Q.No. 34. **Ans.** $x = -\frac{1}{2}$

42. Given, $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$

$$\Rightarrow (\tan^{-1} x)^2 + \left(\frac{\pi}{2} - \tan^{-1} x\right)^2 = \frac{5\pi^2}{8}$$

$\left[\because \cot^{-1} x + \tan^{-1} x = \frac{\pi}{2}; x\in R\right]$

$$\Rightarrow (\tan^{-1} x)^2 + \left(\frac{\pi}{2}\right)^2 + (\tan^{-1} x)^2 - 2\times\frac{\pi}{2}\times\tan^{-1} x = \frac{5\pi^2}{8}$$

$[\because (a-b)^2 = a^2+b^2-2ab]$ **(1)**

$$\Rightarrow 2(\tan^{-1} x)^2 + \frac{\pi^2}{4} - \pi\tan^{-1} x = \frac{5\pi^2}{8}$$

$$\Rightarrow 2(\tan^{-1} x)^2 - \pi\tan^{-1} x = \frac{5\pi^2}{8} - \frac{\pi^2}{4}$$

$$\Rightarrow 2(\tan^{-1} x)^2 - \pi\tan^{-1} x = \frac{3\pi^2}{8}$$

Let $\tan^{-1} x = \theta$, where $\theta\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ **(1)**

Then, $2\theta^2 - \pi\theta = \frac{3\pi^2}{8}$

$$\Rightarrow 16\theta^2 - 8\pi\theta - 3\pi^2 = 0$$

$$\Rightarrow 16\theta^2 - 12\pi\theta + 4\pi\theta - 3\pi^2 = 0$$

$$\Rightarrow 4\theta(4\theta - 3\pi) + \pi(4\theta - 3\pi) = 0$$

$$\Rightarrow (4\theta+\pi)(4\theta - 3\pi) = 0$$

$\therefore \theta = -\frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$ **(1)**

But $\theta\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$, so $\theta\neq\frac{3\pi}{4}$

Now, $\theta = \frac{-\pi}{4} \Rightarrow \tan^{-1} x = -\frac{\pi}{4} \Rightarrow x = \tan\left(-\frac{\pi}{4}\right)$

$\Rightarrow x = -\tan\frac{\pi}{4}$ $[\because \tan(-\theta) = -\tan\theta; x\in R]$

$\therefore x = -1$ $\left[\because \tan\frac{\pi}{4} = 1\right]$ **(1)**

43. To prove,

$$\cot^{-1}\left(\frac{xy+1}{x-y}\right) + \cot^{-1}\left(\frac{yz+1}{y-z}\right) + \cot^{-1}\left(\frac{zx+1}{z-x}\right) = 0;\ [0<xy, yx, zx<1]$$

Let LHS $= \cot^{-1}\left(\frac{xy+1}{x-y}\right) + \cot^{-1}\left(\frac{yz+1}{y-z}\right) + \cot^{-1}\left(\frac{zx+1}{z-x}\right)$

$$= \tan^{-1}\left(\frac{x-y}{1+xy}\right) + \tan^{-1}\left(\frac{y-z}{1+yz}\right) + \tan^{-1}\left(\frac{z-x}{1+zx}\right)$$ **(1)**

$\left[\because \cot^{-1} x = \tan^{-1}\frac{1}{x}; x>0\right]$

$$= (\tan^{-1} x - \tan^{-1} y) + (\tan^{-1} y - \tan^{-1} z) + (\tan^{-1} z - \tan^{-1} x)$$ **(2)**

$\left[\because 0<xy, yz, zx<1 \text{ and } \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right), \text{if } xy>-1\right]$

$= 0 =$ RHS

Hence proved. (1)

44. First, convert each inverse trigonometric function in the form of $\tan^{-1}\left(\frac{x-y}{1+xy}\right)$ and then use the formula $\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y,\ xy > -1$. Further, simplify it and again use the above formula.

Given, $\tan^{-1}\left(\frac{1}{1+1\cdot 2}\right) + \tan^{-1}\left(\frac{1}{1+2\cdot 3}\right)$

$$+ \ldots + \tan^{-1}\left(\frac{1}{1+n(n+1)}\right) = \tan^{-1}\theta$$

$$\Rightarrow \tan^{-1}\left(\frac{2-1}{1+2\cdot 1}\right) + \tan^{-1}\left(\frac{3-2}{1+3\cdot 2}\right)$$

$$+ \ldots + \tan^{-1}\left(\frac{(n+1)-n}{1+n(n+1)}\right) = \tan^{-1}\theta \quad (1)$$

$$\Rightarrow \tan^{-1}(2) - \tan^{-1}(1) + \tan^{-1}(3) - \tan^{-1}(2)$$

$$+ \ldots + \tan^{-1}(n+1) - \tan^{-1}(n) = \tan^{-1}\theta \quad (1)$$

$$\left[\because \tan^{-1}\left(\frac{x-y}{1+x\cdot y}\right) = \tan^{-1}x - \tan^{-1}y;\ xy > -1\right]$$

$$\Rightarrow \tan^{-1}(n+1) - \tan^{-1}(1) = \tan^{-1}\theta \quad (1)$$

$$\Rightarrow \tan^{-1}\left(\frac{n+1-1}{1+(n+1)\cdot 1}\right) = \tan^{-1}\theta$$

$$\left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+x\cdot y}\right);\ xy > -1\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{n}{1+n+1}\right) = \tan^{-1}\theta$$

$$\therefore \theta = \frac{n}{2+n} \quad (1)$$

45. First, use the relation, $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), -1 < x < 1$ and then use $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1$

To prove,

$$2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \sin^{-1}\left(\frac{31}{25\sqrt{2}}\right)$$

$$\text{LHS} = 2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left[\frac{2\times(1/2)}{1-(1/2)^2}\right] + \tan^{-1}\frac{1}{7}$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right); -1 < x < 1\right] \quad (1)$$

$$= \tan^{-1}\left(\frac{1}{1-\frac{1}{4}}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{1}{3/4}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right) \quad (1)$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right); xy < 1\right]$$

$$= \tan^{-1}\left(\frac{\frac{28+3}{21}}{\frac{21-4}{21}}\right) = \tan^{-1}\frac{31}{17} \quad (1)$$

Now, put $\tan^{-1}\frac{31}{17} = \theta$

$$\Rightarrow \tan\theta = \frac{31}{17} \text{ and } \theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

Clearly, $\sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{31}{17}\right)^2 = \frac{1250}{289}$

$$\Rightarrow \sec\theta = \frac{25\sqrt{2}}{17}$$

$$\left[\text{taking positive square root as } \theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\right]$$

$$\Rightarrow \frac{1}{\cos\theta} = \frac{25\sqrt{2}}{17} \qquad \left[\because \sec\theta = \frac{1}{\cos\theta}\right]$$

$$\Rightarrow \cos\theta = \frac{17}{25\sqrt{2}}$$

$$\Rightarrow \sin\theta = \tan\theta\cdot\cos\theta = \frac{31}{17}\cdot\frac{17}{25\sqrt{2}} = \frac{31}{25\sqrt{2}}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{31}{25\sqrt{2}}\right)$$

$$\therefore \tan^{-1}\left(\frac{31}{17}\right) = \sin^{-1}\left(\frac{31}{25\sqrt{2}}\right)$$

Thus, $\text{LHS} = \sin^{-1}\left(\frac{31}{25\sqrt{2}}\right) = \text{RHS}$ (1)

Hence proved.

46. Given, $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x;\ x > 0$

$$\Rightarrow 2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \tan^{-1}x$$

[multiplying both sides by 2]

$$\Rightarrow \quad \tan^{-1}\left[\frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2}\right] = \tan^{-1} x \qquad \textbf{(1½)}$$

$$\left[\because 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right); -1 < x < 1\right]$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{2(1-x)(1+x)}{(1+x)^2-(1-x)^2}\right] = \tan^{-1} x$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2(1-x^2)}{4x}\right) = \tan^{-1} x$$

$$[\because (a-b)(a+b) = a^2 - b^2$$
$$\text{and } (a+b)^2 - (a-b)^2 = 4ab]$$

$$\Rightarrow \quad \frac{2(1-x^2)}{4x} = x$$

$$\Rightarrow \quad \frac{1-x^2}{2x} = x \Rightarrow 1 - x^2 = 2x^2$$

$$\Rightarrow \quad 3x^2 = 1 \Rightarrow x^2 = \frac{1}{3} \qquad \textbf{(1½)}$$

$$\Rightarrow \quad x = \pm\frac{1}{\sqrt{3}} \qquad \textbf{(1)}$$

[$\because x > 0$ given, so we do not take $x = -\frac{1}{\sqrt{3}}$

$\therefore x = \frac{1}{\sqrt{3}}$ is the only solution of the given equation.]

47. Given equation is $\tan^{-1} x + 2\cot^{-1} x = \frac{2\pi}{3}$.

Then, the given equation can be written as

$$\tan^{-1} x + 2\tan^{-1}\left(\frac{1}{x}\right) = \frac{2\pi}{3}$$

$$\left[\because \cot^{-1} x = \tan^{-1}\frac{1}{x}; x > 0\right]$$

$$\Rightarrow \quad \tan^{-1} x + \tan^{-1}\left(\frac{2\times\frac{1}{x}}{1-\frac{1}{x^2}}\right) = \frac{2\pi}{3}$$

$$\left[\because 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right); -1 < x < 1\right]$$

$$\Rightarrow \quad \tan^{-1} x + \tan^{-1}\left(\frac{\frac{2}{x}}{\frac{x^2-1}{x^2}}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \quad \tan^{-1} x + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3} \qquad \textbf{(1½)}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x + \frac{2x}{x^2-1}}{1 - \frac{2x^2}{x^2-1}}\right) = \frac{2\pi}{3}$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right); xy < 1\right]$$

$$\Rightarrow \quad \frac{x^3 - x + 2x}{x^2 - 1 - 2x^2} = \tan\frac{2\pi}{3} \qquad \textbf{(1½)}$$

$$\Rightarrow \quad \frac{x^3 + x}{-1 - x^2} = \tan\left(\pi - \frac{\pi}{3}\right) \Rightarrow \frac{x^3 + x}{-(1 + x^2)} = -\tan\frac{\pi}{3}$$

$$[\because \tan(\pi - \theta) = -\tan\theta]$$

$$\therefore \quad \frac{x(1 + x^2)}{-(1 + x^2)} = -\sqrt{3} \qquad \left[\because \tan\frac{\pi}{3} = \sqrt{3}\right]$$

$$\Rightarrow \quad x = \sqrt{3} \qquad \textbf{(1)}$$

48. To prove,

$$2\tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2\tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}.$$

$$\text{LHS} = 2\tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2\tan^{-1}\left(\frac{1}{8}\right)$$

$$= 2\left\{\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\right\} + \sec^{-1}\frac{5\sqrt{2}}{7}$$

$$= 2\tan^{-1}\left\{\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right\} + \tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2 - 1}$$

$\Big[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right); xy < 1$

and $\sec^{-1}\frac{5\sqrt{2}}{7} = \theta \Rightarrow \sec\theta = \frac{5\sqrt{2}}{7}$

$\Rightarrow \tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2 - 1}$ **(1)**

$\Rightarrow \theta = \tan^{-1}\sqrt{\frac{50}{49} - 1}\Big]$

$$= 2\tan^{-1}\frac{13}{39} + \tan^{-1}\sqrt{\frac{50}{49} - 1}$$

$$= 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} \qquad \textbf{(1)}$$

$$= \tan^{-1}\left\{\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right\} + \tan^{-1}\frac{1}{7}$$

$$\left[\because 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right); -1 < x < 1\right] \textbf{(1)}$$

$$= \tan^{-1}\left(\frac{\frac{2}{3}}{\frac{9-1}{9}}\right) + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right) = \tan^{-1}\left(\frac{21+4}{28-3}\right)$$

$$= \tan^{-1}\left(\frac{25}{25}\right) = \tan^{-1}(1)$$

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4} = \text{RHS} \quad (1)$$

$$\left[\because \tan^{-1}(\tan\theta) = \theta; \forall\theta\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)\right]$$

Hence proved.

49. To prove,

$$\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x$$

$$\text{LHS} = \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

Put $x = \cos 2\theta$, then $\theta = \frac{1}{2}\cos^{-1}x$ (1)

$$\therefore \text{LHS} = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$$

$[\because 1+\cos 2\theta = 2\cos^2\theta, 1-\cos 2\theta = 2\sin^2\theta]$ (1)

$$= \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

[dividing numerator and denominator by $\sqrt{2}\cos\theta$]

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\theta\right)\right] \quad (1)$$

$$\left[\because \frac{1-\tan\theta}{1+\tan\theta} = \tan\left(\frac{\pi}{4}-\theta\right)\right]$$

$$= \frac{\pi}{4}-\theta \left[\because \tan^{-1}(\tan\theta) = \theta; \forall\theta\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)\right]$$

$$= \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x \quad \left[\because \theta = \frac{1}{2}\cos^{-1}x\right]$$

= RHS **Hence proved.** (1)

50. Do same as Q. No. 32. **[Ans. $x = \pm\sqrt{2}$]**

51. To prove,

$$\cos^{-1}(x) + \cos^{-1}\left\{\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right\} = \frac{\pi}{3}$$

Let $\text{LHS} = \cos^{-1}(x) + \cos^{-1}\left\{\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right\}$

Put $\cos^{-1}x = \alpha$,

then $x = \cos\alpha$, where $\alpha \in [0, \pi]$

$$\text{Now, LHS} = \alpha + \cos^{-1}\left[\frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sqrt{1-\cos^2\alpha}\right] \quad (1)$$

$$= \alpha + \cos^{-1}\left[\cos\frac{\pi}{3}\cos\alpha + \sin\frac{\pi}{3}\sin\alpha\right]$$

$$\left[\because \sqrt{1-\cos^2\alpha} = \sqrt{(\sin\alpha)^2} = \sin\alpha; \text{ as } \alpha\in[0,\pi] \text{ and } \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\right] \quad (1)$$

$$= \alpha + \cos^{-1}\left[\cos\left(\frac{\pi}{3}-\alpha\right)\right] \quad (1)$$

$[\because \cos A\cos B + \sin A\sin B = \cos(A-B)]$

$$= \alpha + \frac{\pi}{3} - \alpha = \frac{\pi}{3} = \text{RHS}$$

$[\because \cos^{-1}(\cos\theta) = \theta; \forall\theta\in[0,\pi]]$ (1)

Hence proved.

52. To prove, $\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18 = \cot^{-1}3$

$$\text{LHS} = \cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18$$

$$= \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18}$$

$$\left[\because \cot^{-1}x = \tan^{-1}\frac{1}{x}; x>0\right] \quad (1)$$

$$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}\right) + \tan^{-1}\frac{1}{18}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right); xy<1\right] \quad (1)$$

$$= \tan^{-1}\left(\frac{15}{55}\right) + \tan^{-1}\frac{1}{18}$$

$$= \tan^{-1}\left(\frac{3}{11}\right) + \tan^{-1}\frac{1}{18}$$

$$= \tan^{-1}\left(\frac{\frac{3}{11}+\frac{1}{18}}{1-\frac{3}{11}\times\frac{1}{18}}\right) = \tan^{-1}\left(\frac{65}{195}\right) \quad (1)$$

$$= \tan^{-1}\left(\frac{1}{3}\right) = \cot^{-1}3 = \text{RHS} \quad (1)$$

$$\left[\because \tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x; x>0\right]$$

Hence proved.

53. To prove, $\sin^{-1}\left(\frac{8}{17}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\cos^{-1}\left(\frac{36}{85}\right)$

Let $\sin^{-1}\left(\frac{8}{17}\right)=x$ and $\sin^{-1}\left(\frac{3}{5}\right)=y;$

$\forall x, y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ ...(i)

Then, $\sin x=\frac{8}{17}$ and $\sin y=\frac{3}{5}$ **(1)**

Now, $\cos^2 x=1-\sin^2 x$

$\Rightarrow \cos^2 x=1-\frac{64}{289}=\frac{225}{289}$

$\Rightarrow \cos x=\sqrt{\frac{225}{289}}$

$\left[\text{taking positive square root as } x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right]$

$\therefore \cos x=\frac{15}{17}$ **(1)**

Also, $\cos^2 y=1-\sin^2 y=1-\frac{9}{25}$

$\Rightarrow \cos y=\sqrt{\frac{16}{25}}$ $\left[\text{taking positive square root as } y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right]$

$\therefore \cos y=\frac{4}{5}$ **(1)**

We know that,

$\cos(x+y)=\cos x\cos y-\sin x\sin y$

$\Rightarrow \cos(x+y)=\left(\frac{15}{17}\times\frac{4}{5}\right)-\left(\frac{8}{17}\times\frac{3}{5}\right)$

$\Rightarrow \cos(x+y)=\frac{60}{85}-\frac{24}{85}=\frac{36}{85}$

$\Rightarrow x+y=\cos^{-1}\left(\frac{36}{85}\right)$

$\therefore \sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\cos^{-1}\frac{36}{85}$

[from Eq. (i)] **(1)**

Hence proved.

54. To prove, $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}$

LHS $=\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)$ ...(i)

Let $\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)=\theta$ ...(ii)

Then, $\sin^{-1}\left(\frac{3}{4}\right)=2\theta \Rightarrow \sin 2\theta=\frac{3}{4}$ **(1)**

Also, $2\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Now, $\sin 2\theta=\frac{3}{4}$

$\Rightarrow \frac{2\tan\theta}{1+\tan^2\theta}=\frac{3}{4}$ $\left[\because \sin 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}\right]$

$\Rightarrow 8\tan\theta=3+3\tan^2\theta$

$\Rightarrow 3\tan^2\theta-8\tan\theta+3=0$

Now, by quadratic formula

$\tan\theta=\frac{-(-8)\pm\sqrt{(-8)^2-4\times3\times3}}{2\times3}$

$\Rightarrow \tan\theta=\frac{8\pm\sqrt{64-36}}{6}=\frac{8\pm\sqrt{28}}{6}$ **(1)**

$\Rightarrow \tan\theta=\frac{8\pm2\sqrt{7}}{6}=\frac{4\pm\sqrt{7}}{3}$

As, $-\frac{\pi}{2}\le 2\theta\le\frac{\pi}{2} \Rightarrow -\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$

$\Rightarrow -1\le\tan\theta\le1$

$\therefore \tan\theta=\frac{4-\sqrt{7}}{3}$ $\left[\because \frac{4+\sqrt{7}}{3}>1\right]$

$\Rightarrow \theta=\tan^{-1}\left(\frac{4-\sqrt{7}}{3}\right)$

$[\because \tan\theta=x\Rightarrow\theta=\tan^{-1}x]$

$\Rightarrow \frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)=\tan^{-1}\left(\frac{4-\sqrt{7}}{3}\right)$ **(1)**

[from Eq. (ii)]

On taking tan both sides, we get

$\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\tan\left\{\tan^{-1}\left(\frac{4-\sqrt{7}}{3}\right)\right\}$

$\therefore \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}=$ RHS

$[\because \tan(\tan^{-1}x)=x;\ x\in R]$ **(1)**

Hence proved.

55. Given equation is $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$

$\Rightarrow \sin^{-1}(1-x)=\frac{\pi}{2}+2\sin^{-1}x$ **(1/2)**

$\Rightarrow 1-x=\sin\left(\frac{\pi}{2}+2\sin^{-1}x\right)$

$\Rightarrow 1-x=\cos(2\sin^{-1}x)$

$\left[\because \sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta\right]$ **(1/2)**

Put $\sin^{-1}x=\theta$, then $\Rightarrow 1-x=\cos 2\theta$

$\Rightarrow 1-x=1-2\sin^2\theta$ $[\because \cos 2A=1-2\sin^2 A]$

$\Rightarrow \quad 1-x=1-2x^2 \quad [\because \sin^{-1}x=\theta \Rightarrow x=\sin\theta]$ (1)

$\Rightarrow \quad 2x^2-x=0 \quad \Rightarrow x(2x-1)=0$

$\therefore \quad x=0$ or $x=\frac{1}{2}$ (1)

For $x=\frac{1}{2}$, LHS $=\sin^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(\frac{1}{2}\right)$

$=\frac{\pi}{6}-\frac{2\pi}{6}=\frac{-\pi}{6}\neq\frac{\pi}{2}$

$\therefore x=\frac{1}{2}$ is not a solution of given equation.

Hence, $x=0$ is the only solution. (1)

56. To prove, $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{77}{36}$

Let $\sin^{-1}\frac{8}{17}=x$ and $\sin^{-1}\frac{3}{5}=y;\ \forall x,y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\Rightarrow \quad \sin x=\frac{8}{17}$ and $\sin y=\frac{3}{5}$ (1)

Now, $\cos^2 x=1-\sin^2 x=1-\frac{64}{289}=\frac{225}{289}$

$\Rightarrow \quad \cos x=\sqrt{\frac{225}{289}}=\frac{15}{17}$

[taking positive square root as $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$]

and $\cos^2 y=1-\sin^2 y=1-\frac{9}{25}=\frac{16}{25}$

$\Rightarrow \quad \cos y=\sqrt{\frac{16}{25}}=\frac{4}{5}$

[taking positive square root as $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$] (1)

Clearly, $\tan x=\frac{\sin x}{\cos x}=\frac{8}{15}$ and $\tan y=\frac{3}{4}$ (1)

$\Rightarrow \quad x=\tan^{-1}\frac{8}{15}$ and $y=\tan^{-1}\frac{3}{4}$

Now, LHS $=x+y=\tan^{-1}\frac{8}{15}+\tan^{-1}\frac{3}{4}$

$=\tan^{-1}\left(\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}}\right)=\tan^{-1}\left(\frac{32+45}{60-24}\right)$

$=\tan^{-1}\left(\frac{77}{36}\right)=$ RHS **Hence proved.** (1)

57. Let $x=\tan\theta$ and $y=\tan\phi$, then

$\tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right]$

$=\tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)+\cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right)\right]$ (1)

$=\tan\frac{1}{2}[\sin^{-1}(\sin 2\theta)+\cos^{-1}(\cos 2\phi)]$

$\left[\because \sin 2A=\frac{2\tan A}{1+\tan^2 A}\text{ and }\cos 2A=\frac{1-\tan^2 A}{1+\tan^2 A}\right]$ (1)

$=\tan\frac{1}{2}[2\theta+2\phi]=\tan(\theta+\phi)$

$\left[\because \sin^{-1}(\sin x)=x;\ \forall x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\text{ and }\cos^{-1}(\cos x)=x;\ \forall x\in[0,\pi]\right]$ (1)

$=\tan(\tan^{-1}x+\tan^{-1}y)$

$[\because \theta=\tan^{-1}x$ and $\phi=\tan^{-1}y]$

$=\tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right);\text{ if } xy<1\right]$

$=\frac{x+y}{1-xy}$ $\quad [\because \tan(\tan^{-1}x)=x;\ x\in R]$ (1)

58. Do same as Q. No. 35.

59. To prove, $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\frac{\pi}{4}-\frac{x}{2},\ x\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

LHS $=\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

$=\tan^{-1}\left(\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$

$\left[\because \cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2};\ 1=\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\text{ and }\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}\right]$ (1)

$=\tan^{-1}\left[\frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}\right]$

$=\tan^{-1}\left(\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right)$ (1)

On dividing the numerator and denominator by $\cos x/2$, we get **(1)**

$$\text{LHS} = \tan^{-1}\left(\frac{\dfrac{\cos\frac{x}{2}}{\cos\frac{x}{2}} - \dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{\dfrac{\cos\frac{x}{2}}{\cos\frac{x}{2}} + \dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}\right) = \tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right] \quad \left[\because \frac{1-\tan A}{1+\tan A} = \tan\left(\frac{\pi}{4}-A\right)\right]$$

$$= \frac{\pi}{4}-\frac{x}{2} \quad \left[\because \tan^{-1}(\tan\theta) = \theta;\ \forall \theta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$$

$= \text{RHS}$ **Hence proved. (1)**

60. To prove, $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

Let $\cos^{-1}\frac{4}{5} = x$ and $\cos^{-1}\left(\frac{12}{13}\right) = y;\ \forall x, y \in [0, \pi]$

$\Rightarrow \quad \cos x = \frac{4}{5}$ and $\cos y = \frac{12}{13}$ ...(i)

$\Rightarrow \quad \sin x = \sqrt{1-\cos^2 x}$ and $\sin y = \sqrt{1-\cos^2 y}$ **(1)**

[taking positive sign as $x, y \in [0, \pi]$]

$\Rightarrow \quad \sin x = \sqrt{1-\left(\frac{4}{5}\right)^2}$ and $\sin y = \sqrt{1-\left(\frac{12}{13}\right)^2}$

$\Rightarrow \quad \sin x = \sqrt{1-\frac{16}{25}}$ and $\sin y = \sqrt{1-\frac{144}{169}}$

$\Rightarrow \quad \sin x = \sqrt{\frac{25-16}{25}} = \sqrt{\frac{9}{25}}$ **(1)**

and $\quad \sin y = \sqrt{\frac{169-144}{169}} = \sqrt{\frac{25}{169}}$

$\Rightarrow \quad \sin x = \frac{3}{5}$ and $\sin y = \frac{5}{13}$

We know that,

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\Rightarrow \quad \cos(x+y) = \left(\frac{4}{5}\times\frac{12}{13}\right) - \left(\frac{3}{5}\times\frac{5}{13}\right)$

$\Rightarrow \quad \cos(x+y) = \frac{48}{65} - \frac{15}{65}$

$\Rightarrow \quad \cos(x+y) = \frac{33}{65}$ **(1)**

$\Rightarrow \quad x + y = \cos^{-1}\frac{33}{65}$

$\therefore \quad \cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$ [from Eq. (i)]

Hence proved. (1)

61. To prove, $\cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}$

Let $\sin^{-1}\frac{3}{5} = x$ and $\cot^{-1}\left(\frac{3}{2}\right) = y;\ \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

and $\quad y \in (0, \pi)$ ...(i)

$\Rightarrow \quad \sin x = \frac{3}{5}$ and $\cot y = \frac{3}{2}$ **(1)**

$\Rightarrow \quad \cos x = \sqrt{1-\sin^2 x}$ and $\operatorname{cosec} y = \sqrt{1+\cot^2 y}$

$\left[\text{taking positive sign as } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ and } y \in (0, \pi)\right]$

$\Rightarrow \quad \cos x = \sqrt{1-\left(\frac{3}{5}\right)^2}$ and $\operatorname{cosec} y = \sqrt{1+\left(\frac{3}{2}\right)^2}$ **(1)**

$\Rightarrow \quad \cos x = \sqrt{1-\frac{9}{25}}$ and $\operatorname{cosec} y = \sqrt{1+\frac{9}{4}}$

$\Rightarrow \quad \cos x = \sqrt{\frac{25-9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

and $\quad \operatorname{cosec} y = \sqrt{\frac{4+9}{4}} = \sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{2}$

$\Rightarrow \quad \cos x = \frac{4}{5}$ and $\frac{1}{\sin y} = \frac{\sqrt{13}}{2}$

$\Rightarrow \quad \cos x = \frac{4}{5}$ and $\sin y = \frac{2}{\sqrt{13}}$

Also, $\cos y = \sin y \cdot \cot y = \frac{2}{\sqrt{13}} \times \frac{3}{2} = \frac{3}{\sqrt{13}}$ **(1)**

Now, $\cos(x+y) = \cos x \cos y - \sin x \sin y$

$= \frac{4}{5}\times\frac{3}{\sqrt{13}} - \frac{3}{5}\times\frac{2}{\sqrt{13}}$

$= \frac{12}{5\sqrt{13}} - \frac{6}{5\sqrt{13}} = \frac{6}{5\sqrt{13}} = \text{RHS}$

Hence proved. (1)

62. To prove, $\sin^{-1}\left(\frac{63}{65}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$

Let $\sin^{-1}\frac{5}{13} = x$ and $\cos^{-1}\frac{3}{5} = y,\ \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

and $\quad y \in [0, \pi]$ ...(i)

$\Rightarrow \quad \sin x = \frac{5}{13}$ and $\cos y = \frac{3}{5}$

$\Rightarrow \quad \cos x = \sqrt{1-\sin^2 x}$ and $\sin y = \sqrt{1-\cos^2 y}$

[taking positive sign as $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $y \in [0, \pi]$]

$\Rightarrow \quad \cos x = \sqrt{1-\left(\frac{5}{13}\right)^2}$ and $\sin y = \sqrt{1-\left(\frac{3}{5}\right)^2}$ **(2)**

$\Rightarrow \quad \cos x = \sqrt{1-\frac{25}{169}} = \sqrt{\frac{169-25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$

and $\sin y = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{25-9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

Now, $\sin(x+y) = \sin x\cos y + \cos x \sin y$

$\therefore \quad \sin(x+y) = \frac{5}{13}\cdot\frac{3}{5}+\frac{12}{13}\cdot\frac{4}{5} = \frac{15}{65}+\frac{48}{65} = \frac{63}{65}$ **(1)**

$\Rightarrow \quad x+y = \sin^{-1}\left(\frac{63}{65}\right)$

$\therefore \sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{63}{65}\right)$ [from Eq. (i)]

Hence proved. (1)

63. Do same as Q. No. 36. $\left[\textbf{Ans.} \frac{\pi}{4}\right]$

64. We have, $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$

$$= \tan^{-1}\left(\frac{\frac{x}{y}-\frac{x-y}{x+y}}{1+\frac{x}{y}\cdot\frac{x-y}{x+y}}\right) \quad \textbf{(1½)}$$

$\left[\because \tan^{-1} a - \tan^{-1} b = \tan^{-1}\left(\frac{a-b}{1+ab}\right)\right]$

$$= \tan^{-1}\left[\frac{x(x+y)-y(x-y)}{y(x+y)+x(x-y)}\right]$$

$$= \tan^{-1}\left(\frac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right) \quad \textbf{(1½)}$$

$$= \tan^{-1}\left(\frac{x^2+y^2}{y^2+x^2}\right) = \tan^{-1}(1)$$

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$$

$\left[\because \tan^{-1}(\tan\theta) = \theta, \theta \in \left(\frac{-\pi}{2},\frac{\pi}{2}\right)\right]$ **(1)**

65. Do same as Q. No. 45.

66. To prove, $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

Let $\sin^{-1}\left(\frac{1}{3}\right) = x$

and $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = y; \forall x, y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Then, $\sin x = \frac{1}{3}$ and $\sin y = \frac{2\sqrt{2}}{3}$

Now, $\cos^2 x = 1-\sin^2 x = 1-\frac{1}{9} = \frac{8}{9}$

$\Rightarrow \quad \cos x = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

[taking positive square root as $x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$] **(1)**

Similarly, $\cos^2 y = 1-\sin^2 y = 1-\frac{8}{9} = \frac{1}{9}$

$\Rightarrow \quad \cos y = \sqrt{\frac{1}{9}} = \frac{1}{3}$

[taking positive square root as $x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$] **(1/2)**

Now, $\sin(x+y) = \sin x\cos y + \cos x \sin y$

$$= \frac{1}{3}\times\frac{1}{3}+\frac{2\sqrt{2}}{3}\times\frac{2\sqrt{2}}{3}$$

$$= \frac{1}{9}+\frac{8}{9} = \frac{9}{9} = 1 \quad \textbf{(1)}$$

$\Rightarrow x+y = \sin^{-1}(1) = \sin^{-1}\left(\sin\frac{\pi}{2}\right)$

$\Rightarrow \quad x+y = \frac{\pi}{2} \quad \left[\because \sin^{-1}(\sin\theta) = \theta; \forall\theta \in \left(\frac{-\pi}{2},\frac{\pi}{2}\right)\right]$

$\Rightarrow \quad \sin^{-1}\left(\frac{1}{3}\right)+\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{\pi}{2}$ **(1/2)**

$\left[\because x = \sin^{-1}\left(\frac{1}{3}\right) \text{ and } y = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right]$

$\Rightarrow \quad \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)+\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{9\pi}{8}$

[multiplying both sides by 9/4]

$\therefore \quad \frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ **(1)**

Hence proved.

Alternate Method

LHS $= \frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$ **(1)**

$= \frac{9}{4}\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{1}{3}\right)\right] = \frac{9}{4}\left[\cos^{-1}\left(\frac{1}{3}\right)\right]$

$\left[\because \cos^{-1} x = \frac{\pi}{2}-\sin^{-1} x; x \in [-1,1]\right]$ **(1)**

$= \frac{9}{4}\sin^{-1}\left(\sqrt{1-\frac{1}{9}}\right)$

$\left[\begin{array}{l}\text{let } \cos^{-1}\frac{1}{3} = y \Rightarrow \cos y = \frac{1}{3}\text{, then} \\ \sin y = \sqrt{1-\cos^2 y} \Rightarrow y = \sin^{-1}\sqrt{1-\left(\frac{1}{3}\right)^2} \\ \Rightarrow \cos^{-1}\frac{1}{3} = \sin^{-1}\sqrt{1-\left(\frac{1}{9}\right)}\end{array}\right]$ **(1)**

$$= \frac{9}{4}\sin^{-1}\left(\sqrt{\frac{8}{9}}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

= RHS **Hence proved. (1)**

67. To prove,

$$\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) = \frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right) \quad \ldots(i)$$

Eq. (i) can be rewritten as

$$2\left[\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)\right] = \tan^{-1}\left(\frac{4}{3}\right) \ldots(ii)$$

(1/2)

$$\text{LHS} = 2\left[\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)\right]$$

$$= 2\left[\tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right)\right] \quad \textbf{(1)}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right); xy<1\right]$$

$$= 2\tan^{-1}\left(\frac{\frac{9+8}{36}}{\frac{36-2}{36}}\right) = 2\tan^{-1}\left(\frac{17}{34}\right)$$

$$= 2\tan^{-1}\left(\frac{1}{2}\right) = \tan^{-1}\left[\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right] \quad \textbf{(1)}$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right); -1<x<1\right]$$

$$= \tan^{-1}\left(\frac{1}{1-\frac{1}{4}}\right) = \tan^{-1}\left(\frac{4}{3}\right) = \text{RHS} \quad \textbf{(1½)}$$

Hence proved.

68. Given equation is

$$\cos(2\sin^{-1}x) = \frac{1}{9}, x>0 \quad \ldots(i)$$

Put $\sin^{-1}x = y$

$\Rightarrow$ $x = \sin y$ **(1/2)**

Then, Eq. (i) becomes, $\cos 2y = \frac{1}{9}$

$\Rightarrow$ $1 - 2\sin^2 y = \frac{1}{9}$ $[\because \cos 2\theta = 1 - 2\sin^2\theta]$ **(1)**

$\Rightarrow$ $2\sin^2 y = 1 - \frac{1}{9} = \frac{8}{9}$ **(1/2)**

$\Rightarrow$ $\sin^2 y = \frac{4}{9}$

$\Rightarrow$ $x^2 = \frac{4}{9}$ $[\because \sin y = x]$

$\therefore$ $x = \pm\frac{2}{3}$ [taking square root] **(1)**

But it is given that, $x > 0$.

$\therefore$ $x = \frac{2}{3}$ **(1)**

69.

> First, use the relation $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ $-1<x<1$ and then use the relation $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$, $xy>-1$ and get the required result.

To prove, $2\tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \frac{\pi}{4}$

$$\text{LHS} = 2\tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\frac{9}{16}}\right) - \tan^{-1}\left(\frac{17}{31}\right) \quad \textbf{(1)}$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right); -1<x<1\right]$$

$$= \tan^{-1}\left(\frac{3/2}{7/16}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\left(\frac{24}{7}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\times\frac{17}{31}}\right) \quad \textbf{(1)}$$

$$\left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right); xy>-1\right]$$

$$= \tan^{-1}\left(\frac{24\times31 - 17\times7}{7\times31 + 24\times17}\right)$$

$$= \tan^{-1}\left(\frac{744-119}{217+408}\right)$$

$$= \tan^{-1}\left(\frac{625}{625}\right) = \tan^{-1}(1)$$

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) \quad \left[\because 1 = \tan\frac{\pi}{4}\right]$$

$$= \frac{\pi}{4} = \text{RHS}$$

$$\left[\because \tan^{-1}(\tan\theta) = \theta, \theta\in\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\right]$$

Hence proved. (2)

70. Given equation is

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3};-1<x<1$$

We know that, $\cot^{-1}x=\tan^{-1}\frac{1}{x};\forall x>0$, so by using this result, we may write

$$\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\quad\textbf{(1/2)}$$

Then, given equation becomes

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}\quad\textbf{(1/2)}$$

$$\Rightarrow\quad 2\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}\quad\textbf{(1/2)}$$

$$\Rightarrow\quad \tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{6}$$

$$\Rightarrow\quad \frac{2x}{1-x^2}=\tan\frac{\pi}{6}$$

$$\Rightarrow\quad \frac{2x}{1-x^2}=\frac{1}{\sqrt{3}}\quad\left[\because \tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}\right]$$

$$\Rightarrow\quad 2\sqrt{3}x=1-x^2\quad\textbf{(1)}$$

$$\Rightarrow\quad x^2+2\sqrt{3}x-1=0$$

$$\therefore\quad x=\frac{-2\sqrt{3}\pm\sqrt{12+4}}{2}\quad\text{[by quadratic formula]}$$

$$\Rightarrow\quad x=\frac{-2\sqrt{3}\pm 4}{2}=\frac{4-2\sqrt{3}}{2},\frac{-4-2\sqrt{3}}{2}\quad\textbf{(1)}$$

$$\therefore\quad x=2-\sqrt{3}\text{ or }-(2+\sqrt{3})$$

But it is given that $-1<x<1$, so $x=-(2+\sqrt{3})$ is rejected, hence $x=2-\sqrt{3}$. **(1/2)**

71.

> First, put $\sqrt{x}=\tan\theta\Rightarrow\theta=\tan^{-1}\sqrt{x}$
>
> and then use $\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}$.

To prove, $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x\in(0,1)$.

$$\text{RHS}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\cos^{-1}\left[\frac{1-(\sqrt{x})^2}{1+(\sqrt{x})^2}\right]\quad\textbf{(1)}$$

On substituting $\sqrt{x}=\tan\theta$, we get

$$\text{RHS}=\frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)\quad\textbf{(1)}$$

$$=\frac{1}{2}\cos^{-1}(\cos 2\theta)\quad\left[\because\frac{1-\tan^2A}{1+\tan^2A}=\cos 2A\right]\textbf{(1)}$$

$$=\frac{1}{2}(2\theta)=\theta\quad[\because\cos^{-1}(\cos\theta)=\theta;\forall\theta\in[0,\pi]]$$

$$=\tan^{-1}\sqrt{x}\quad[\because\theta=\tan^{-1}\sqrt{x}]\ \textbf{(1)}$$

$$=\text{LHS}$$

Hence proved.

72. Do same as Q. No. 62.

73. To prove,

$$\tan^{-1}(1)+\tan^{-1}(2)+\tan^{-1}(3)=\pi$$

$$\text{LHS}=\tan^{-1}(1)+\tan^{-1}(2)+\tan^{-1}(3)$$

$$=\tan^{-1}\left(\tan\frac{\pi}{4}\right)+\frac{\pi}{2}-\cot^{-1}(2)+\frac{\pi}{2}-\cot^{-1}(3)\ \textbf{(1)}$$

$$\left[\because\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2},x\in R\right]$$

$$=\frac{\pi}{4}+\pi-[\cot^{-1}(2)+\cot^{-1}(3)]$$

$$\left[\because\tan^{-1}(\tan\theta)=\theta;\forall\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]\textbf{(1/2)}$$

$$=\frac{5\pi}{4}-\left[\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)\right]$$

$$\left[\because\cot^{-1}x=\tan^{-1}\frac{1}{x},x>0\right]\textbf{(1)}$$

$$=\frac{5\pi}{4}-\left[\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)\right]$$

$$\left[\because\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right),\text{if } xy<1\right]$$

$$=\frac{5\pi}{4}-\tan^{-1}\left(\frac{5/6}{5/6}\right)\quad\textbf{(1)}$$

$$=\frac{5\pi}{4}-\tan^{-1}(1)=\frac{5\pi}{4}-\frac{\pi}{4}=\frac{4\pi}{4}=\pi=\text{RHS}$$

Hence proved. **(1/2)**

74. To prove,

$$\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\text{LHS}=\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$=\tan^{-1}\left[\frac{x+\frac{2x}{1-x^2}}{1-x\left(\frac{2x}{1-x^2}\right)}\right]\quad\textbf{(1½)}$$

$$\left[\because\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right);xy<1\right]$$

$$= \tan^{-1}\left(\frac{x - x^3 + 2x}{1 - x^2 - 2x^2}\right) \quad \textbf{(1½)}$$

$$= \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) \quad \textbf{(1)}$$

$= \text{RHS}$ **Hence proved.**

Alternate Method

Let $\tan^{-1} x = \theta \Rightarrow x = \tan\theta$ **(1/2)**

Then, $\text{RHS} = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

$$= \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right) \quad [\because x = \tan\theta] \ \textbf{(1½)}$$

$$= \tan^{-1}(\tan 3\theta) \quad \left[\because \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \tan 3\theta\right]$$

$= 3\theta = 3\tan^{-1} x$ $[\because \theta = \tan^{-1} x]$ **(1)**

$= \tan^{-1} x + 2\tan^{-1} x$

$$= \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1 - x^2}\right) = \text{LHS} \quad \textbf{(1)}$$

$$\left[\because 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right); -1 < x < 1\right]$$

Hence proved.

75. To prove, $\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}] = \sqrt{\frac{1 + x^2}{2 + x^2}}$

$\text{LHS} = \cos[\tan^{-1}\{\sin(\cot^{-1} x)\}]$

Put $\cot^{-1} x = \theta \Rightarrow x = \cot\theta$ **(1/2)**

Then, $\text{LHS} = \cos[\tan^{-1}(\sin\theta)]$

$$= \cos\left[\tan^{-1}\left(\frac{1}{\operatorname{cosec}\theta}\right)\right] \quad \textbf{(1/2)}$$

$$\left[\because \sin\theta = \frac{1}{\operatorname{cosec}\theta}\right]$$

$$= \cos\left[\tan^{-1}\left(\frac{1}{\sqrt{1 + \cot^2\theta}}\right)\right]$$

$$[\because \operatorname{cosec}^2\theta = 1 + \cot^2\theta]$$

$$= \cos\left[\tan^{-1}\left(\frac{1}{\sqrt{1 + x^2}}\right)\right] \quad [\because \cot\theta = x]$$

$= \cos\phi$

where, $\tan^{-1}\left(\frac{1}{\sqrt{1 + x^2}}\right) = \phi$ or $\tan\phi = \frac{1}{\sqrt{1 + x^2}}$ **(1)**

Now, $\text{LHS} = \cos\phi = \frac{1}{\sec\phi}$ $\left[\because \cos\theta = \frac{1}{\sec\theta}\right]$

$$= \frac{1}{\sqrt{1 + \tan^2\phi}} \quad [\because \tan^2\theta + 1 = \sec^2\theta]$$

$$= \frac{1}{\sqrt{1 + \frac{1}{1 + x^2}}} \quad \left[\because \tan\phi = \frac{1}{\sqrt{1 + x^2}}\right] \quad \textbf{(1)}$$

$$= \frac{1}{\sqrt{\frac{1 + x^2 + 1}{1 + x^2}}} = \sqrt{\frac{1 + x^2}{2 + x^2}} = \text{RHS} \quad \textbf{(1)}$$

Hence proved.

76. Given, $\cos^{-1} x + \sin^{-1}\left(\frac{x}{2}\right) = \frac{\pi}{6}$

$$\Rightarrow \cos^{-1} x = \frac{\pi}{6} - \sin^{-1}\frac{x}{2}$$

$$\Rightarrow x = \cos\left(\frac{\pi}{6} - \sin^{-1}\frac{x}{2}\right)$$

$$\Rightarrow x = \cos\frac{\pi}{6}\cos\left(\sin^{-1}\frac{x}{2}\right) + \sin\frac{\pi}{6}\sin\left(\sin^{-1}\frac{x}{2}\right) \quad \textbf{(1)}$$

$$[\because \cos(x - y) = \cos x\cos y + \sin x \sin y]$$

$$\Rightarrow x = \frac{\sqrt{3}}{2}\cos\left(\sin^{-1}\frac{x}{2}\right) + \frac{1}{2}\cdot\frac{x}{2}$$

$$[\because \sin(\sin^{-1} x) = x; \forall x \in [-1, 1]]$$

$$\Rightarrow x = \frac{\sqrt{3}}{2}\cos\left(\cos^{-1}\sqrt{1 - \frac{x^2}{4}}\right) + \frac{x}{4}$$

$$\left[\begin{array}{l}\text{let } \sin^{-1} y = \theta \Rightarrow \sin\theta = y \\ \text{Now, } \cos\theta = \sqrt{1 - \sin^2\theta} \\ \Rightarrow \sin^{-1} y = \cos^{-1}\sqrt{1 - y^2}\end{array}\right]$$

$$\Rightarrow x = \frac{\sqrt{3}}{2}\left(\sqrt{1 - \frac{x^2}{4}}\right) + \frac{x}{4}$$

$$[\because \cos(\cos^{-1} x) = x; \forall x \in [-1, 1]]$$

$$\Rightarrow x - \frac{x}{4} = \frac{\sqrt{3}}{2}\left(\sqrt{1 - \frac{x^2}{4}}\right)$$

$$\Rightarrow \frac{3x}{4} = \frac{\sqrt{3}}{2}\left(\sqrt{1 - \frac{x^2}{4}}\right) \quad \textbf{(1)}$$

On squaring both sides, we get

$$\frac{9x^2}{16} = \frac{3}{4}\left(1 - \frac{x^2}{4}\right)$$

$$\Rightarrow \frac{3}{4}x^2 = 1 - \frac{x^2}{4}$$

$$\Rightarrow \frac{3}{4}x^2 + \frac{x^2}{4} = 1$$

$\Rightarrow \quad \frac{4x^2}{4} = 1$

$\therefore \quad x^2 = 1 \Rightarrow x = \pm 1$ **(1)**

But $x = -1$, does not satisfy the given equation.

Hence, $x = 1$ satisfy the given equation. **(1)**

77. Do same as Q. No. 69.

78. Given, $\tan^{-1}\frac{x}{2} + \tan^{-1}\frac{x}{3} = \frac{\pi}{4}, \sqrt{6} > x > 0$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{x}{2}+\frac{x}{3}}{1-\frac{x^2}{6}}\right) = \frac{\pi}{4}$$

$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right); xy < 1\right]$ **(1½)**

$$\Rightarrow \quad \frac{\frac{3x+2x}{6}}{\frac{6-x^2}{6}} = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{5x}{6-x^2} = 1 \qquad \left[\because \tan\frac{\pi}{4} = 1\right]$$

$\Rightarrow \quad 5x = 6 - x^2$

$\Rightarrow \quad x^2 + 5x - 6 = 0$

$\Rightarrow \quad x^2 + 6x - x - 6 = 0$

$\Rightarrow \quad x(x+6) - 1(x+6) = 0$

$\Rightarrow \quad (x-1)(x+6) = 0$

$\therefore \quad x = 1$ or -6 **(1½)**

But it is given that, $\sqrt{6} > x > 0 \Rightarrow x > 0$

$\therefore x = -6$ is rejected.

Hence, $x = 1$ is the only solution of the given equation. **(1)**

79. Do same as Q. No. 28. $\left[\textbf{Ans. } x = \frac{1}{4}\right]$

Solutions Objectives

1. (*c*) We know that,

$\sin^{-1} x + \cos^{-1} x = \pi/2, \forall x \in [-1, 1]$

$\therefore \quad \cos^{-1} x > \sin^{-1} x \Rightarrow \frac{\pi}{2} - \sin^{-1} x > \sin^{-1} x$

$\Rightarrow \frac{\pi}{2} > 2\sin^{-1} x \Rightarrow \sin^{-1} x < \frac{\pi}{4} \Rightarrow x < \frac{1}{\sqrt{2}}$

$\therefore \quad -1 \le x < \frac{1}{\sqrt{2}}$

2. (*a*) Given, $\tan^{-1}\left(\frac{a}{x}\right) + \tan^{-1}\left(\frac{b}{x}\right) = \frac{\pi}{2}$

$$\Rightarrow \tan^{-1}\left[\frac{\left(\frac{a}{x}+\frac{b}{x}\right)}{1-\frac{ab}{x^2}}\right] = \frac{\pi}{2} \Rightarrow \frac{\frac{a+b}{x}}{\frac{x^2-ab}{x^2}} = \tan\frac{\pi}{2} = \infty$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1\right]$$

$$\Rightarrow \frac{a+b}{x} \times \frac{x^2}{x^2-ab} = \frac{1}{0} \Rightarrow x^2 = ab$$

$\therefore \quad x = \sqrt{ab}$

3. (*d*) Let $E = \sin(2\sin^{-1} 0.8)$

Put $\sin^{-1} 0.8 = \theta \Rightarrow \sin\theta = 0.8$

$\therefore \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-0.64} = \sqrt{0.36} = 0.6$

$\therefore \quad E = \sin(2\theta) = 2\sin\theta\cos\theta$

$= 2 \times 0.8 \times 0.6 = 0.96$

4. (*d*) $\sin^{-1}\left[\left(\sin\frac{2\pi}{3}\right)\right] = \sin^{-1}\left[\sin\left(\pi - \frac{\pi}{3}\right)\right]$

$= \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$

$\left[\because \sin^{-1}(\sin\theta) = \theta, \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right]$

5. (*d*) $\sin(2\tan^{-1} x) = \sin\left(\sin^{-1}\frac{2x}{1+x^2}\right)$

$\left[\because 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), -1 < x < 1\right]$

$= \frac{2x}{1+x^2}, -1 \le x \le 1$

$[\because \sin(\sin^{-1} x) = x, x \in [-1, 1]]$

6. (*b*) We have, $\tan^{-1} x = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{3}\right)$

$\Rightarrow \quad \tan^{-1} x + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\left[\frac{x+\frac{1}{3}}{1-x\left(\frac{1}{3}\right)}\right] = \frac{\pi}{4} \Rightarrow \frac{3x+1}{3-x} = 1$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1\right]$$

$\Rightarrow \quad 3x + 1 = 3 - x \Rightarrow 4x = 2$

$\therefore \quad x = 1/2$

7. (*a*) Consider,

$\sin\left[\tan^{-1}\left(\frac{1-x^2}{2x}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$

Now, substitute $x = \tan\theta$, we get

$\sin\left[\tan^{-1}\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) + \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)\right]$

$= \sin[\tan^{-1}(\cot 2\theta) + \cos^{-1}(\cos 2\theta)]$

$$=\sin\left[\tan^{-1}\left\{\tan\left(\frac{\pi}{2}-2\theta\right)\right\}+2\theta\right]$$

$[\because \cos^{-1}(\cos\theta)=\theta, \theta\in[0,\pi]]$

$$=\sin\left(\frac{\pi}{2}-2\theta+2\theta\right)=1$$

$$\left[\because \tan^{-1}(\tan\theta)=\theta, \theta\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)\right]$$

8. (c) Given that, $\sin^{-1}\frac{1}{3}+\sin^{-1}\frac{2}{3}=\sin^{-1}x$

$$\Rightarrow \sin^{-1}\left(\frac{1}{3}\sqrt{1-\frac{4}{9}}+\frac{2}{3}\sqrt{1-\frac{1}{9}}\right)=\sin^{-1}x$$

$$\left[\because \sin^{-1}x+\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2}), \text{ if } |x|,|y|\le 1 \text{ and } x^2+y^2\le 1\right]$$

$$\Rightarrow \sin^{-1}\left(\frac{1}{3}\cdot\frac{\sqrt{5}}{3}+\frac{2}{3}\cdot\frac{\sqrt{8}}{3}\right)=\sin^{-1}x$$

$$\Rightarrow \sin^{-1}\left(\frac{\sqrt{5}+4\sqrt{2}}{9}\right)=\sin^{-1}x$$

$$\therefore \quad x=\left(\frac{\sqrt{5}+4\sqrt{2}}{9}\right)$$

9. (a) Let $x=-y, y>0$

$\therefore \sin^{-1}x=\sin^{-1}(-y)$

$=-\sin^{-1}y$

$=-\cos^{-1}\sqrt{1-y^2}=-\cos^{-1}\sqrt{1-x^2}$

10. (c) $\tan(\sin^{-1}x)=\tan\left(\tan^{-1}\frac{x}{\sqrt{1-x^2}}\right), x\in(-1,1)$

$$=\frac{x}{\sqrt{1-x^2}}$$

$[\because \tan(\tan^{-1}x)=x, x\in R]$

11. (b) $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3}+\tan^{-1}\frac{2}{3}\right)$

$$=\cot\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)$$

$$=\cot\left[\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{1}{2}}\right)\right]=\cot\left[\tan^{-1}\left(\frac{17}{6}\right)\right]=\frac{6}{17}$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), xy<1, \tan^{-1}\frac{1}{x}=\cot^{-1}, x>0\right]$$

12. (a) Given, $\sin^{-1}x+\sin^{-1}y=\frac{\pi}{2}$

$$\therefore \frac{\pi}{2}-\cos^{-1}x+\frac{\pi}{2}-\cos^{-1}y=\frac{\pi}{2}$$

$$\Rightarrow \cos^{-1}x+\cos^{-1}y=\frac{\pi}{2}$$

13. (b) Given, $\sin^{-1}x-\cos^{-1}x=\frac{\pi}{6}$

$$\Rightarrow \left(\frac{\pi}{2}-\cos^{-1}x\right)-\cos^{-1}x=\frac{\pi}{6} \Rightarrow \cos^{-1}x=\frac{\pi}{6}$$

$$\therefore \quad x=\frac{\sqrt{3}}{2}$$

14. (a) $\tan\left\{\cos^{-1}\left(-\frac{2}{7}\right)-\frac{\pi}{2}\right\}$

$$=\tan\left\{\pi-\cos^{-1}\left(\frac{2}{7}\right)-\frac{\pi}{2}\right\}$$

$$=\tan\left\{\frac{\pi}{2}-\cos^{-1}\left(\frac{2}{7}\right)\right\}=\tan\left\{\sin^{-1}\left(\frac{2}{7}\right)\right\}$$

$[\because \cos^{-1}(-x)=\pi-\cos^{-1}x, x\in[-1,1]]$

$$=\tan\left\{\tan^{-1}\left(\frac{2}{3\sqrt{5}}\right)\right\}=\frac{2}{3\sqrt{5}}$$

$[\because \tan(\tan^{-1}x)=x, x\in R]$