DCAM classes

Dynamic Classes for Academic Mastery

vectors:CBSE(Solutions)

Solutions

1. Let given position vectors are $\overrightarrow{OA} = \vec{a} - 2\vec{b}$ and $\overrightarrow{OB} = 2\vec{a} + \vec{b}$.

Let $\overrightarrow{OC}$ be the position vector of a point C whi[ch] divides the join of points, with position vecto[rs] $\overrightarrow{OA}$ and $\overrightarrow{OB}$, externally in the ratio 2 : 1.

$$\therefore \quad \overrightarrow{OC} = \frac{2\,\overrightarrow{OB} - 1\,\overrightarrow{OA}}{2-1} = \frac{2(2\vec{a} + \vec{b}) - 1(\vec{a} - 2\vec{b}}{1}$$

[by external section formula]

$$= 4\vec{a} + 2\vec{b} - \vec{a} + 2\vec{b} = 3\vec{a} + 4\vec{b}$$ **(1)**

2. Given vectors are

$$\vec{a} = 4\hat{i} - \hat{j} + \hat{k}, \vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}.$$

Now, $\vec{a} + \vec{b} = (4\hat{i} - \hat{j} + \hat{k}) + (2\hat{i} - 2\hat{j} + \hat{k})$

$= 6\hat{i} - 3\hat{j} + 2\hat{k}$

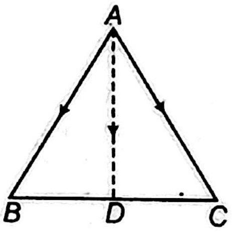
and $|\vec{a} + \vec{b}| = \sqrt{(6)^2 + (-3)^2 + (2)^2}$

$= \sqrt{36 + 9 + 4} = \sqrt{49} = 7$ unit **(1/2)**

The unit vector parallel to the vector $\vec{a}+\vec{b}$ is

$$\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}=\frac{6\hat{i}-3\hat{j}+2\hat{k}}{7}$$ **(1/2)**

3. Given, $\overrightarrow{AB}=\hat{j}+\hat{k}$ and $\overrightarrow{AC}=3\hat{i}-\hat{j}+4\hat{k}$.



Clearly, median vector, $\overrightarrow{AD}=\frac{\overrightarrow{AB}+\overrightarrow{AC}}{2}$

$$=\frac{(\hat{j}+\hat{k})+(3\hat{i}-\hat{j}+4\hat{k})}{2}=\frac{3\hat{i}+5\hat{k}}{2}$$ **(1/2)**

Now, length of median $=|\overrightarrow{AD}|$

$$=\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{5}{2}\right)^2}=\frac{1}{2}\sqrt{9+25}=\frac{\sqrt{34}}{2}$$

$$=\frac{\sqrt{17}\times\sqrt{2}}{2}=\sqrt{\frac{17}{2}}\text{ units}$$ **(1/2)**

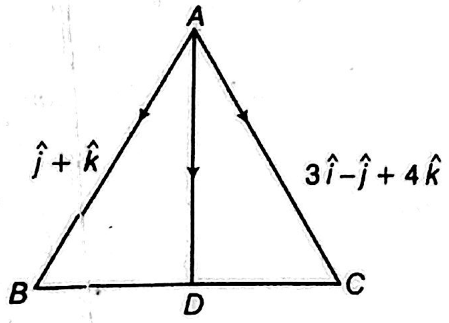
Alternate Method

Given $\overrightarrow{AB}=\hat{j}+\hat{k}$ and $\overrightarrow{AC}=3\hat{i}-\hat{j}+4\hat{k}$

Now, $\overrightarrow{BC}=\overrightarrow{BA}+\overrightarrow{AC}$

$$=-\overrightarrow{AB}+\overrightarrow{AC}$$

$$=-(\hat{j}+\hat{k})+(3\hat{i}-\hat{j}+4\hat{k})=3\hat{i}-2\hat{j}+3\hat{k}$$



Since, AD is median.

$$\therefore \overrightarrow{BD}=\frac{1}{2}\overrightarrow{BC}=\frac{1}{2}(3\hat{i}-2\hat{j}+3\hat{k})$$

$$=\frac{3}{2}\hat{i}-\hat{j}+\frac{3}{2}\hat{k}$$

Now, $\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BD}$

$$=\hat{j}+\hat{k}+\frac{3}{2}\hat{i}-\hat{j}+\frac{3}{2}\hat{k}=\frac{3}{2}\hat{i}+\frac{5}{2}\hat{k}$$

$\therefore$ Length of $\overrightarrow{AD}=\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{5}{2}\right)^2}$

$$=\frac{1}{2}\sqrt{9+25}=\frac{\sqrt{34}}{2}$$

$$=\frac{\sqrt{17}\times\sqrt{2}}{2}=\sqrt{\frac{17}{2}}\text{ units}$$ **(1)**

4. Clearly, $3\vec{a}+2\vec{b}=3(\hat{i}+\hat{j}-2\hat{k})+2(2\hat{i}-4\hat{j}+5\hat{k})$

$$=(3\hat{i}+3\hat{j}-6\hat{k})+(4\hat{i}-8\hat{j}+10\hat{k})$$

$$=7\hat{i}-5\hat{j}+4\hat{k}$$ **(1/2)**

Hence, direction ratios of vectors $3\vec{a}+2\vec{b}$ are 7, -5 and 4. **(1/2)**

5. Let $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=4\hat{i}-3\hat{j}+2\hat{k}$

Now, sum of two vectors,

$$\vec{a}+\vec{b}=(2\hat{i}+3\hat{j}-\hat{k})+(4\hat{i}-3\hat{j}+2\hat{k})=6\hat{i}+\hat{k}$$ **(1/2)**

$\therefore$ Required unit vector $=\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}$

$$=\frac{6\hat{i}+\hat{k}}{\sqrt{6^2+1^2}}=\frac{6\hat{i}+\hat{k}}{\sqrt{36+1}}=\frac{6\hat{i}+\hat{k}}{\sqrt{37}}=\frac{6}{\sqrt{37}}\hat{i}+\frac{\hat{k}}{\sqrt{37}}$$ **(1/2)**

6.

> To find a vector in the direction of given vector, first of all we find unit vector in the direction of given vector and then multiply it with given magnitude.

Let $\vec{a}=2\hat{i}-3\hat{j}+6\hat{k}$

Then, $|\vec{a}|=\sqrt{(2)^2+(-3)^2+(6)^2}$

$$=\sqrt{4+9+36}=\sqrt{49}=7\text{ units}$$ **(1/2)**

The unit vector in the direction of the given vector $\vec{a}$ is

$$\hat{a}=\frac{\vec{a}}{|\vec{a}|}=\frac{1}{7}(2\hat{i}-3\hat{j}+6\hat{k})$$

$$=\frac{2}{7}\hat{i}-\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}$$

Now, the vector of magnitude equal to 21 units and in the direction of $\vec{a}$ is given by

$$21\hat{a}=21\left(\frac{2}{7}\hat{i}-\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}\right)=6\hat{i}-9\hat{j}+18\hat{k}$$ **(1/2)**

7. Here, we have $l=\cos\frac{\pi}{4}$, $m=\cos\frac{\pi}{2}$ and $n=\cos\theta$

$$\Rightarrow l=\frac{1}{\sqrt{2}}, m=0 \text{ and } n=\cos\theta$$

We know that, $l^2 + m^2 + n^2 = 1$

$$\Rightarrow \left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2 + n^2 = 1 \Rightarrow \frac{1}{2} + n^2 = 1$$

$$\Rightarrow n^2 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow n = \pm\frac{1}{\sqrt{2}} \Rightarrow n = \frac{1}{\sqrt{2}}$$

[$\because \theta$ is an acute angle with Z-axis]

$$\therefore \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$

Thus, direction cosines of a line are $\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$. (1/2)

$\therefore$ Vector $\vec{a} = |\vec{a}|(l\hat{i} + m\hat{j} + n\hat{k})$

$$= 5\sqrt{2}\left(\frac{1}{\sqrt{2}}\hat{i} + (0)\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\right) = 5\hat{i} + 5\hat{k} \quad (1/2)$$

[$\because |\vec{a}| = 5\sqrt{2}$, given]

8. Do same as Q. No. 5. $\left[\text{Ans. } \frac{1}{13}(4\hat{i} + 3\hat{j} - 12\hat{k})\right]$

9. Given, $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are two parallel vectors, so their direction ratios will be proportional.

$$\therefore \frac{3}{1} = \frac{2}{-2p} = \frac{9}{3} \Rightarrow \frac{2}{-2p} = \frac{3}{1}$$

$$\Rightarrow -6p = 2 \Rightarrow p = \frac{2}{-6} \Rightarrow p = -\frac{1}{3} \quad (1)$$

10. Given, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

Now, unit vector in the direction of $\vec{a}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{(1)^2 + (1)^2 + (1)^2}}$$

$$= \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

$\therefore$ Cosine of the angle which given vector makes with Y-axis is $\frac{1}{\sqrt{3}}$. (1)

11. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

Now, unit vector in the direction of $\vec{a}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{1^2 + 1^2 + 1^2}} \Rightarrow \hat{a} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$$

$$\Rightarrow \hat{a} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

So, angle between X-axis and the vector $\hat{i} + \hat{j} + \hat{k}$ is $\cos\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

[$\because \hat{a} = l\hat{i} + m\hat{j} + n\hat{k}$ and $\cos\alpha = l$ $\Rightarrow \alpha = \cos^{-1} l$] (1)

12. Do same as Q. No. 6. [Ans. $3\hat{i} - 6\hat{j} + 6\hat{k}$]

13.

> First, find the vector $\vec{PQ}$ by using the formula $(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$, then required unit vector is given by $\frac{\vec{PQ}}{|\vec{PQ}|}$.

Given points are $P(1, 3, 0)$ and $Q(4, 5, 6)$.

Here, $x_1 = 1, y_1 = 3, z_1 = 0$

and $x_2 = 4, y_2 = 5, z_2 = 6$

So, vector $\vec{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$$= (4-1)\hat{i} + (5-3)\hat{j} + (6-0)\hat{k}$$

$$= 3\hat{i} + 2\hat{j} + 6\hat{k} \quad (1/2)$$

$\therefore$ Magnitude of given vector

$$|\vec{PQ}| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{9 + 4 + 36}$$

$$= \sqrt{49} = 7 \text{ units}$$

Hence, the unit vector in the direction of $\vec{PQ}$ is

$$\frac{\vec{PQ}}{|\vec{PQ}|} = \frac{3\hat{i} + 2\hat{j} + 6\hat{k}}{7} = \frac{3}{7}\hat{i} + \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}. \quad (1/2)$$

14. Here, we have

$$l = \cos\frac{\pi}{3}, m = \cos\frac{\pi}{4} \text{ and } n = \cos\theta$$

$$\Rightarrow l = \frac{1}{2}, m = \frac{1}{\sqrt{2}} \text{ and } n = \cos\theta$$

We know that, $l^2 + m^2 + n^2 = 1$

$$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + n^2 = 1$$

$$\Rightarrow n^2 = 1 - \frac{1}{4} - \frac{1}{2}$$

$$\Rightarrow n^2 = \frac{4-1-2}{4} = \frac{1}{4} \Rightarrow n = \pm\frac{1}{2} \quad (1/2)$$

$$\Rightarrow \cos\theta = \pm\frac{1}{2}$$

But θ is an acute angle, therefore $\cos\theta = \frac{1}{2}$

$$\Rightarrow \theta = \frac{\pi}{3} \quad (1/2)$$

15. Do same as Q. No. 5. $\left[\textbf{Ans. } \frac{1}{\sqrt{26}}\hat{i} + \frac{5}{\sqrt{26}}\hat{k}\right]$

16. Two vectors are equal, if coefficients of their components are equal.

Given, $\vec{a} = \vec{b} \Rightarrow x\hat{i} + 2\hat{j} - z\hat{k} = 3\hat{i} - y\hat{j} + \hat{k}$

On comparing the coefficient of components, we get

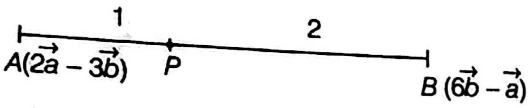
$$x = 3, y = -2, z = -1$$

Now, $x + y + z = 3 - 2 - 1 = 0$ **(1)**

17. Do same as Q. No. 1. $[\textbf{Ans. } -\vec{a} + 4\vec{b}]$

18. Do same as Q. No. 1. $[\textbf{Ans. } 5\vec{b}]$

19. Given, A and B are two points with position vectors $2\vec{a} - 3\vec{b}$ and $6\vec{b} - \vec{a}$, respectively. Also, point P divides the line segment AB in the ratio 1 : 2 internally.



∴ Position vector of a point P

$$= \frac{1 \times (6\vec{b} - \vec{a}) + 2 \times (2\vec{a} - 3\vec{b})}{1 + 2}$$

[by internal section formula]

$$= \frac{6\vec{b} - \vec{a} + 4\vec{a} - 6\vec{b}}{3} = \frac{3\vec{a}}{3} = \vec{a}$$ **(1)**

20. Given vectors are $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$,

$\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

Sum of the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is

$$\vec{a} + \vec{b} + \vec{c} = (\hat{i} - 2\hat{j} + \hat{k}) + (-2\hat{i} + 4\hat{j} + 5\hat{k}) + (\hat{i} - 6\hat{j} - 7\hat{k})$$

$$= -4\hat{j} - \hat{k}$$ **(1)**

21. Do same as Q. No. 20. $[\textbf{Ans. } 3\hat{i} - \hat{j} - 2\hat{k}]$

22. Do same as Q. No. 20. $[\textbf{Ans. } 5\hat{i} - 5\hat{j} + 3\hat{k}]$

23. Given initial point is A (2, 1) and terminal point is B (– 5, 7), then scalar component of $\overrightarrow{AB}$ are $x_2 - x_1 = -5 - 2 = -7$ and $y_2 - y_1 = 7 - 1 = 6$. **(1)**

24. If $\vec{a}$ and $\vec{b}$ are collinear, then use the condition $\vec{a} = \lambda\vec{b}$, where λ is some scalar.

Let given vectors are $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$

and $\vec{b} = a\hat{i} + 6\hat{j} - 8\hat{k}$.

We know that, vectors $\vec{a}$ and $\vec{b}$ are said to be collinear, if

$\vec{a} = k \cdot \vec{b}$, where k is a scalar.

∴ $2\hat{i} - 3\hat{j} + 4\hat{k} = k(a\hat{i} + 6\hat{j} - 8\hat{k})$

On comparing the coefficients of $\hat{i}$ and $\hat{j}$, we get

$$2 = ka \text{ and } -3 = 6k \Rightarrow k = -\frac{1}{2}$$

∴ $2 = -\frac{1}{2}a \Rightarrow a = -4$ **(1)**

Alternate Method

Do same as Q. No. 9. $[\textbf{Ans. } a = -4]$

25. Direction cosines of the vector $a\hat{i} + b\hat{j} + c\hat{k}$ are $\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}$.

Let $\vec{a} = -2\hat{i} + \hat{j} - 5\hat{k}$

∴ Direction cosines of $\vec{a}$ are

$$\frac{-2}{\sqrt{(-2)^2 + (1)^2 + (-5)^2}}, \frac{1}{\sqrt{(-2)^2 + (1)^2 + (-5)^2}} \text{ and } \frac{-5}{\sqrt{(-2)^2 + (1)^2 + (-5)^2}}$$

i.e. $\frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{-5}{\sqrt{30}}$ **(1)**

26. Mid-point of the position vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is $\frac{\vec{a} + \vec{b}}{2}$

or $\frac{(a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}}{2}$.

Given points are $P(2, 3, 4)$ and $Q(4, 1, -2)$ whose position vectors are $\overrightarrow{OP} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\overrightarrow{OQ} = 4\hat{i} + \hat{j} - 2\hat{k}$.

Now, position vector of mid-point of vector joining points $P(2, 3, 4)$ and $Q(4, 1, -2)$ is

$$\overrightarrow{OR} = \frac{\overrightarrow{OP} + \overrightarrow{OQ}}{2} = \frac{(2\hat{i} + 3\hat{j} + 4\hat{k}) + (4\hat{i} + \hat{j} - 2\hat{k})}{2}$$

$$\therefore \overrightarrow{OR} = \frac{6\hat{i} + 4\hat{j} + 2\hat{k}}{2} = 3\hat{i} + 2\hat{j} + \hat{k}$$ **(1)**

27. We know that, unit vector in the direction of $\vec{a}$ is $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$.

$\therefore$ Required unit vector in the direction of vector $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{(2)^2 + (1)^2 + (2)^2}} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{9}}$$

$$= \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k} \quad \textbf{(1)}$$

28.

> Magnitude of a vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.

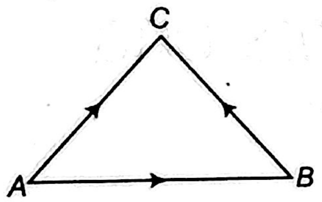
Given vector is $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$.

$\therefore$ Magnitude of $\vec{a} = |\vec{a}| = \sqrt{(3)^2 + (-2)^2 + (6)^2}$

$= \sqrt{9 + 4 + 36} = \sqrt{49} = 7$ units **(1)**

29. Do same as Q. No. 27. $\left[\textbf{Ans. } \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right]$

30. Let ΔABC be the given triangle.



Now, by triangle law of vector addition, we have

$$\vec{AB} + \vec{BC} = \vec{AC}$$

$$\Rightarrow \vec{AB} + \vec{BC} + \vec{CA} = \vec{CA} + \vec{AC}$$

[adding $\vec{CA}$ on both sides]

$$\Rightarrow \vec{AB} + \vec{BC} + \vec{CA} = \vec{CA} - \vec{CA} \quad [\because \vec{AC} = -\vec{CA}]$$

$$\therefore \vec{AB} + \vec{BC} + \vec{CA} = \vec{0} \quad \textbf{(1)}$$

31. Do same as Q. No. 27. $\left[\textbf{Ans. } \frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right]$

32. Do same as Q. No. 6. $[\textbf{Ans. } 4\hat{i} - 2\hat{j} + 4\hat{k}]$

33. Do same as Q. No. 26. $[\textbf{Ans. } 2\hat{i} + 3\hat{j} + \hat{k}]$

34. Do same as Q. No. 6. $[\textbf{Ans. } -6\hat{i} + 3\hat{j} + 6\hat{k}]$

35. Do same as Q. No. 6. $[\textbf{Ans. } 5\hat{i} - 10\hat{j} + 10\hat{k}]$

36. Do same as Q. No. 10. $\left[\textbf{Ans. } \frac{1}{2}\right]$

37.

> First, find resultant of the vectors $\vec{a}$ and $\vec{b}$, which is $\vec{a} + \vec{b}$. Then, find a unit vector in the direction of $\vec{a} + \vec{b}$. After this, the unit vector is multiplying by 5.

Given, $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Now, resultant of above vectors $= \vec{a} + \vec{b}$

$= (2\hat{i} + 3\hat{j} - \hat{k}) + (\hat{i} - 2\hat{j} + \hat{k}) = 3\hat{i} + \hat{j}$ **(1)**

Let $\vec{a} + \vec{b} = \vec{c}$

$\therefore \vec{c} = 3\hat{i} + \hat{j}$

Now, unit vector $\hat{c}$ in the direction of $\vec{c}$ is $\dfrac{\vec{c}}{|\vec{c}|}$

$$= \frac{3\hat{i} + \hat{j}}{\sqrt{(3)^2 + (1)^2}} \quad \textbf{(1)}$$

$$= \frac{3\hat{i} + \hat{j}}{\sqrt{10}} = \frac{3}{\sqrt{10}}\hat{i} + \frac{1}{\sqrt{10}}\hat{j} \quad \textbf{(1)}$$

Hence, vector of magnitude 5 units and parallel to resultant of $\vec{a}$ and $\vec{b}$ is

$$5\left(\frac{3}{\sqrt{10}}\hat{i} + \frac{1}{\sqrt{10}}\hat{j}\right) = \frac{15}{\sqrt{10}}\hat{i} + \frac{5}{\sqrt{10}}\hat{j}. \quad \textbf{(1)}$$

38.

> First, find the vector $2\vec{a} - \vec{b} + 3\vec{c}$, then find a unit vector in the direction of $2\vec{a} - \vec{b} + 3\vec{c}$. After this, the unit vector is multiplying by 6.

Given, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$

$\therefore 2\vec{a} - \vec{b} + 3\vec{c}$

$= 2(\hat{i} + \hat{j} + \hat{k}) - (4\hat{i} - 2\hat{j} + 3\hat{k}) + 3(\hat{i} - 2\hat{j} + \hat{k})$

$= 2\hat{i} + 2\hat{j} + 2\hat{k} - 4\hat{i} + 2\hat{j} - 3\hat{k} + 3\hat{i} - 6\hat{j} + 3\hat{k}$

$\Rightarrow 2\vec{a} - \vec{b} + 3\vec{c} = \hat{i} - 2\hat{j} + 2\hat{k}$ **(1½)**

Now, a unit vector in the direction of vector

$$\widehat{2a - b + 3c} = \frac{2\vec{a} - \vec{b} + 3\vec{c}}{|2\vec{a} - \vec{b} + 3\vec{c}|}$$

$$= \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{(1)^2 + (-2)^2 + (2)^2}} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{9}}$$

$$= \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3} = \frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k} \quad \textbf{(1½)}$$

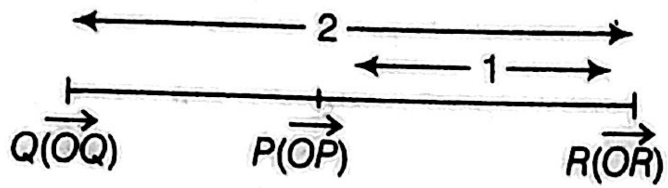
Hence, vector of magnitude 6 units parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c} = 6\left(\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$

$$= 2\hat{i} - 4\hat{j} + 4\hat{k}$$ **(1)**

39. Given, $\overrightarrow{OP}$ = Position vector of $P = 2\vec{a} + \vec{b}$

and $\overrightarrow{OQ}$ = Position vector of $Q = \vec{a} - 3\vec{b}$

Let $\overrightarrow{OR}$ be the position vector of point R, which divides PQ in the ratio 1 : 2 externally



$$\therefore \quad \overrightarrow{OR} = \frac{1(\vec{a} - 3\vec{b}) - 2(2\vec{a} + \vec{b})}{1 - 2}$$ **(1)**

[by external section formula]

$$= \frac{\vec{a} - 3\vec{b} - 4\vec{a} - 2\vec{b}}{-1}$$

$$= \frac{-3\vec{a} - 5\vec{b}}{-1} = 3\vec{a} + 5\vec{b}$$

Hence, $\overrightarrow{OR} = 3\vec{a} + 5\vec{b}$ **(1½)**

Now, we have to show that P is the mid-point of RQ,

i.e. $\overrightarrow{OP} = \frac{\overrightarrow{OR} + \overrightarrow{OQ}}{2}$

We have, $\overrightarrow{OR} = 3\vec{a} + 5\vec{b}$, $\overrightarrow{OQ} = \vec{a} - 3\vec{b}$

$$\therefore \quad \frac{\overrightarrow{OR} + \overrightarrow{OQ}}{2} = \frac{(3\vec{a} + 5\vec{b}) + (\vec{a} - 3\vec{b})}{2}$$

$$= \frac{4\vec{a} + 2\vec{b}}{2}$$

$$= \frac{2(2\vec{a} + \vec{b})}{2}$$

$$= 2\vec{a} + \vec{b}$$

$$= \overrightarrow{OP} \qquad [\because \overrightarrow{OP} = 2\vec{a} + \vec{b}]$$

Hence, P is the mid-point of line segment RQ. **(1½)**

Solutions

1. Given, two vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{a}|=|\vec{b}|$, $\vec{a}\cdot\vec{b}=\frac{9}{2}$ and angle between them is 60°.

We know that

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta,$$

where θ is angle between $\vec{a}$ and $\vec{b}$.

$$\therefore \quad \frac{9}{2}=|\vec{a}|\cdot|\vec{a}|\cos 60^\circ \qquad \textbf{(1/2)}$$

$$\Rightarrow \quad \frac{1}{2}\cdot|\vec{a}|^2=\frac{9}{2} \qquad \left[\because \cos 60^\circ=\frac{1}{2}\right]$$

$$\Rightarrow \quad |\vec{a}|^2=9$$

$$\Rightarrow \quad |\vec{a}|=3$$

[∵ magnitude cannot be negative]

Thus, $|\vec{a}|=|\vec{b}|=3$ **(1/2)**

2. $[\hat{i}\ \hat{k}\ \hat{j}]=\hat{i}\cdot(\hat{k}\times\hat{j})=-\hat{i}\cdot(\hat{j}\times\hat{k})$

$$=-[\hat{i}\ \hat{j}\ \hat{k}]=-\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}=-1 \qquad \textbf{(1)}$$

3. Given, $(\hat{i} + 3\hat{j} + 9\hat{k}) \times (3\hat{i} - \lambda\hat{j} + \mu\hat{k}) = \vec{0}$

$$\therefore \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 9 \\ 3 & -\lambda & \mu \end{vmatrix} = \vec{0}$$

$$\Rightarrow \hat{i}(3\mu + 9\lambda) - \hat{j}(\mu - 27) + \hat{k}(-\lambda - 9) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

On comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$3\mu + 9\lambda = 0, -\mu + 27 = 0$ and $-\lambda - 9 = 0$ **(1/2)**

$\Rightarrow \quad \mu = 27$ and $-\lambda = 9$

$\Rightarrow \quad \mu = 27$ and $\lambda = -9$

Also, the values of μ and λ satisfy the equation $3\mu + 9\lambda = 0$.

Hence, $\mu = 27$ and $\lambda = -9$. **(1/2)**

4. We know that, unit vectors perpendicular to $\vec{a}$ and $\vec{b}$ are $\pm\left(\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}\right)$.

So, there are two unit vectors perpendicular to the given vectors. **(1)**

5. Given, $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$ and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.

Consider, $\quad \vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \quad (\vec{a} + \vec{b} + \vec{c})^2 = (\vec{0})^2$

$\Rightarrow \quad (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = \vec{0} \cdot \vec{0}$

$\Rightarrow \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} + \vec{c} \cdot \vec{c} = 0$ **(1/2)**

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$

[$\because \vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$ and $\vec{x} \cdot \vec{x} = |\vec{x}|^2$]

$\Rightarrow 1 + 1 + 1 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$

[$\because |\vec{a}| = |\vec{b}| = |\vec{c}| = 1$]

$\Rightarrow \quad \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = \dfrac{-3}{2}$ **(1/2)**

6. Given, $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = 400$ and $|\vec{a}| = 5$

Consider, $\quad |\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = 400$

$\Rightarrow \quad |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta + |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta = 400$

[$\because |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$ and $|\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}||\cos\theta|$]

$\Rightarrow \quad |\vec{a}|^2 |\vec{b}|^2 (\sin^2\theta + \cos^2\theta) = 400$ **(1/2)**

$\Rightarrow \quad 25 \cdot |\vec{b}|^2 = 400 \quad [\because |\vec{a}| = 5]$

$\Rightarrow \quad |\vec{b}|^2 = 16 \Rightarrow |\vec{b}| = 4$

[$\because$ length cannot be negative] **(1/2)**

7. Given vectors are $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$,

$\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{j} + 3\hat{k}$.

Since, the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar,

Therefore, we have $\quad [\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$\Rightarrow \begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & -1 \\ 0 & \lambda & 3 \end{vmatrix} = 0$$

$\Rightarrow \quad 1(-3 + \lambda) - 3(6) + 1(2\lambda) = 0$

[expanding along R_1]

$\Rightarrow \quad -3 + \lambda - 18 + 2\lambda = 0$

$\Rightarrow \quad 3\lambda = 21$

$\therefore \quad \lambda = 7$ **(1)**

8. Given vectors are $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$

and $\quad \vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$

Now, the projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

$$= \frac{(7\hat{i} + \hat{j} - 4\hat{k}) \cdot (2\hat{i} + 6\hat{j} + 3\hat{k})}{\sqrt{2^2 + 6^2 + 3^2}} = \frac{14 + 6 - 12}{\sqrt{49}} = \frac{8}{7}$$ **(1)**

9. Given, $\hat{a}$, $\hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors, i.e.

$\hat{a} \cdot \hat{b} = \hat{b} \cdot \hat{c} = \hat{c} \cdot \hat{a} = 0 \quad \ldots(i)$

and $\quad |\hat{a}| = |\hat{b}| = |\hat{c}| = 1 \quad \ldots(ii)$

Now, $|2\hat{a} + \hat{b} + \hat{c}|^2 = (2\hat{a} + \hat{b} + \hat{c}) \cdot (2\hat{a} + \hat{b} + \hat{c})$

[$\because |\vec{a}|^2 = \vec{a} \cdot \vec{a}$]

$= 4(\hat{a} \cdot \hat{a}) + 2(\hat{a} \cdot \hat{b}) + 2(\hat{a} \cdot \hat{c}) + 2(\hat{b} \cdot \hat{a}) + (\hat{b} \cdot \hat{b}) + (\hat{b} \cdot \hat{c}) + 2(\hat{c} \cdot \hat{a}) + (\hat{c} \cdot \hat{b}) + (\hat{c} \cdot \hat{c})$

[$\because$ dot product is distributive over addition] **(1/2)**

$= 4(|\hat{a}|^2) + 2(0) + 2(0) + 2(0) + |\hat{b}|^2 + (0) + 2(0) + (0) + |\hat{c}|^2$

[from Eq. (i) and $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$]

$= 4(1) + 1 + 1 = 4 + 1 + 1 = 6 \quad$ [from Eq. (ii)]

$\therefore \quad |2\hat{a} + \hat{b} + \hat{c}| = \sqrt{6}$ **(1/2)**

[$\because$ length cannot be negative]

10. First, determine perpendicular vectors of $\vec{a}$ and $\vec{b}$, i.e. $\vec{a}\times\vec{b}$. Further, determine perpendicular unit vector by using formula $\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.

Given vectors are $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$

As we know that, the vectors $\vec{a}\times\vec{b}$ is perpendicular to both the vectors, so let us first evaluate $\vec{a}\times\vec{b}$.

$$\text{Then, } \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \hat{i}(0-1) - \hat{j}(0-1) + \hat{k}(1-1)$$

$$= -\hat{i} + \hat{j} \qquad \textbf{(1/2)}$$

Then, the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is given by

$$\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{-\hat{i}+\hat{j}}{\sqrt{(-1)^2+(1)^2}} = \frac{-\hat{i}+\hat{j}}{\sqrt{2}} = \frac{-\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}} \qquad \textbf{(1/2)}$$

11. Let adjacent sides of a parallelogram be $\vec{a} = 2\hat{i} - 3\hat{k}$ and $\vec{b} = 4\hat{j} + 2\hat{k}$.

$\therefore$ Area of parallelogram $= |\vec{a}\times\vec{b}|$

$$= |(2\hat{i} - 3\hat{k})\times(4\hat{j} + 2\hat{k})|$$

$$= \left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -3 \\ 0 & 4 & 2 \end{matrix} \right\|$$

$$= |\hat{i}(0+12) - \hat{j}(4+0) + \hat{k}(8-0)|$$

$$= |12\hat{i} - 4\hat{j} + 8\hat{k}|$$

$$= \sqrt{12^2 + (-4)^2 + (8)^2}$$

$$= \sqrt{144+16+64}$$

$$= \sqrt{224} = 4\sqrt{14} \text{ sq units} \qquad \textbf{(1)}$$

12. Given, $|\vec{a} + \vec{b}| = 13$, and $|\vec{a}| = 5$

Now, $(\vec{a} + \vec{b})\cdot(\vec{a} + \vec{b}) = \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$

$$\Rightarrow \quad |\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + 0 + 0 + |\vec{b}|^2$$

$$[\because \vec{x}\cdot\vec{x} = |\vec{x}|^2, \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} = 0 \text{ as } \vec{a}\perp\vec{b}]$$

$$\Rightarrow \quad (13)^2 = (5)^2 + |\vec{b}|^2$$

$$\Rightarrow \quad 169 = 25 + |\vec{b}|^2 \Rightarrow 169 - 25 = |\vec{b}|^2$$

$$\Rightarrow \quad 144 = |\vec{b}|^2 \quad \Rightarrow \quad |\vec{b}| = 12 \qquad \textbf{(1)}$$

[$\because$ length cannot be negative]

13. Given, $|\vec{a}| = 1, |\vec{b}| = 1$ and $|\vec{a} + \vec{b}| = 1$

Now, $|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b})\cdot(\vec{a} + \vec{b})$

$$= \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}$$

$$\Rightarrow \quad |\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2$$

$$[\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} \text{ and } \vec{x}\cdot\vec{x} = |\vec{x}|^2]$$

$$\Rightarrow \quad 1 = 1 + 2\vec{a}\cdot\vec{b} + 1 \qquad \text{[given]}$$

$$\Rightarrow \quad 2\vec{a}\cdot\vec{b} = -1 \qquad \textbf{(1/2)}$$

$$\Rightarrow |\vec{a}||\vec{b}|\cos\theta = -\frac{1}{2} \qquad [\because \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta]$$

$$\Rightarrow \quad \cos\theta = -\frac{1}{2} \qquad [\because |\vec{a}| = |\vec{b}| = 1]$$

$$\Rightarrow \quad \cos\theta = \cos\frac{2\pi}{3} \Rightarrow \theta = \frac{2\pi}{3}$$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{2\pi}{3}$. **(1/2)**

14. Do same as Q. No. 8. [Ans. ...]

[Hint Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{b} = 2\hat{i} - 3\hat{j} + 6\hat{k}$]

15. Do same as Q. No. 8. [Ans. ...]

16. We have, $\hat{i}\times(\hat{j}+\hat{k}) + \hat{j}\times(\hat{k}+\hat{i}) + \hat{k}\times(\hat{i}+\hat{j})$

$$= \hat{i}\times\hat{j} + \hat{i}\times\hat{k} + \hat{j}\times\hat{k} + \hat{j}\times\hat{i} + \hat{k}\times\hat{i} + \hat{k}\times\hat{j}$$

[$\because$ cross product is distributive over addition]

$$= \hat{k} - \hat{j} + \hat{i} - \hat{k} + \hat{j} - \hat{i} = \vec{0}$$

$$[\because \hat{i}\times\hat{j} = \hat{k}, \hat{i}\times\hat{k} = -\hat{j}, \hat{j}\times\hat{k} = \hat{i}, \hat{j}\times\hat{i} = -\hat{k}, \hat{k}\times\hat{i} = \hat{j}, \hat{k}\times\hat{j} = -\hat{i}] \qquad \textbf{(1)}$$

17. Given, $|\vec{a}| = 3$ and $|\vec{b}| = 2/3$

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Also, given $|\vec{a}\times\vec{b}| = 1$

$$\Rightarrow \quad |\vec{a}||\vec{b}|\sin\theta = 1 \Rightarrow 3\times\frac{2}{3}\sin\theta = 1 \qquad \textbf{(1/2)}$$

$$\Rightarrow \quad 2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \qquad \textbf{(1/2)}$$

18. Given, $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}, \vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$

and $\quad \vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$

$$\text{Now, } \vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= 2(4-1) - 1(-2-3) + 3(-1-6)$$

$$= 2\times3 - 1\times(-5) + 3\times(-7)$$

$$= 6 + 5 - 21 = 11 - 21 = -10 \qquad \textbf{(1)}$$

19. Given, $\vec{a}$ and $\vec{b}$ are two unit vectors, then $|\vec{a}|=|\vec{b}|=1$.

Also, $(\sqrt{3}\vec{a}-\vec{b})$ is a unit vector.

$\therefore \quad |\sqrt{3}\vec{a}-\vec{b}|=1 \Rightarrow |\sqrt{3}\vec{a}-\vec{b}|^2=1^2$

$\Rightarrow (\sqrt{3}\vec{a}-\vec{b})\cdot(\sqrt{3}\vec{a}-\vec{b})=1 \quad [\because |\vec{a}|^2=\vec{a}\cdot\vec{a}]$

$\Rightarrow 3(\vec{a}\cdot\vec{a})-\sqrt{3}(\vec{a}\cdot\vec{b})-\sqrt{3}(\vec{b}\cdot\vec{a})+\vec{b}\cdot\vec{b}=1$

$\Rightarrow 3|\vec{a}|^2-\sqrt{3}|\vec{a}||\vec{b}|\cos\theta$

$-\sqrt{3}|\vec{b}||\vec{a}|\cos\theta+|\vec{b}|^2=1$ **(1/2)**

where, θ is the angle between $\vec{a}$ and $\vec{b}$.

$\Rightarrow 3\times1-\sqrt{3}\times1\times1\times\cos\theta-\sqrt{3}\times1\times1\times\cos\theta+1=1$

$\Rightarrow \quad 3=2\sqrt{3}\cos\theta \Rightarrow \cos\theta=\dfrac{3}{2\sqrt{3}}$

$\Rightarrow \cos\theta=\dfrac{\sqrt{3}}{2} \Rightarrow \theta=\dfrac{\pi}{6}$

Hence, the required angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{6}$. **(1/2)**

20. Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Given, $\quad |\vec{a}|=8, |\vec{b}|=3$ and $|\vec{a}\times\vec{b}|=12$

We know that, $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta$

$\therefore \quad |\vec{a}||\vec{b}|\sin\theta=12$

$\Rightarrow \quad \sin\theta=\dfrac{12}{|\vec{a}||\vec{b}|}$

$\Rightarrow \quad \sin\theta=\dfrac{12}{8\times3}$

$\Rightarrow \quad \sin\theta=\dfrac{1}{2} \Rightarrow \theta=\dfrac{\pi}{6}$

Hence, the required angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{6}$. **(1)**

21. Do same as Q. No. 8. $\left[\textbf{Ans. } \dfrac{2}{3}\right]$

22. Given vectors are $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$

and $\quad \vec{b}=\hat{i}-2\hat{j}+3\hat{k}$

Since, vectors are perpendicular.

$\therefore \quad \vec{a}\cdot\vec{b}=0$ **(1/2)**

$\Rightarrow \quad (2\hat{i}+\lambda\hat{j}+\hat{k})\cdot(\hat{i}-2\hat{j}+3\hat{k})=0$

$\Rightarrow \quad 2-2\lambda+3=0$

$\therefore \quad \lambda=5/2$ **(1/2)**

23. We have, $\vec{a}=2\hat{i}-2\hat{j}+\hat{k}, \vec{b}=\hat{i}+2\hat{j}-2\hat{k}$ and

$\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$

Consider, $(\vec{b}+\vec{c})=(\hat{i}+2\hat{j}-2\hat{k})+(2\hat{i}-\hat{j}+4\hat{k})$

$=3\hat{i}+\hat{j}+2\hat{k}$ **(1/2)**

Now, the projection of $\vec{b}+\vec{c}$ on $\vec{a}$ is given by

$$\frac{(\vec{b}+\vec{c})\cdot\vec{a}}{|\vec{a}|}=\frac{(3\hat{i}+\hat{j}+2\hat{k})\cdot(2\hat{i}-2\hat{j}+\hat{k})}{\sqrt{2^2+(-2)^2+(1)^2}}$$

$$=\frac{6-2+2}{\sqrt{4+4+1}}=\frac{6}{3}=2$$ **(1/2)**

24. Do same as Q. No. 8. $\left[\textbf{Ans. } \dfrac{8}{7}\right]$

25. To prove, $(2\vec{a}+\vec{b})\perp\vec{b}$

Given, $\quad |\vec{a}+\vec{b}|=|\vec{a}|$

On squaring both sides, we get

$|\vec{a}+\vec{b}|^2=|\vec{a}|^2$

$\Rightarrow \quad |\vec{a}|^2+2\vec{a}\cdot\vec{b}+|\vec{b}|^2=|\vec{a}|^2$

$\Rightarrow \quad 2\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}=0 \quad [\because |\vec{x}|^2=\vec{x}\cdot\vec{x}]$

$\Rightarrow \quad (2\vec{a}+\vec{b})\cdot\vec{b}=0$

$\therefore \quad (2\vec{a}+\vec{b})\perp\vec{b}$

$[\because \text{if } \vec{a}\perp\vec{b} \Leftrightarrow \vec{a}\cdot\vec{b}=0]$ **(1)**

Hence proved.

26. Given, $\hat{a}$ is a unit vector. Then, $|\hat{a}|=1$.

Now, we have $(\vec{x}-\hat{a})\cdot(\vec{x}+\hat{a})=15$

$\Rightarrow \quad \vec{x}\cdot\vec{x}-\hat{a}\cdot\vec{x}+\vec{x}\cdot\hat{a}-\hat{a}\cdot\hat{a}=15$

$\Rightarrow \quad \vec{x}\cdot\vec{x}-\hat{a}\cdot\vec{x}+\hat{a}\cdot\vec{x}-\hat{a}\cdot\hat{a}=15$

[$\because$ scalar product is commutative, i.e. $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$]

$\Rightarrow \quad |\vec{x}|^2-|\hat{a}|^2=15 \quad [\because \vec{z}\cdot\vec{z}=|\vec{z}|^2]$

$\Rightarrow \quad |\vec{x}|^2-1=15 \quad [\text{given}, |\hat{a}|=1]$

$\Rightarrow \quad |\vec{x}|^2=16$

$\therefore \quad |\vec{x}|=4$

[$\because$ length cannot be negative] **(1)**

27. Given, $\vec{a}=\lambda\hat{i}+\hat{j}+4\hat{k}, \vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$ and projection of $\vec{a}$ on $\vec{b}=4$.

$\therefore \quad \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = 4 \quad \left[\because \text{projection of } \vec{a} \text{ on } \vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right]$

$\Rightarrow \quad \dfrac{(\lambda\hat{i}+\hat{j}+4\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k})}{\sqrt{(2)^2+(6)^2+(3)^2}} = 4$

$\Rightarrow \quad \dfrac{2\lambda+6+12}{\sqrt{49}} = 4$

$\Rightarrow \quad \dfrac{2\lambda+18}{7} = 4$

$\Rightarrow \quad 2\lambda + 18 = 28$

$\Rightarrow \quad 2\lambda = 10$

$\therefore \quad \lambda = 5$ **(1)**

28. Use the results $\hat{k}\times\hat{j} = -\hat{i}$, $\hat{j}\cdot\hat{k} = 0$ and $\hat{i}\cdot\hat{i} = 1$ and simplify it.

Given, $(\hat{k}\times\hat{j})\cdot\hat{i} + \hat{j}\cdot\hat{k} = (-\hat{i})\cdot\hat{i} + \hat{j}\cdot\hat{k}$

$= -(\hat{i}\cdot\hat{i}) + 0 = -1 \quad [\because (\hat{i}\cdot\hat{i}) = 1]$ **(1)**

29. Given, $\vec{a}\cdot\vec{a} = 0 \Rightarrow |\vec{a}|^2 = 0$

$\Rightarrow \quad |\vec{a}| = 0 \quad \ldots(\text{i})$

and $\quad \vec{a}\cdot\vec{b} = 0$

$\Rightarrow \quad |\vec{a}||\vec{b}|\cos\theta = 0 \quad \ldots(\text{ii})$

From Eqs. (i) and (ii), it may be concluded that $\vec{b}$ is either zero or non-zero perpendicular vector. **(1)**

30. Do same as Q. No. 8. **[Ans. 0]**

31. Let θ be the angle between $\vec{a}$ and $\vec{b}$, then use the following formula

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}.$$

Given, $|\vec{a}| = \sqrt{3}, |\vec{b}| = 2$ and $\vec{a}\cdot\vec{b} = \sqrt{6}$.

Now, the angle between $\vec{a}$ and $\vec{b}$ is given by

$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{\sqrt{6}}{\sqrt{3}\times 2} = \dfrac{\sqrt{3}\times\sqrt{2}}{\sqrt{3}\times 2} = \dfrac{\sqrt{2}}{2}$

$\Rightarrow \quad \cos\theta = \dfrac{1}{\sqrt{2}} = \cos\dfrac{\pi}{4} \quad \left[\because \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}\right]$

$\therefore \quad \theta = \dfrac{\pi}{4}$ **(1)**

32. Do some as Q. No. 22. **[Ans. 1/2]**

33. We know that, $\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cos\theta$

On putting $|\vec{a}| = \sqrt{3}, |\vec{b}| = 2$ and $\theta = 60°$, we get

$\vec{a}\cdot\vec{b} = \sqrt{3}\times 2\cos 60°$

$\Rightarrow \quad \vec{a}\cdot\vec{b} = \dfrac{1}{2}\times 2\sqrt{3} = \sqrt{3} \quad \left[\because \cos 60° = \dfrac{1}{2}\right]$

$\therefore \quad \vec{a}\cdot\vec{b} = \sqrt{3}$ **(1)**

34. Do same as Q. No. 22. **[Ans. 3]**

35. Given, $|\vec{a}| = 2, |\vec{b}| = 3$ and $\vec{a}\cdot\vec{b} = 3$

$\therefore \quad$ Projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|}$

$= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \quad [\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}]$

$= \dfrac{3}{2} \quad [\because \vec{a}\cdot\vec{b} = 3 \text{ and } |\vec{a}| = 2]$ **(1)**

36. Use the following formulae:

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

and $\quad |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

where, θ is the angle between $\vec{a}$ and $\vec{b}$.

Given, $\quad |\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$

$\Rightarrow \quad |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\sin\theta$

$\Rightarrow \quad \cos\theta = \sin\theta$

$\tan\theta = 1$

$\Rightarrow \quad \tan\theta = \tan\dfrac{\pi}{4} \quad \left[\because 1 = \tan\dfrac{\pi}{4}\right]$

$\therefore \quad \theta = \dfrac{\pi}{4}$

So, the angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{4}$. **(1)**

37. Do same as Q. No. 3. **[Ans. –3]**

38. Let $\vec{c} = \hat{a} + \hat{b}$. Then, according to given condition $\vec{c}$ is a unit vector, i.e. $|\vec{c}| = 1$. **(1)**

To show $\quad |\hat{a} - \hat{b}| = \sqrt{3}$ **(1/2)**

Consider, $\quad \vec{c} = \hat{a} + \hat{b}$

$\Rightarrow \quad |\vec{c}| = |\hat{a} + \hat{b}|$

$\Rightarrow \quad 1 = |\hat{a} + \hat{b}|$

$\Rightarrow \quad |\hat{a} + \hat{b}|^2 = 1$

$$\Rightarrow \quad (\hat{a}+\hat{b})\cdot(\hat{a}+\hat{b})=1$$

$$\Rightarrow \quad |\hat{a}|^2 + 2\hat{a}\cdot\hat{b} + |\hat{b}|^2 = 1$$

$$\Rightarrow \quad 1 + 2\hat{a}\cdot\hat{b} + 1 = 1 \Rightarrow 2\hat{a}\cdot\hat{b} = -1 \quad \text{...(i)} \quad \text{(1)}$$

Now consider, $|\hat{a}-\hat{b}|^2 = (\hat{a}-\hat{b})\cdot(\hat{a}-\hat{b})$

$$= |\hat{a}|^2 - 2\hat{a}\cdot\hat{b} + |\hat{b}|^2$$

$$= 1-(-1)+1 \text{ [using Eq. (i)]}$$

$$= 3 \quad \text{(1)}$$

$$\Rightarrow \quad |\hat{a}-\hat{b}| = \sqrt{3}$$

[taking positive square root, as magnitude cannot be negative]

Hence proved. (1/2)

39. Given, $\vec{a} = 2\hat{i}+3\hat{j}+\hat{k}$, $\vec{b} = \hat{i}-2\hat{j}+\hat{k}$

and $\vec{c} = -3\hat{i}+\hat{j}+2\hat{k}$

We know that, $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \\ -3 & 1 & 2 \end{vmatrix} \quad \text{(1)}$$

$$= 2(-4-1)-3(2+3)+1(1-6)$$

$$= -10-15-5 = -30 \quad \text{(1)}$$

40. Let θ be the angle between $\vec{a}$ and $\vec{b}$.

We have, $\vec{a}\times\vec{b} = 3\hat{i}+2\hat{j}+6\hat{k}$

Now, $|\vec{a}\times\vec{b}| = \sqrt{3^2+2^2+6^2} = \sqrt{49} = 7$

$\Rightarrow |\vec{a}||\vec{b}|\sin\theta = 7 \quad [\because |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta]$

$$\Rightarrow \quad \sin\theta = \frac{7}{|\vec{a}||\vec{b}|} = \frac{7}{2\times 7} = \frac{1}{2} \quad \text{(1)}$$

$$\Rightarrow \quad \sin\theta = \sin\left(\frac{\pi}{6}\right) \Rightarrow \theta = \frac{\pi}{6}$$

Hence, the required angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$ (1)

41. Let $\vec{a} = -3\hat{i}+7\hat{j}+5\hat{k}$, $\vec{b} = -5\hat{i}+7\hat{j}-3\hat{k}$

and $\vec{c} = 7\hat{i}-5\hat{j}-3\hat{k}$

Then, the volume of cuboid whose edges are given by $\vec{a}, \vec{b}$ and $\vec{c}$ is

$$|[\vec{a}, \vec{b}, \vec{c}]| = \left\| \begin{matrix} -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{matrix} \right\| \quad \text{(1)}$$

$$= |-3(-21-15)-7(15+21)+5(25-49)|$$

$$= |108-252-120|$$

$$= 264 \text{ cubic units} \quad \text{(1)}$$

42. Given, points are $A(-2\hat{i}+3\hat{j}+5\hat{k})$, $B(\hat{i}+2\hat{j}+3\hat{k})$ and $C(7\hat{i}-\hat{k})$

Here, $\vec{AB} = \vec{b}-\vec{a} = (\hat{i}+2\hat{j}+3\hat{k})-(-2\hat{i}+3\hat{j}+5\hat{k})$

$$= 3\hat{i}-\hat{j}-2\hat{k} \quad \text{(1)}$$

and $\vec{BC} = \vec{c}-\vec{b} = (7\hat{i}-\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})$

$$= 6\hat{i}-2\hat{j}-4\hat{k} = 2(3\hat{i}-\hat{j}-2\hat{k})$$

Since, $\vec{AB} = \lambda\,\vec{BC}$, where $\lambda = 2$

So, given points are collinear. (1)

43. We have, $\vec{a} = 2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b} = 3\hat{i}+5\hat{j}-2\hat{k}$

$$\therefore \quad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix} \quad \text{(1)}$$

$$= \hat{i}(-2-15)-\hat{j}(-4-9)+\hat{k}(10-3)$$

$$= -17\hat{i}+13\hat{j}+7\hat{k} \quad \text{(1)}$$

44. Let $\vec{a} = \hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b} = 3\hat{i}-2\hat{j}+\hat{k}$

Then, $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|}$ ...(i)

$$\left[\because |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta\right]$$

Here, $|\vec{a}| = \sqrt{1^2+(-2)^2+3^2} = \sqrt{1+4+9} = \sqrt{14}$

$$|\vec{b}| = \sqrt{3^2+(-2)^2+1^2} = \sqrt{9+4+1} = \sqrt{14} \quad \text{(1/2)}$$

and $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 3 & -2 & 1 \end{vmatrix}$

$$= \hat{i}(-2+6)-\hat{j}(1-9)+\hat{k}(-2+6)$$

$$= 4\hat{i}+8\hat{j}+4\hat{k} = 4(\hat{i}+2\hat{j}+\hat{k})$$

$$\Rightarrow \quad |\vec{a}\times\vec{b}| = 4\sqrt{1^2+2^2+1^2}$$

$$= 4\sqrt{1+4+1} = 4\sqrt{6} \quad \text{(1/2)}$$

Now, from Eq. (i), we get

$$\sin\theta = \frac{4\sqrt{6}}{\sqrt{14}\cdot\sqrt{14}} = \frac{4\sqrt{6}}{14} = \frac{2\sqrt{6}}{7} \quad \text{(1)}$$

45. Given, $\vec{a}+\vec{b}+\vec{c}=0$

$$\Rightarrow \quad \vec{a}+\vec{b}=-\vec{c}$$

$$\Rightarrow \quad (\vec{a}+\vec{b})^2=(-\vec{c})^2$$

[by squaring on both sides]

$$\Rightarrow \quad (\vec{a}+\vec{b})(\vec{a}+\vec{b})=(-\vec{c})\cdot(-\vec{c})$$

$$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}=\vec{c}\cdot\vec{c}$$

$$\Rightarrow \quad |\vec{a}|^2+2\vec{a}\cdot\vec{b}+|\vec{b}|^2=|\vec{c}|^2 \qquad [\because \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}]$$

$$\Rightarrow |\vec{a}|^2+2|\vec{a}||\vec{b}|\cos\theta+|\vec{b}|^2=|\vec{c}|^2 \qquad \ldots(i)$$

$[\because \vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta]$ **(1)**

Putting the values of $|\vec{a}|=5$, $|\vec{b}|=6$ and $|\vec{c}|=9$ in Eq. (i), we get

$$(5)^2+2\times5\times6\times\cos\theta+(6)^2=(9)^2$$

$$\Rightarrow \quad 25+60\cos\theta+36=81$$

$$\Rightarrow \quad 60\cos\theta=81-61=20$$

$$\Rightarrow \quad \cos\theta=\frac{20}{60}=\frac{1}{3}$$

$$\Rightarrow \quad \theta=\cos^{-1}\left(\frac{1}{3}\right) \qquad \textbf{(1)}$$

46. Given $\overrightarrow{OA}=(\hat{i}+\hat{j}+\hat{k})$, $\overrightarrow{OB}=(2\hat{i}+5\hat{j})$, $\overrightarrow{OC}=(3\hat{i}+2\hat{j}-3\hat{k})$ and $\overrightarrow{OD}=(\hat{i}-6\hat{j}-\hat{k})$

Angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is given by

$$\cos\theta=\frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}|\cdot|\overrightarrow{CD}|} \qquad \ldots(i)\ \textbf{(1)}$$

Here, $\overrightarrow{AB}=(2-1)\hat{i}+(5-1)\hat{j}+(0-1)\hat{k}$

$$=\hat{i}+4\hat{j}-\hat{k},$$

$$\overrightarrow{CD}=(1-3)\hat{i}+(-6-2)\hat{j}+(-1-(-3))\hat{k}$$

$$=-2\hat{i}-8\hat{j}+2\hat{k},$$

$$|\overrightarrow{AB}|=\sqrt{1^2+4^2+(-1)^2}=\sqrt{18}=\sqrt{9\times2}=3\sqrt{2},$$

and $|\overrightarrow{CD}|=\sqrt{(-2)^2+(-8)^2+2^2}$

$$=\sqrt{72}=\sqrt{36\times2}=6\sqrt{2}$$

Now, $\cos\theta=\dfrac{(\hat{i}+4\hat{j}-\hat{k})\cdot(-2\hat{i}-8\hat{j}+2\hat{k})}{3\sqrt{2}\times6\sqrt{2}}$

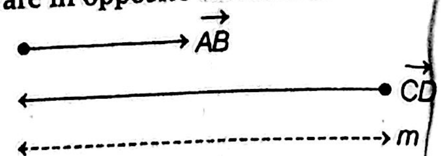
[from Eq. (i)] **(1)**

$$=\frac{1(-2)+4(-8)+(-1)(2)}{3\times6\times2}=-1$$

$\cos\theta=-1\Rightarrow\theta=180°=\pi$

So angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is π. **(1)**

Also, since angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is 180°, they are in opposite directions.



Since, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel to the same line m, they are collinear. **(1)**

47. Given, $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$

and $\quad \vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$

$\therefore \vec{b}+\vec{c}=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$ **(1)**

Let $\hat{r}$ denote the unit vector along $\vec{b}+\vec{c}$.

Then, $\hat{r}=\dfrac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}=\dfrac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+36+4}}$

$$=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+40}} \qquad \ldots(i)\ \textbf{(1)}$$

Now, according to given condition, we have

$$(\hat{i}+\hat{j}+\hat{k})\cdot\hat{r}=1 \qquad \text{[given]}$$

$$\Rightarrow \quad (\hat{i}+\hat{j}+\hat{k})\cdot\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+40}}=1$$

$$\Rightarrow \quad (\hat{i}+\hat{j}+\hat{k})\cdot\{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}\}$$

$$=\sqrt{(2+\lambda)^2+40} \qquad \textbf{(1)}$$

$$\Rightarrow 2+\lambda+6-2=\sqrt{(2+\lambda)^2+40}$$

$$\Rightarrow \quad (\lambda+6)^2=(2+\lambda)^2+40$$

$$\Rightarrow \quad 8\lambda=8\Rightarrow\lambda=1$$

Putting $\lambda=1$ in Eq. (i), we get

$$\hat{r}=\frac{1}{7}(3\hat{i}+6\hat{j}-2\hat{k}) \qquad \textbf{(1)}$$

48. We have, $\vec{a}=4\hat{i}+5\hat{j}-\hat{k}$, $\vec{b}=\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}-\hat{k}$

Since, $\vec{d}$ is perpendicular to both $\vec{c}$ and $\vec{b}$.

$$\therefore \vec{d} = \lambda(\vec{c} \times \vec{b}) = \lambda \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -1 \\ 1 & -4 & 5 \end{vmatrix} \quad (1)$$

$$= \lambda[\hat{i}(5-4) - \hat{j}(15+1) + \hat{k}(-12-1)]$$

$$= \lambda(\hat{i} - 16\hat{j} - 13\hat{k}) \quad \text{...(i) (1)}$$

Also, it is given that $\vec{d} \cdot \vec{a} = 21$

$$\therefore \lambda(\hat{i} - 16\hat{j} - 13\hat{k}) \cdot (4\hat{i} + 5\hat{j} - \hat{k}) = 21$$

$$\Rightarrow \lambda(4 - 80 + 13) = 21$$

$$\Rightarrow \lambda(-63) = 21$$

$$\Rightarrow \lambda = \frac{-1}{3} \quad (1)$$

Now from Eq. (i), we get

$$\vec{d} = -\frac{1}{3}(\hat{i} - 16\hat{j} - 13\hat{k}) \quad (1)$$

49. Given points are $A(4,4,4)$, $B(5,x,8)$, $C(5,4,1)$ and $D(7,7,2)$, then position vectors of A, B, C and D respectively, are

$$\overrightarrow{OA} = 4\hat{i} + 4\hat{j} + 4\hat{k},\ \overrightarrow{OB} = 5\hat{i} + x\hat{j} + 8\hat{k},$$

$$\overrightarrow{OC} = 5\hat{i} + 4\hat{j} + \hat{k} \text{ and } \overrightarrow{OD} = 7\hat{i} + 7\hat{j} + 2\hat{k}$$

$$\therefore \overrightarrow{AB} = (5\hat{i} + x\hat{j} + 8\hat{k}) - (4\hat{i} + 4\hat{j} + 4\hat{k})$$

$$= \hat{i} + (x-4)\hat{j} + 4\hat{k}$$

$$\overrightarrow{AC} = (5\hat{i} + 4\hat{j} + \hat{k}) - (4\hat{i} + 4\hat{j} + 4\hat{k})$$

$$= \hat{i} - 3\hat{k}$$

and $\overrightarrow{AD} = (7\hat{i} + 7\hat{j} + 2\hat{k}) - (4\hat{i} + 4\hat{j} + 4\hat{k})$

$$= 3\hat{i} + 3\hat{j} - 2\hat{k}$$

Given points are coplanar, if vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are coplanar.

$$\Rightarrow [\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = 0$$

$$\Rightarrow \begin{vmatrix} 1 & x-4 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0$$

$$\Rightarrow 1(0+9) - (x-4)(-2+9) + 4(3-0) = 0$$

$$\Rightarrow 9 - (x-4)(7) + 12 = 0$$

$$\Rightarrow 9 - 7x + 28 + 12 = 0$$

$$\Rightarrow 49 - 7x = 0$$

$$\Rightarrow 7x = 49$$

$$\Rightarrow x = 7$$

$\Rightarrow$ [Ans. 5]

50. Do same as Q. No. 49.

51.

> If three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular to each other, then $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$ and if all three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are equally inclined with the vector $(\vec{a} + \vec{b} + \vec{c})$, that means each vector $\vec{a}, \vec{b}$ and $\vec{c}$ makes equal angle with $(\vec{a} + \vec{b} + \vec{c})$ by using formula
> $$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}.$$

Given, $|\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$ (say) ...(i)

and $\vec{a} \cdot \vec{b} = 0, \vec{b} \cdot \vec{c} = 0$ and $\vec{c} \cdot \vec{a} = 0$ ...(ii) (1/2)

Now, $|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$

$$+2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$$

$$= \lambda^2 + \lambda^2 + \lambda^2 + 2(0+0+0) = 3\lambda^2$$

$$\Rightarrow |\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}\lambda$$

[length cannot be negative] (1)

Suppose $(\vec{a} + \vec{b} + \vec{c})$ is inclined at angles θ_1, θ_2 and θ_3 respectively with vectors $\vec{a}, \vec{b}$ and $\vec{c}$, then

$$(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{a} = |\vec{a} + \vec{b} + \vec{c}|\ |\vec{a}|\cos\theta_1$$

$$[\because \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta]$$

$$\Rightarrow |\vec{a}|^2 + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = \sqrt{3}\lambda \times \lambda\cos\theta_1$$

$$\Rightarrow \lambda^2 + 0 + 0 = \sqrt{3}\lambda^2\cos\theta_1$$

[from Eqs. (i) and (ii)]

$$\therefore \cos\theta_1 = \frac{1}{\sqrt{3}}$$

$$(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{b} = |\vec{a} + \vec{b} + \vec{c}||\vec{b}|\cos\theta_2 \quad (1)$$

$$\Rightarrow \vec{a} \cdot \vec{b} + |\vec{b}|^2 + \vec{c} \cdot \vec{b} = \sqrt{3}\lambda \cdot \lambda\cos\theta_2$$

$$\Rightarrow 0 + \lambda^2 + 0 = \sqrt{3}\lambda^2\cos\theta_2$$

[from Eqs. (i) and (ii)]

$$\Rightarrow \cos\theta_2 = \frac{1}{\sqrt{3}}$$

Similarly, $(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{c} = |\vec{a} + \vec{b} + \vec{c}||\vec{c}|\cos\theta_3$

$$\Rightarrow \cos\theta_1 = \frac{1}{\sqrt{3}} \quad (1)$$

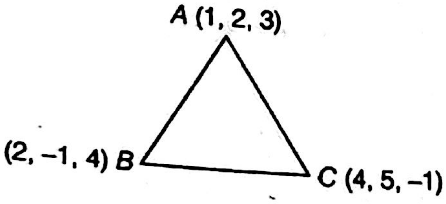
Thus, $\cos\theta_1 = \cos\theta_2 = \cos\theta_3 = \frac{1}{\sqrt{3}}$.

Hence, it is proved that $(\vec{a} + \vec{b} + \vec{c})$ is equally inclined with the vectors $\vec{a}, \vec{b}$ and $\vec{c}$. (1/2)

52. Let the position vectors of the vertices A, B and C of ΔABC be

$$\overrightarrow{OA} = \hat{i} + 2\hat{j} + 3\hat{k},\ \overrightarrow{OB} = 2\hat{i} - \hat{j} + 4\hat{k}$$

and $\overrightarrow{OC} = 4\hat{i} + 5\hat{j} - \hat{k}$, respectively.



(1)

Then, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= (2\hat{i} - \hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$
$$= \hat{i} - 3\hat{j} + \hat{k}$$

and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$$= (4\hat{i} + 5\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$
$$= (3\hat{i} + 3\hat{j} - 4\hat{k})$$ (1)

Now, $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix}$

$$= \hat{i}(12-3) - \hat{j}(-4-3) + \hat{k}(3+9)$$
$$= 9\hat{i} + 7\hat{j} + 12\hat{k}$$ (1)

and $|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(9)^2 + (7)^2 + (12)^2}$

$$= \sqrt{81 + 49 + 144} = \sqrt{274}$$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$$= \frac{1}{2}\sqrt{274} \text{ sq units}$$ (1)

53. Given, $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = \hat{i} + 0\cdot\hat{j} + 0\cdot\hat{k}$

and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

The given vectors are coplanar iff $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 \quad \text{... (i)}$$

(a) If $c_1 = 1$ and $c_2 = 2$

Then, from Eq. (i), we get

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & c_3 \end{vmatrix} = 0$$ (1)

$$\Rightarrow 1\begin{vmatrix} 0 & 0 \\ 2 & c_3 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & c_3 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 0$$

$$\Rightarrow -1(c_3 - 0) + 1(2 - 0) = 0$$
$$\Rightarrow -c_3 + 2 = 0$$
$$\Rightarrow -c_3 = -2$$
$$\Rightarrow c_3 = 2$$ (1)

(b) If $c_2 = -1$ and $c_3 = 1$, then from Eq. (i), we get

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & -1 & 1 \end{vmatrix} = 0$$ (1)

$$\Rightarrow 1(0) - 1(1 - 0) + 1(-1 - 0) = 0$$
$$\Rightarrow 0 - 1 - 1 = 0 \Rightarrow -2 \neq 0$$

$\therefore$ No value of c_1 can make $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar.

Hence proved. (1)

54. We have,

$\overrightarrow{AB}$ = (Position vector of B) – (Position vector of A)

$$= (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$$
$$= -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\overrightarrow{BC} = (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k})$$
$$= 2\hat{i} - \hat{j} + \hat{k}$$

and $\overrightarrow{CA} = (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k})$

$$= -\hat{i} + 3\hat{j} + 5\hat{k}$$

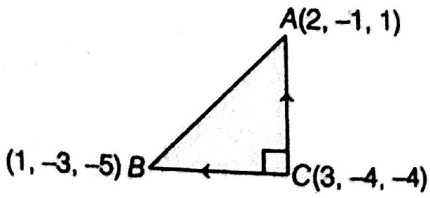
Here, $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$

$\Rightarrow$ A, B and C are the vertices of a triangle. (1)

Now, $\overrightarrow{BC} \cdot \overrightarrow{CA} = (2\hat{i} - \hat{j} + \hat{k}) \cdot (-\hat{i} + 3\hat{j} + 5\hat{k})$

$$= -2 - 3 + 5 = 0$$

$\Rightarrow \overrightarrow{BC} \perp \overrightarrow{CA} \Rightarrow \angle C = 90°$



Now, area of $\Delta ABC = \frac{1}{2}|\overrightarrow{CB} \times \overrightarrow{CA}|$

$$= \frac{1}{2}|(-8\hat{i} - 11\hat{j} + 5\hat{k})| = \frac{1}{2}\sqrt{210} \text{ sq units}$$

55. Consider $[(\vec{a} + \vec{b})\ (\vec{b} + \vec{c})\ (\vec{c} + \vec{a})]$

$$= (\vec{a} + \vec{b}) \cdot \{(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})\}$$
$$= (\vec{a} + \vec{b}) \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a})$$

$= (\vec{a} + \vec{b}) \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a})$ $\quad [\because \vec{c} \times \vec{c} = \vec{0}]$

$= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{b} \times \vec{a}) + \vec{a} \cdot (\vec{c} \times \vec{a})$

$\quad + \vec{b} \cdot (\vec{b} \times \vec{c}) + \vec{b} \cdot (\vec{b} \times \vec{a}) + \vec{b} \cdot (\vec{c} \times \vec{a})$ **(1)**

$= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{a}] + [\vec{a}\ \vec{c}\ \vec{a}] + [\vec{b}\ \vec{b}\ \vec{c}]$

$\quad + [\vec{b}\ \vec{b}\ \vec{a}] + [\vec{b}\ \vec{c}\ \vec{a}]$

$= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{b}\ \vec{c}\ \vec{a}]$

$[\because [\vec{a}\ \vec{b}\ \vec{a}] = [\vec{b}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{b}\ \vec{a}] = [\vec{a}\ \vec{c}\ \vec{a}] = 0]$

$= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{c}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$ ...(i)

$[\because [\vec{b}\ \vec{c}\ \vec{a}] = -[\vec{b}\ \vec{a}\ \vec{c}] = -(-[\vec{a}\ \vec{b}\ \vec{c}]) = [\vec{a}\ \vec{b}\ \vec{c}]]$ **(1)**

Now, if $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$

$\Rightarrow \quad 2[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$\Rightarrow [\vec{a} + \vec{b}\quad \vec{b} + \vec{c}\quad \vec{c} + \vec{a}] = 0$ [from Eq. (i)]

$\Rightarrow \vec{a} + \vec{b}, \vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar. **(1)**

56. Let the position vector of points A, B, C and D are

$\vec{OA} = 4\hat{i} + 5\hat{j} + \hat{k}$, $\vec{OB} = 0\hat{i} - \hat{j} - \hat{k}$,

$\vec{OC} = 3\hat{i} + 9\hat{j} + 4\hat{k}$ and $\vec{OD} = -4\hat{i} + 4\hat{j} + 4\hat{k}$

Now, $\quad \vec{AB} = \vec{OB} - \vec{OA}$

$= 0\hat{i} - \hat{j} - \hat{k} - (4\hat{i} + 5\hat{j} + \hat{k})$

$= -4\hat{i} - 6\hat{j} - 2\hat{k}$ **(1/2)**

$\vec{AC} = \vec{OC} - \vec{OA}$

$= 3\hat{i} + 9\hat{j} + 4\hat{k} - (4\hat{i} + 5\hat{j} + \hat{k})$

$= -\hat{i} + 4\hat{j} + 3\hat{k}$ **(1/2)**

and $\quad \vec{AD} = \vec{OD} - \vec{OA}$

$= -4\hat{i} + 4\hat{j} + 4\hat{k} - (4\hat{i} + 5\hat{j} + \hat{k})$

$= -8\hat{i} - \hat{j} + 3\hat{k}$ **(1/2)**

Use the condition that the four points are coplanar, if $[\vec{AB}\ \ \vec{AC}\ \ \vec{AD}] = 0$. **(1/2)**

Now, $\quad [\vec{AB}\ \ \vec{AC}\ \ \vec{AD}] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$ **(1)**

$= -4(12 + 3) + 6(-3 + 24) - 2(1 + 32)$

$= -60 + 126 - 66 = 0$

Hence, the four points A, B, C and D are coplanar. **(1)**

57. Let $ABCD$ be the given parallelogram with $\vec{AB} = 2\hat{i} - 4\hat{j} - 5\hat{k}$ and $\vec{AD} = 2\hat{i} + 2\hat{j} + 3\hat{k}$.

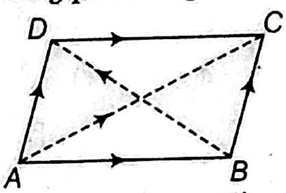
Clearly, the diagonal $\vec{AC}$ is given by $\vec{AB} + \vec{AD}$

$= 4\hat{i} - 2\hat{j} - 2\hat{k}$

and the diagonal $\vec{BD}$ is given by $\vec{BC} + \vec{BA}$

$= \vec{AD} - \vec{AB} = 6\hat{j} + 8\hat{k}$

[using parallelogram law of addition] **(1)**



Now, the unit vector along $\vec{AC}$ is given by

$$\frac{\vec{AC}}{|\vec{AC}|} = \frac{4\hat{i} - 2\hat{j} - 2\hat{k}}{\sqrt{16 + 4 + 4}}$$

$$= \frac{4\hat{i} - 2\hat{j} - 2\hat{k}}{\sqrt{24}}$$

$$= \frac{4\hat{i} - 2\hat{j} - 2\hat{k}}{2\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}(2\hat{i} - \hat{j} - \hat{k})$$

and the unit vector along $\vec{BD}$ is given by

$$\frac{\vec{BD}}{|\vec{BD}|} = \frac{6\hat{j} + 8\hat{k}}{\sqrt{36 + 64}} = \frac{6\hat{j} + 8\hat{k}}{10} = \frac{1}{5}(3\hat{j} + 4\hat{k})$$ **(1)**

Now, area of parallelogram $ABCD$

$= \frac{1}{2}|\vec{AC} \times \vec{BD}|$ **(1/2)**

Here, $\vec{AC} \times \vec{BD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & -2 \\ 0 & 6 & 8 \end{vmatrix}$

$= \hat{i}(-16 + 12) - \hat{j}(32 - 0) + \hat{k}(24 - 0)$

$= -4\hat{i} - 32\hat{j} + 24\hat{k}$ **(1/2)**

and $|\vec{AC} \times \vec{BD}| = \sqrt{(-4)^2 + (-32)^2 + (24)^2}$

$= \sqrt{4^2(1 + 8^2 + 6^2)}$

$= 4\sqrt{1 + 64 + 36} = 4\sqrt{101}$ **(1/2)**

$\therefore$ Area of parallelogram $ABCD = \frac{1}{2} \times 4\sqrt{101}$

$= 2\sqrt{101}$ sq units **(1/2)**

58. Use the result, if two vectors are parallel, then their cross-product will be a zero vector.

Given, $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ ...(i)

and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$ ...(ii) **(1)**

On subtracting Eq. (ii) from Eq. (i), we get

$$(\vec{a}\times\vec{b})-(\vec{a}\times\vec{c})=(\vec{c}\times\vec{d})-(\vec{b}\times\vec{d})$$

$$\Rightarrow (\vec{a}\times\vec{b})-(\vec{a}\times\vec{c})+(\vec{b}\times\vec{d})-(\vec{c}\times\vec{d})=\vec{0}$$

$$\Rightarrow \vec{a}\times(\vec{b}-\vec{c})+(\vec{b}-\vec{c})\times\vec{d}=\vec{0}$$

$$\Rightarrow \vec{a}\times(\vec{b}-\vec{c})-\vec{d}\times(\vec{b}-\vec{c})=\vec{0}$$

$[\because \vec{a}\times\vec{b}=-\vec{b}\times\vec{a}]$ **(1)**

$\therefore \quad (\vec{a}-\vec{d})\times(\vec{b}-\vec{c})=\vec{0}$

$[\because \vec{a}\neq\vec{d}$ and $\vec{b}\neq\vec{c}$, given] **(1/2)**

Thus, we have that cross-product of vectors $\vec{a}-\vec{d}$ and $\vec{b}-\vec{c}$ is a zero vector, so $\vec{a}-\vec{d}$ is parallel to $\vec{b}-\vec{c}$. **(1½)**

59. Given, $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

Now, $\vec{r}\times\hat{i}=(x\hat{i}+y\hat{j}+z\hat{k})\times\hat{i}$

$$=x(\hat{i}\times\hat{i})+y(\hat{j}\times\hat{i})+z(\hat{k}\times\hat{i})$$
$$=x\cdot 0+y(-\hat{k})+z(\hat{j})$$
$$=-y\hat{k}+z\hat{j}$$ **(1)**

$[\because \vec{a}\times\vec{a}=\vec{0};\ \hat{j}\times\hat{i}=-\hat{k},\ \hat{k}\times\hat{i}=\hat{j}]$

and $(\vec{r}\times\hat{j})=(x\hat{i}+y\hat{j}+z\hat{k})\times\hat{j}$

$$=x(\hat{i}\times\hat{j})+y(\hat{j}\times\hat{j})+z(\hat{k}\times\hat{j})$$
$$=x\hat{k}+y\cdot 0+z(-\hat{i})$$
$$=x\hat{k}-z\hat{i}$$

$[\because \vec{a}\times\vec{a}=0;\ \hat{i}\times\hat{j}=\hat{k}$ and $\hat{k}\times\hat{j}=-\hat{i}]$ **(1)**

$\therefore \quad (\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j})=(-y\hat{k}+z\hat{j})\cdot(x\hat{k}-z\hat{i})$

$$=-yx+yz\cdot 0+0\cdot zx-z^2\cdot 0$$

$[\because \hat{k}\cdot\hat{k}=1,\ \hat{k}\cdot\hat{i}=0,\ \hat{j}\cdot\hat{k}=0,\ \hat{j}\cdot\hat{i}=0]$

$$=-xy$$

$\therefore \quad (\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j})+xy=-xy+xy=0$ **(2)**

60. Given vectors are $\vec{a}=\hat{i}+2\hat{j}+\hat{k},\ \vec{b}=2\hat{i}+\hat{j}$

and $\vec{c}=3\hat{i}-4\hat{j}-5\hat{k}$

Now, $\vec{a}-\vec{b}=(\hat{i}+2\hat{j}+\hat{k})-(2\hat{i}+\hat{j})$
$$=-\hat{i}+\hat{j}+\hat{k}$$ **(1)**

and $\vec{c}-\vec{b}=(3\hat{i}-4\hat{j}-5\hat{k})-(2\hat{i}+\hat{j})$
$$=\hat{i}-5\hat{j}-5\hat{k}$$ **(1)**

Now, a vector perpendicular to $(\vec{a}-\vec{b})$ and $(\vec{c}-\vec{b})$ is given by

$$(\vec{a}-\vec{b})\times(\vec{c}-\vec{b})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & 1\\ 1 & -5 & -5\end{vmatrix}$$

$$=\hat{i}(-5+5)-\hat{j}(5-1)+\hat{k}(5-1)$$
$$=\hat{i}(0)-\hat{j}(4)+\hat{k}(4)=-4\hat{j}+4\hat{k}$$ **(1)**

and unit vector along $(\vec{a}-\vec{b})\times(\vec{c}-\vec{b})$ is given by

$$\frac{-4\hat{j}+4\hat{k}}{|-4\hat{j}+4\hat{k}|}=\frac{-4\hat{j}+4\hat{k}}{\sqrt{(-4)^2+4^2}}=\frac{-4\hat{j}+4\hat{k}}{\sqrt{32}}$$

$$=\frac{-4\hat{j}+4\hat{k}}{4\sqrt{2}}=-\frac{\hat{j}}{\sqrt{2}}+\frac{\hat{k}}{\sqrt{2}}$$ **(1)**

61. Use the condition that four points with position vectors $\vec{A},\vec{B},\vec{C}$ and $\vec{D}$ are coplanar, if $[\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}]=\vec{0}$.

Given, $\overrightarrow{OA}=4\hat{i}+5\hat{j}+\hat{k},\ \overrightarrow{OB}=-\hat{j}-\hat{k}$,

$\overrightarrow{OC}=3\hat{i}+\lambda\hat{j}+4\hat{k}$ and $\overrightarrow{OD}=-4\hat{i}+4\hat{j}+4\hat{k}$

Now, $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=-\hat{j}-\hat{k}-(4\hat{i}+5\hat{j}+\hat{k})$
$$=-4\hat{i}-6\hat{j}-2\hat{k}$$ **(1)**

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=3\hat{i}+\lambda\hat{j}+4\hat{k}-(4\hat{i}+5\hat{j}+\hat{k})$
$$=-\hat{i}+(\lambda-5)\hat{j}+3\hat{k}$$

and $\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}$
$$=-4\hat{i}+4\hat{j}+4\hat{k}-(4\hat{i}+5\hat{j}+\hat{k})$$
$$=-8\hat{i}-\hat{j}+3\hat{k}$$ **(1)**

Since, vectors $\overrightarrow{OA},\overrightarrow{OB},\overrightarrow{OC}$ and $\overrightarrow{OD}$ are coplanar.

$$\therefore \quad [\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]=0$$

$$\therefore \quad \begin{vmatrix}-4 & -6 & -2\\ -1 & (\lambda-5) & 3\\ -8 & -1 & 3\end{vmatrix}=0$$ **(1)**

On expanding along R_1, we get

$$-4(3\lambda-15+3)+6(-3+24)-2(1+8\lambda-40)=0$$
$$\Rightarrow -4(3\lambda-12)+6(21)-2(8\lambda-39)=0$$
$$\Rightarrow -12\lambda+48+126-16\lambda+78=0$$
$$\Rightarrow -28\lambda+252=0$$
$$\Rightarrow \lambda=9$$ **(1)**

62. To prove $[\vec{a}\ \vec{b}+\vec{c}\ \vec{d}]=[\vec{a}\ \vec{b}\ \vec{d}]+[\vec{a}\ \vec{c}\ \vec{d}]$

Let LHS $=[\vec{a}\ \vec{b}+\vec{c}\ \vec{d}]=\vec{a}\cdot\left\{(\vec{b}+\vec{c})\times\vec{d}\right\}$

[by definition of scalar triple product] **(1)**

$=\vec{a}\cdot\left(\vec{b}\times\vec{d}+\vec{c}\times\vec{d}\right)$ **(1)**

$=\vec{a}(\vec{b}\times\vec{d})+\vec{a}(\vec{c}\times\vec{d})=[\vec{a}\ \vec{b}\ \vec{d}]$

$+[\vec{a}\ \vec{c}\ \vec{d}]$ **(2)**

Hence proved.

63. Given, $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}, \vec{b}=-\hat{i}+\hat{k}$ and $\vec{c}=2\hat{j}-\hat{k}$.

Let $\vec{d_1}=\vec{a}+\vec{b}$ and $\vec{d_2}=\vec{b}+\vec{c}$.

Then, $\vec{d_1}=(2\hat{i}-3\hat{j}+\hat{k})+(-\hat{i}+\hat{k})=\hat{i}-3\hat{j}+2\hat{k}$

and $\vec{d_2}=(-\hat{i}+\hat{k})+(2\hat{j}-\hat{k})=-\hat{i}+2\hat{j}$ **(1)**

Clearly, area of given parallelogram with diagonals $\vec{d_1}$ and $\vec{d_2}$ is given by $\frac{1}{2}|\vec{d_1}\times\vec{d_2}|$. **(1)**

Here, $\vec{d_1}\times\vec{d_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 2\\ -1 & 2 & 0\end{vmatrix}$

$=\hat{i}(-4)-\hat{j}(0+2)+\hat{k}(2-3)$

$=-4\hat{i}-2\hat{j}-\hat{k}$ **(1)**

So, area of parallelogram $=\frac{1}{2}|-4\hat{i}-2\hat{j}-\hat{k}|$

$=\frac{1}{2}\sqrt{(-4)^2+(-2)^2+(-1)^2}$

$=\frac{1}{2}\sqrt{16+4+1}$

$=\frac{1}{2}\sqrt{21}$ sq units **(1)**

64. Do same as Q. No. 45. $\left[\textbf{Ans. } \frac{\pi}{3}\right]$

65. First, determine the unit vector of $\vec{b}+\vec{c}$, i.e. $\frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}$. Further put $\vec{a}\cdot\frac{(\vec{b}+\vec{c})}{|\vec{b}+\vec{c}|}=1$ and then determine the value of λ.

Given, $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$

and $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$.

Now, $\vec{b}+\vec{c}=2\hat{i}+4\hat{j}-5\hat{k}+\lambda\hat{i}+2\hat{j}+3\hat{k}$

$=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$

$\therefore\quad |\vec{b}+\vec{c}|=\sqrt{(2+\lambda)^2+(6)^2+(-2)^2}$

$=\sqrt{4+\lambda^2+4\lambda+36+4}$

$=\sqrt{\lambda^2+4\lambda+44}$ **(1)**

Now, the unit vector along $\vec{b}+\vec{c}$

$$=\frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}\quad \text{...(i)}$$

Given scalar product of $(\hat{i}+\hat{j}+\hat{k})$ with unit vector $\vec{b}+\vec{c}$ is 1. **(1)**

$\therefore\quad (\hat{i}+\hat{j}+\hat{k})\cdot\frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}=1$

$\Rightarrow\quad (\hat{i}+\hat{j}+\hat{k})\cdot\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}=1$

$\Rightarrow\quad \frac{1(2+\lambda)+1(6)+1(-2)}{\sqrt{\lambda^2+4\lambda+44}}=1$

$\Rightarrow\quad \frac{(2+\lambda)+6-2}{\sqrt{\lambda^2+4\lambda+44}}=1$

$\Rightarrow\quad \lambda+6=\sqrt{\lambda^2+4\lambda+44}$

$\Rightarrow\quad (\lambda+6)^2=\lambda^2+4\lambda+44$

[squaring both sides]

$\Rightarrow\quad \lambda^2+36+12\lambda=\lambda^2+4\lambda+44$

$\Rightarrow\quad 8\lambda=8$

$\Rightarrow\quad \lambda=1$

Hence, the value of λ is 1. **(1)**

On substituting the value of λ in Eq. (i), we get

Unit vector along $\vec{b}+\vec{c}$

$=\frac{(2+1)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(1)^2+4(1)+44}}$

$=\frac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{1+4+44}}$

$=\frac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{49}}$

$=\frac{3}{7}\hat{i}+\frac{6}{7}\hat{j}-\frac{2}{7}\hat{k}$ **(1)**

66. Given, $\vec{\alpha} = 4\hat{i} + 5\hat{j} - \hat{k}$,

$\vec{\beta} = \hat{i} - 4\hat{j} + 5\hat{k}$

and $\vec{q} = 3\hat{i} + \hat{j} - \hat{k}$

Also, vector $\vec{p}$ is perpendicular to α and β.

Then, $\vec{p} = \lambda(\vec{\alpha} \times \vec{\beta})$ ...(i)

Now, $\vec{\alpha} \times \vec{\beta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & -1 \\ 1 & -4 & 5 \end{vmatrix}$ **(1)**

$= \hat{i}(25-4) - \hat{j}(20+1) + \hat{k}(-16-5)$

$= \hat{i}(21) - \hat{j}(21) + \hat{k}(-21)$

$= 21\hat{i} - 21\hat{j} - 21\hat{k}$

So, $\vec{p} = 21\lambda\hat{i} - 21\lambda\hat{j} - 21\lambda\hat{k}$ [from Eq. (i)] ...(ii) **(1)**

Also, given that $\vec{p} \cdot \vec{q} = 21$

$\therefore$ $(21\lambda\hat{i} - 21\lambda\hat{j} - 21\lambda\hat{k}) \cdot (3\hat{i} + \hat{j} - \hat{k}) = 21$

$\Rightarrow$ $63\lambda - 21\lambda + 21\lambda = 21$

$\Rightarrow$ $63\lambda = 21$

$\Rightarrow$ $\lambda = 1/3$ **(1)**

On putting $\lambda = \frac{1}{3}$ in Eq. (ii), we get

$$\vec{p} = 21 \times \frac{1}{3}\hat{i} - 21 \times \frac{1}{3}\hat{j} - 21 \times \frac{1}{3}\hat{k}$$

$\therefore$ $\vec{p} = 7\hat{i} - 7\hat{j} - 7\hat{k}$

which is the required vector. **(1)**

67. Given, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

Let the required unit vector be

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k},$$

then $\sqrt{x^2 + y^2 + z^2} = 1$

$\Rightarrow$ $x^2 + y^2 + z^2 = 1$ ...(i)

Now, $\vec{a} + \vec{b} = (\hat{i} + \hat{j} + \hat{k}) + (\hat{i} + 2\hat{j} + 3\hat{k})$

$= 2\hat{i} + 3\hat{j} + 4\hat{k}$

and $\vec{a} - \vec{b} = (\hat{i} + \hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$= -\hat{j} - 2\hat{k}$ **(1)**

Since, $\vec{r}$ is perpendicular to $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$,

$\therefore$ $\vec{r} \cdot (\vec{a} + \vec{b}) = 0$ and $\vec{r} \cdot (\vec{a} - \vec{b}) = 0$

i.e. $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 0$

$\Rightarrow$ $2x + 3y + 4z = 0$ ...(ii)

and $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (-\hat{j} - 2\hat{k}) = 0$

$\Rightarrow$ $-y - 2z = 0$

$\Rightarrow$ $y = -2z$

On putting the value of y in Eq. (ii), we get

$2x + 3(-2z) + 4z = 0$

$\Rightarrow$ $x = z$ **(1)**

On substituting the value of x and y in Eq. (i), we get

$z^2 + 4z^2 + z^2 = 1$

$\Rightarrow$ $6z^2 = 1$

$\Rightarrow$ $z = \pm\frac{1}{\sqrt{6}}$

then, $x = \pm\frac{1}{\sqrt{6}}$

and $y = \mp\frac{2}{\sqrt{6}}$ **(1)**

Hence, the required vectors are

$\frac{1}{\sqrt{6}}\hat{i} - \frac{2}{\sqrt{6}}\hat{j} + \frac{1}{\sqrt{6}}\hat{k}$ and $\frac{-1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k}$. **(1)**

68. A unit vector perpendicular to plane ABC is

$$\frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|}.$$

Let O be the origin of reference.

Then, given $\overrightarrow{OA} = 2\hat{i} - \hat{j} + \hat{k}$,

$\overrightarrow{OB} = \hat{i} + \hat{j} + 2\hat{k}$

and $\overrightarrow{OC} = 2\hat{i} + 3\hat{k}$

Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \hat{i} + \hat{j} + 2\hat{k} - 2\hat{i} + \hat{j} - \hat{k}$

$= -\hat{i} + 2\hat{j} + \hat{k}$ **(1)**

and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= 2\hat{i} + 3\hat{k} - 2\hat{i} + \hat{j} - \hat{k}$

$= \hat{j} + 2\hat{k}$

Now, $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$

$= \hat{i}(4-1) - \hat{j}(-2-0) + \hat{k}(-1-0)$

$= 3\hat{i} + 2\hat{j} - \hat{k}$ **(1)**

and $|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(3)^2 + (2)^2 + (-1)^2}$

$= \sqrt{9 + 4 + 1} = \sqrt{14}$ **(1)**

∴ Unit vector perpendicular to the plane ABC

$$= \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|}$$

$$= \frac{3\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{14}}$$

$$= \frac{3}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} - \frac{1}{\sqrt{14}}\hat{k}$$ **(1)**

69. Let the required vector is $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Also, let $\vec{b} = \hat{i} - \hat{j} + \hat{k}$,

$\vec{c} = 2\hat{i} + \hat{j} - 3\hat{k}$

and $\vec{d} = \hat{i} + \hat{j} + \hat{k}$

Given, $\vec{a} \cdot \vec{b} = 4$,

$\vec{a} \cdot \vec{c} = 0$

and $\vec{a} \cdot \vec{d} = 2$ **(1/2)**

Now, $\vec{a} \cdot \vec{b} = 4$

$\Rightarrow \quad a_1 - a_2 + a_3 = 4$ ...(i)

$\vec{a} \cdot \vec{c} = 0$

$\Rightarrow \quad 2a_1 + a_2 - 3a_3 = 0$ ...(ii)

and $\vec{a} \cdot \vec{d} = 2$

$\Rightarrow \quad a_1 + a_2 + a_3 = 2$ ...(iii) **(1)**

On subtracting Eq. (iii) from Eq. (i), we get

$-2a_2 = 2$

$\Rightarrow \quad a_2 = -1$ **(1/2)**

On substituting $a_2 = -1$ in Eq. (ii) and (iii), we get

$2a_1 - 3a_3 = 1$ ...(iv)

$a_1 + a_3 = 3$ ...(v)

On multiplying Eq. (v) by 3 and then adding with Eq. (iv), we get

$5a_1 = 1 + 9 = 10$

$\Rightarrow \quad a_1 = 2$ **(1)**

On substituting $a_1 = 2$ in Eq. (v), we get

$a_3 = 1$

Hence, the vector is $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$. **(1)**

70. Let θ be the obtuse angle between the vectors $\vec{a} = 2\lambda^2\hat{i} + 4\lambda\hat{j} + \hat{k}$ and $\vec{b} = 7\hat{i} - 2\hat{j} + \lambda\hat{k}$ **(1/2)**

Then, $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$$\Rightarrow \quad \cos\theta = \frac{14\lambda^2 - 8\lambda + \lambda}{\sqrt{4\lambda^4 + 16\lambda^2 + 1}\sqrt{49 + 4 + \lambda^2}}$$ **(1)**

∵ θ is an obtuse angle.

∴ $\cos\theta < 0$

$$\Rightarrow \quad \frac{14\lambda^2 - 7\lambda}{\sqrt{4\lambda^4 + 16\lambda^2 + 1}\sqrt{53 + \lambda^2}} < 0$$

$\Rightarrow \quad 14\lambda^2 - 7\lambda < 0$

$\Rightarrow \quad 2\lambda^2 - \lambda < 0$

$\Rightarrow \quad \lambda(2\lambda - 1) < 0$ **(1)**

$\Rightarrow$ Either $\lambda < 0,\ 2\lambda - 1 > 0$ or $\lambda > 0,\ 2\lambda - 1 < 0$

$\Rightarrow$ Either $\lambda < 0,\ \lambda > \frac{1}{2}$ or $\lambda > 0,\ \lambda < \frac{1}{2}$ **(1/2)**

Clearly, first option is impossible.

∴ $\lambda > 0,\ \lambda < \frac{1}{2}$

$\Rightarrow \quad 0 < \lambda < \frac{1}{2}$

$\Rightarrow \quad \lambda \in \left(0, \frac{1}{2}\right)$ **(1)**

71. Given, $\vec{a} \perp (\vec{b} + \vec{c}),\ \vec{b} \perp (\vec{c} + \vec{a}),\ \vec{c} \perp (\vec{a} + \vec{b})$

and $|\vec{a}| = 3,\ |\vec{b}| = 4,\ |\vec{c}| = 5$

To Prove $|\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2}$ **(1/2)**

Consider, $|\vec{a} + \vec{b} + \vec{c}|^2 = (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c})$

$[\because |\vec{x}|^2 = \vec{x} \cdot \vec{x}]$ **(1)**

$= \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} + \vec{c} \cdot \vec{c}$ **(1/2)**

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + \vec{a} \cdot (\vec{b} + \vec{c}) + \vec{b} \cdot (\vec{a} + \vec{c}) + \vec{c} \cdot (\vec{a} + \vec{b})$ **(1/2)**

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 0 + 0 + 0$ **(1)**

$\left[\begin{array}{l} \because \vec{a} \perp (\vec{b} + \vec{c}), \text{therefore} \\ \vec{a} \cdot (\vec{b} + \vec{c}) = 0 \\ \text{Similarly, } \vec{b} \cdot (\vec{a} + \vec{c}) = 0 \\ \text{and } \vec{c} \cdot (\vec{a} + \vec{b}) = 0 \end{array}\right]$

$= 3^2 + 4^2 + 5^2 = 9 + 16 + 25$ [given]

$\Rightarrow \quad |\vec{a} + \vec{b} + \vec{c}|^2 = 50$

$\Rightarrow \quad |\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2}$

[length cannot be negative] **(1/2)**

72. Given, $\vec{a} = 3\hat{i} - \hat{j}$

and $\vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}$

Let $\vec{b_1} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b_2} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ are two vectors such that $\vec{b_1} + \vec{b_2} = \vec{b}$, $\vec{b_1} \parallel \vec{a}$ and $\vec{b_2} \perp \vec{a}$. **(1)**

Consider, $\vec{b_1} + \vec{b_2} = \vec{b}$

$\Rightarrow \quad (x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j} + (z_1 + z_2)\hat{k} = 2\hat{i} + \hat{j} - 3\hat{k}$

On comparing the coefficient of $\hat{i}$, $\hat{j}$ and $\hat{k}$ both sides, we get

$\Rightarrow \quad x_1 + x_2 = 2 \quad \ldots(i)$

$y_1 + y_2 = 1 \quad \ldots(ii)$

and $\quad z_1 + z_2 = -3 \quad \ldots(iii)$ **(1/2)**

Now, consider $\vec{b_1} \parallel \vec{a}$

$\Rightarrow \quad \frac{x_1}{3} = \frac{y_1}{-1} = \frac{z_1}{0} = \lambda$ (say)

$\Rightarrow \quad x_1 = 3\lambda,\ y_1 = -\lambda$ and $z_1 = 0 \quad \ldots(iv)$ **(1/2)**

On substituting the values of x, y and z, from Eq. (iv) to Eq. (i), (ii) and (iii), respectively, we get

$x_2 = 2 - 3\lambda,\ y_2 = 1 + \lambda$ and $z_2 = -3 \quad \ldots(v)$ **(1/2)**

Since, $\vec{b_2} \perp \vec{a}$, therefore $\vec{b_2} \cdot \vec{a} = 0$

$\Rightarrow \quad 3x_2 - y_2 = 0$ **(1/2)**

$\Rightarrow \quad 3(2 - 3\lambda) - (1 + \lambda) = 0 \quad$ [from Eq. (v)]

$\Rightarrow \quad 6 - 9\lambda - 1 - \lambda = 0$

$\Rightarrow \quad 5 - 10\lambda = 0$

$\Rightarrow \quad \lambda = \frac{1}{2}$ **(1/2)**

On substituting $\lambda = \frac{1}{2}$ in Eqs. (iv) and (v), we get

$x_1 = \frac{3}{2},\ y_1 = \frac{-1}{2},\ z_1 = 0$

and $\quad x_2 = \frac{1}{2},\ y_2 = \frac{3}{2}$ and $z_2 = -3$

Hence, $\vec{b_1} + \vec{b_2} = \left(\frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}\right) + \left(\frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}\right)$

$= 2\hat{i} + \hat{j} - 3\hat{k} = \vec{b}$ **(1/2)**

where, $\vec{b_1} \parallel \vec{a}$ and $\vec{b_2} \perp \vec{a}$.

73. Given, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$

Let $\quad \vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore \quad \vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix}$

$= \hat{i}(z - y) - \hat{j}(z - x) + \hat{k}(y - x)$ **(1)**

Now, $\vec{a} \times \vec{c} = \vec{b}$ [given]

$\Rightarrow \quad \hat{i}(z - y) + \hat{j}(x - z) + \hat{k}(y - x)$

$= 0\hat{i} + 1\hat{j} + (-1)\hat{k} \quad [\because \vec{b} = \hat{j} - \hat{k}]$

On comparing the coefficients from both sides, we get

$z - y = 0,\ x - z = 1,\ y - x = -1$

$\Rightarrow \quad y = z$ and $x - y = 1 \quad \ldots(i)$

Also given, $\quad \vec{a} \cdot \vec{c} = 3$

$\Rightarrow (\hat{i} + \hat{j} + \hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = 3$

$\Rightarrow \quad x + y + z = 3$ **(1)**

$\Rightarrow \quad x + 2y = 3 \quad [\because y = z] \ldots(ii)$

On subtracting Eq. (i) from Eq. (ii), we get

$3y = 2$

$\Rightarrow \quad y = \frac{2}{3} = z \quad [\because y = z]$

From Eq. (i),

$x = 1 + y = 1 + \frac{2}{3} = \frac{5}{3}$ **(1)**

Hence, $\quad \vec{c} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$ **(1)**

74. Use the result that if $\vec{a}$ and $\vec{b}$ are perpendicular then their dot product should be zero and simplify it.

Given, $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$

Then, $\vec{a} + \vec{b} = (\hat{i} - \hat{j} + 7\hat{k}) + (5\hat{i} - \hat{j} + \lambda\hat{k})$

$= 6\hat{i} - 2\hat{j} + (7 + \lambda)\hat{k}$ **(1)**

and $\quad \vec{a} - \vec{b} = (\hat{i} - \hat{j} + 7\hat{k}) - (5\hat{i} - \hat{j} + \lambda\hat{k})$

$= -4\hat{i} + (7 - \lambda)\hat{k}$ **(1)**

Since, $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are perpendicular vectors, then $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$

$\Rightarrow [6\hat{i} - 2\hat{j} + (7 + \lambda)\hat{k}] \cdot [-4\hat{i} + (7 - \lambda)\hat{k}] = 0$ **(1)**

$\Rightarrow \quad -24 + (7 + \lambda)(7 - \lambda) = 0$

$\Rightarrow \quad 49 - \lambda^2 = 24$

$\Rightarrow \quad \lambda^2 = 25$

$\therefore \quad \lambda = \pm 5$ **(1)**

75. Do same as Q. No. 74. [Ans. $\lambda = \pm 1$]

76. Do same as Q. No. 5. [Ans. −169]

77. Given vectors are $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$,

$$\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$$

and $$\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$$

Let $\vec{p} = x\hat{i} + y\hat{j} + z\hat{k}$

We have, $\vec{p}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

$$\vec{p} \cdot \vec{a} = 0$$

$$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + 4\hat{j} + 2\hat{k}) = 0$$

$$\Rightarrow x + 4y + 2z = 0 \quad \text{...(i)}$$

and $$\vec{p} \cdot \vec{b} = 0$$

$$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (3\hat{i} - 2\hat{j} + 7\hat{k}) = 0$$

$$\Rightarrow 3x - 2y + 7z = 0 \quad \text{...(ii)}$$

Also, given $$\vec{p} \cdot \vec{c} = 18 \quad (1)$$

$$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} - \hat{j} + 4\hat{k}) = 0$$

$$\Rightarrow 2x - y + 4z = 18 \quad \text{...(iii)}$$

On multiplying Eq. (i) by 3 and subtracting it from Eq. (ii), we get

$$-14y + z = 0 \quad \text{...(iv)}$$

Now, multiplying Eq. (i) by 2 and subtracting it from Eq. (iii), we get

$$-9y = 18$$

$$\Rightarrow y = -2 \quad (1)$$

On putting $y = -2$ in Eq. (iv), we get

$$-14(-2) + z = 0$$

$$\Rightarrow 28 + z = 0$$

$$\Rightarrow z = -28$$

On putting $y = -2$ and $z = -28$ in Eq. (i), we get

$$x + 4(-2) + 2(-28) = 0$$

$$\Rightarrow x - 8 - 56 = 0$$

$$\Rightarrow x = 64 \quad (1\tfrac{1}{2})$$

Hence, the required vector is

$$\vec{p} = x\hat{i} + y\hat{j} + z\hat{k}$$

i.e. $$\vec{p} = 64\hat{i} - 2\hat{j} - 28\hat{k} \quad (1/2)$$

78. Do same as Q. No. 60. $\left[\text{Ans. } \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}\right]$

79. Given, $|\vec{a}| = 2, |\vec{b}| = 1$ and $\vec{a} \cdot \vec{b} = 1$ ...(i)

Now, $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$

$$= 6\vec{a} \cdot \vec{a} + 21\vec{a} \cdot \vec{b} - 10\vec{b} \cdot \vec{a} - 35\vec{b} \cdot \vec{b}$$

$$= 6|\vec{a}|^2 + 21\vec{a} \cdot \vec{b} - 10\vec{a} \cdot \vec{b} - 35|\vec{b}|^2 \quad (1)$$

$$[\because \vec{x} \cdot \vec{x} = |\vec{x}|^2 \text{ and } \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]$$

$$= 6|\vec{a}|^2 + 11\vec{a} \cdot \vec{b} - 35|\vec{b}|^2$$

$$= 6(2)^2 + 11(1) - 35(1)^2 \quad \text{[from Eq. (i)]}$$

$$= 24 + 11 - 35 = 0 \quad (1)$$

Hence, $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b}) = 0$ (2)

80. Given, $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$,

$$\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$$

and $$\vec{c} = 3\hat{i} + \hat{j}$$

Also, $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$.

$$\therefore (\vec{a} + \lambda\vec{b}) \cdot \vec{c} = 0 \quad \text{...(i) (1)}$$

$$[\because \text{when } \vec{a} \perp \vec{b}, \text{ then } \vec{a} \cdot \vec{b} = 0]$$

Now, $\vec{a} + \lambda\vec{b} = (2\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + 2\hat{j} + \hat{k})$

$$\Rightarrow \vec{a} + \lambda\vec{b} = \hat{i}(2 - \lambda) + \hat{j}(2 + 2\lambda) + \hat{k}(3 + \lambda) \quad (1)$$

Then, from Eq. (i), we get

$$[\hat{i}(2 - \lambda) + \hat{j}(2 + 2\lambda) + \hat{k}(3 + \lambda)] \cdot [3\hat{i} + \hat{j}] = 0 \quad (1)$$

$$\Rightarrow 3(2 - \lambda) + 1(2 + 2\lambda) = 0$$

$$\Rightarrow 8 - \lambda = 0$$

$$\therefore \lambda = 8 \quad (1)$$

81. Do same as Q. No. 52. [Ans. $\frac{1}{2}\sqrt{61}$ sq units]

82. Do same as Q. No. 52. $\left[\text{Ans. } \frac{1}{2}\sqrt{61} \text{ sq units}\right]$

Solutions (objective)

1. (*d*) Now, $|(3\hat{i} + 2\hat{j} - 6\hat{k})| = \sqrt{3^2 + 2^2 + (-6)^2}$

$= \sqrt{9 + 4 + 36} = \sqrt{49} = 7$

Since, $\lambda\,(3\hat{i} + 2\hat{j} - 6\hat{k})$ is a unit vector.

$$\therefore \quad \lambda = \pm \frac{1}{|3\hat{i} + 2\hat{j} - 6\hat{k}|} = \pm \frac{1}{7}$$

2. (*c*) Let the vertices be $\overrightarrow{OA} = \hat{i} + \hat{j} + \hat{k}$,

$\overrightarrow{OB} = 2\hat{i} + 3\hat{j}$, $\overrightarrow{OC} = 3\hat{i} + 5\hat{j} - 2\hat{k}$ and $\overrightarrow{OD} = \hat{k} - \hat{j}$

$\therefore \overrightarrow{AB} = \hat{i} + 2\hat{j} - \hat{k}$, $\overrightarrow{BC} = \hat{i} + 2\hat{j} - 2\hat{k}$

$\overrightarrow{CD} = -3\hat{i} - 6\hat{j} + 3\hat{k} = -3(\hat{i} + 2\hat{j} - \hat{k})$

$\overrightarrow{DA} = \hat{i} + 2\hat{j} + \hat{k}$

It is clear that, $|\overrightarrow{AB}| \neq |\overrightarrow{BC}| \neq |\overrightarrow{CA}| \neq |\overrightarrow{DA}|$

Also, $\overrightarrow{AB} \parallel \overrightarrow{CD}$

Hence, figure formed by four points is a trapezium.

3. (*b*) Given, $|\vec{a}| = |\vec{b}| = 1$ and $\theta = \pi/3$

Now, $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta$

$= 1^2 + 1^2 + 2 \times 1 \times 1 \times \cos\frac{\pi}{3}$

$= 1 + 1 + 2 \times \frac{1}{2} = 1 + 1 + 1 = 3$

$\Rightarrow \quad |\vec{a} + \vec{b}| = \sqrt{3}$

$\therefore \quad |\vec{a} + \vec{b}| > 1$

4. (*d*) Given, $\vec{a} \cdot \vec{b} = 0$

Now, $(\vec{a} + \vec{b}) \cdot \vec{a} = |\vec{a} + \vec{b}||\vec{a}|\cos 60°$

$\Rightarrow \quad \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} = |\vec{a} + \vec{b}||\vec{a}|\frac{1}{2}$

$\Rightarrow \quad |\vec{a}|^2 + 0 = \frac{|\vec{a} + \vec{b}||\vec{a}|}{2}$

$\Rightarrow \quad 2|\vec{a}| = |\vec{a} + \vec{b}|$

On squaring both sides, we get

$4|\vec{a}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\,\vec{b}$

$\Rightarrow \quad 3|\vec{a}|^2 = |\vec{b}|^2 + 0 \quad [\because |\vec{a}||\vec{b}|\cos\theta = \vec{a} \cdot \vec{b} = 0]$

$\therefore \quad \sqrt{3}|\vec{a}| = |\vec{b}|$

5. (*a*) Given, $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

$\vec{a} \cdot \vec{b} = |\vec{b}|^2$

and $|\vec{a} - \vec{b}| = \sqrt{7} \Rightarrow |\vec{a} - \vec{b}|^2 = 7$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b} = 7$

$\Rightarrow \quad (\sqrt{1 + 4 + 9})^2 + |\vec{b}|^2 - 2|\vec{b}|^2 = 7$

$\Rightarrow \quad 14 - |\vec{b}|^2 = 7$

$\Rightarrow \quad |\vec{b}|^2 = 7$

$\therefore \quad |\vec{b}| = \sqrt{7}$

6. (*b*) Given, $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$

$\Rightarrow \quad |\vec{a}||\vec{b}|\cos\theta = -|\vec{a}||\vec{b}|$

$\Rightarrow \quad \cos\theta = -1 \Rightarrow \theta = 180°$

7. (*a*) Given, $\vec{a} = \lambda\hat{i} - 7\hat{j} + 3\hat{k}$, $\vec{b} = \lambda\hat{i} + \hat{j} + 2\lambda\hat{k}$

$$\therefore \quad \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{\lambda^2 - 7 + 6\lambda}{\sqrt{\lambda^2 + 49 + 9}\,\sqrt{\lambda^2 + 1 + 4\lambda^2}} < 0$$

$\Rightarrow \quad (\lambda + 7)(\lambda - 1) < 0$

$\Rightarrow \quad -7 < \lambda < 1$

8. (*d*) Given, $|\vec{x}| = |\vec{y}| = 1$ and $\vec{x} \cdot \vec{y} = 0$

$|\vec{x} + \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 + 2(\vec{x} \cdot \vec{y})$

$\Rightarrow \quad |\vec{x} + \vec{y}|^2 = 1 + 1 + 0$

$\Rightarrow \quad |\vec{x} + \vec{y}| = \sqrt{2}$

9. (*c*) The projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

$$= \frac{(3\hat{i} - \hat{j} + 5\hat{k}) \cdot (2\hat{i} + 3\hat{j} + \hat{k})}{\sqrt{2^2 + 3^2 + 1^2}} = \frac{8}{\sqrt{14}}$$

10. (*b*) $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 = 16 - 4 = 12$

and $\quad |\vec{c}|^2 = (2\vec{a} \times \vec{b} - 3\vec{b})^2$

$= 4|\vec{a} \times \vec{b}|^2 + 9|\vec{b}|^2 = 4 \cdot 12 + 9 \cdot 16$

$[\because \vec{b}\ \vec{b}\ \vec{c} = 0]$

$= 192 \Rightarrow |\vec{c}| = 8\sqrt{3}$

Now, $\vec{b} \cdot \vec{c} = \vec{b} \cdot (2\vec{a} \times \vec{b} - 3\vec{b}) = -3|\vec{b}|^2 = -48$

$$\therefore \quad \cos\theta = \frac{\vec{b} \cdot \vec{c}}{|\vec{b}||\vec{c}|} = -\frac{48}{4 \cdot 8\sqrt{3}} = -\frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{5\pi}{6}$$

11. (*b*) Given, $\vec{a} + \vec{b} + \vec{c} = 0$

Taking cross product of both sides, we get

$$(\vec{a} + \vec{b} + \vec{c}) \times \vec{a} = 0 \times \vec{a}$$

$$\Rightarrow \quad \vec{a} \times \vec{a} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a} = 0$$

$$\Rightarrow \quad 0 + \vec{b} \times \vec{a} + \vec{c} \times \vec{a} = 0$$

$$\Rightarrow \quad -\vec{a} \times \vec{b} + \vec{c} \times \vec{a} = 0$$

$$\vec{a} \times \vec{b} = \vec{c} \times \vec{a}$$

Similarly, $\quad \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

$$\therefore \quad \vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a} \neq 0$$

12. (*c*) $\because |\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = 144$

$$\Rightarrow |\vec{a}|^2 \cdot |\vec{b}|^2 (\sin^2\theta + \cos^2\theta) = 144$$

$$\Rightarrow \quad 16\,|\vec{b}|^2 = 144$$

$$\therefore \quad |\vec{b}| = 3$$

13. (*c*) Given, area $= |\vec{a} \times \vec{b}| = 15$

If the sides are $(3\vec{a} + 2\vec{b})$ and $(\vec{a} + 3\vec{b})$, then

Area of parallelogram

$$= |(3\vec{a} + 2\vec{b}) \times (\vec{a} + 3\vec{b})|$$

$$= 7|\vec{a} \times \vec{b}|$$

$$= 7 \times 15 = 105 \text{ sq units}$$

14. (*c*) Given, $\vec{r} \cdot \vec{a} = 0 \Rightarrow \vec{r} \perp \vec{a}$

$$\vec{r} \cdot \vec{b} = 0 \Rightarrow \vec{r} \perp \vec{b}$$

and $\quad \vec{r} \cdot \vec{c} = 0 \Rightarrow \vec{r} \perp \vec{c}$

So, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

$$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] = 0$$

15. (*b*) Since, volume of parallelopiped $= 34$

$$\therefore \quad \begin{vmatrix} 4 & 5 & 1 \\ 0 & -1 & 1 \\ 3 & 9 & p \end{vmatrix} = 34$$

$$\Rightarrow \quad 4(-p - 9) - 5(-3) + 1(3) = 34$$

$$\Rightarrow \quad -4p - 36 + 15 + 3 = 34$$

$$\Rightarrow \quad 4p = -52$$

$$\therefore \quad p = -13$$