integration cbse(solution)

Solutions

1. Let $I = \int \frac{\sin^2 x - \cos^2 x}{\sin x \cdot \cos x} dx$

$= \int \left[\frac{\sin^2 x}{\sin x \cdot \cos x} - \frac{\cos^2 x}{\sin x \cdot \cos x} \right] dx$

$= \int \left[\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x} \right] dx$ **(1/2)**

$= \int (\tan x - \cot x)\, dx = \int \tan x\, dx - \int \cot x\, dx$

$= \log|\sec x| - [-\log|\operatorname{cosec} x|] + C$

$= \log|\sec x| + \log|\operatorname{cosec} x| + C$

$= \log|\sec x \cdot \operatorname{cosec} x| + C$ **(1/2)**

2. Let $I = \int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$

$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx - \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$

$= \int \sec^2 x\, dx - \int \operatorname{cosec}^2 x\, dx$

$= \tan x + \cot x + C$ **(1)**

3. Let $I = \int \frac{\sin^6 x}{\cos^8 x} dx = \int \tan^6 x \sec^2 x\, dx$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\therefore\ I = \int t^6\, dt = \frac{t^7}{7} + C = \frac{\tan^7 x}{7} + C$ [put $t = \tan x$] **(1)**

4. Let $I = \int \frac{dx}{\sin^2 x \cos^2 x}$

$= \int \frac{(\sin^2 x + \cos^2 x)}{\sin^2 x \cdot \cos^2 x} dx$ $[\because \sin^2\theta + \cos^2\theta = 1]$ **(1/2)**

$= \int (\sec^2 x + \text{cosec}^2 x) dx$

$= \int \sec^2 x dx + \int \text{cosec}^2 x \, dx$

$= \tan x - \cot x + C$ **(1/2)**

Alternate Method

On dividing the numerator and denominator by $\cos^4 x$, we get

$I = \int \frac{\sec^2 x \cdot \sec^2 x}{\tan^2 x} dx \Rightarrow I = \int \frac{(1 + \tan^2 x) \cdot \sec^2 x}{\tan^2 x} dx$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

$\therefore \quad I = \int \frac{1 + t^2}{t^2} dt = \int 1 dt + \int \frac{1}{t^2} dt$

$\Rightarrow \quad I = t - \frac{1}{t} + C$

$\Rightarrow \quad I = \tan x - \cot x + C$ [put $t = \tan x$] **(1)**

5. Let $I = \int \cos^{-1}(\sin x) \, dx$

$= \int \cos^{-1}\left[\cos\left(\frac{\pi}{2} - x\right)\right] dx$

$= \int \left(\frac{\pi}{2} - x\right) dx$ $[\because \cos^{-1}(\cos\theta) = \theta]$ **(1/2)**

$= \frac{\pi}{2} \int dx - \int x \, dx$

$= \frac{\pi}{2} x - \frac{x^2}{2} + C$ **(1/2)**

6. Anti-derivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

$= \int \left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx = 3\int \sqrt{x} \, dx + \int \frac{1}{\sqrt{x}} dx$

$= 3\left(\frac{x^{1/2+1}}{1/2+1}\right) + \left[\frac{x^{-1/2+1}}{-1/2+1}\right] + C = 2(x^{3/2} + x^{1/2}) + C$ **(1)**

7. First, multiply the two functions and then use $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1.$

Let $I = \int (1 - x)\sqrt{x} \, dx = \int (\sqrt{x} - x\sqrt{x}) dx$

$= \int (x^{1/2} - x^{3/2}) dx = \frac{2}{3} x^{3/2} - \frac{2}{5} x^{5/2} + C$ **(1)**

$\left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1\right]$

8. Use the relation $\int e^x [f(x) + f'(x)] dx = e^x f(x) + C$ and simplify it.

Given that $\int e^x (\tan x + 1)\sec x \, dx = e^x \cdot f(x) + C$

$\Rightarrow \quad \int e^x (\sec x + \sec x \tan x) dx = e^x f(x) + C$

$\Rightarrow \quad e^x \cdot \sec x + C = e^x f(x) + C$

$\left[\because \int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C \text{ and here } \frac{d}{dx}(\sec x) = \sec x \tan x\right]$

On comparing both sides, we get

$f(x) = \sec x$ **(1)**

9. Let $I = \int \frac{2}{1 + \cos 2x} dx$

$= \int \frac{2}{2\cos^2 x} dx$ $[\because \cos 2\theta = 2\cos^2\theta - 1]$

$= \int \sec^2 x \, dx = \tan x + C$ **(1)**

10. Let $I = \int \frac{x + \cos 6x}{3x^2 + \sin 6x} dx$

Put $3x^2 + \sin 6x = t$

$\Rightarrow \quad 6x + 6\cos 6x = \frac{dt}{dx}$

$\Rightarrow \quad (x + \cos 6x) dx = \frac{dt}{6}$

$\therefore \quad I = \int \frac{dt}{6t} = \frac{1}{6} \log|t| + C$ $\left[\because \int \frac{1}{x} dx = \log|x| + C\right]$

$= \frac{1}{6}[\log|(3x^2 + \sin 6x)|] + C$ **(1)**

[put $t = 3x^2 + \sin 6x$]

11. Let $I = \int \frac{\sec^2 x}{\text{cosec}^2 x} dx = \int \frac{\left(\frac{1}{\cos^2 x}\right)}{\left(\frac{1}{\sin^2 x}\right)} dx$

$= \int \frac{\sin^2 x}{\cos^2 x} dx$

$= \int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx$

$[\because \tan^2 x = \sec^2 x - 1]$

$= \int \sec^2 x \, dx - \int 1 dx = \tan x - x + C$ **(1)**

12. Let $I = \int \frac{dx}{x^2 + 16} = \int \frac{dx}{x^2 + (4)^2}$

$= \frac{1}{4} \tan^{-1} \frac{x}{4} + C$ **(1)**

$\left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C\right]$

13. Let $I = \int \frac{2-3\sin x}{\cos^2 x} dx = \int \left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x} \right) dx$

$= \int (2\sec^2 x - 3\sec x \tan x) dx$

$= 2\int \sec^2 x \, dx - 3\int \sec x \tan x \, dx$

$= 2\tan x - 3\sec x + C$ **(1)**

14. Let $I = \int \sec x(\sec x + \tan x) dx$

$= \int (\sec^2 x + \sec x \tan x) dx$

$= \int \sec^2 x \, dx + \int \sec x \tan x \, dx$

$= \tan x + \sec x + C$ **(1)**

15. Let $I = \int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dx}{\sqrt{(1)^2 - x^2}}$

$= \sin^{-1} x + C \left[\because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C \right]$ **(1)**

16. Let $I = \int \frac{(\log x)^2}{x} dx$

Put $\log x = t \Rightarrow \frac{1}{x} dx = dt$

$\therefore \quad I = \int \frac{(\log x)^2}{x} dx = \int t^2 \, dt$

$= \frac{t^3}{3} + C = \frac{(\log x)^3}{3} + C$ [put $t = \log x$] **(1)**

17. Let $I = \int \frac{e^{\tan^{-1} x}}{1+x^2} dx$

Put $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

$\therefore \; I = \int \frac{e^{\tan^{-1} x}}{1+x^2} dx = \int e^t \, dt$

$= e^t + C$ $[\because \int e^x \, dx = e^x + C]$

$= e^{\tan^{-1} x} + C$ [put $t = \tan^{-1} x$] **(1)**

18. Let $I = \int (ax+b)^3 dx$

Put $t = ax+b \Rightarrow \frac{dt}{dx} = a \Rightarrow \frac{dt}{a} = dx$

$\therefore \; I = \int \frac{t^3}{a} dt = \frac{1}{a} \cdot \frac{t^4}{4} + C = \frac{(ax+b)^4}{4a} + C$

[put $t = ax + b$] **(1)**

19. Let $I = \int \frac{(1 + \log x)^2}{x} dx$

Put $1 + \log x = t \Rightarrow \frac{1}{x} dx = dt$

$\therefore \quad I = \int \frac{(1+\log x)^2}{x} dx = \int t^2 \, dt = \frac{t^3}{3} + C$

$= \frac{(1+\log x)^3}{3} + C$ [put $t = 1 + \log x$] **(1)**

20. Let $I = \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx$

Put $e^{2x} + e^{-2x} = t$

$\Rightarrow \quad (2e^{2x} - 2e^{-2x}) \, dx = dt \quad \left[\because \frac{d}{dx}(e^{ax}) = ae^{ax} \right]$

$\Rightarrow \quad (e^{2x} - e^{-2x}) \, dx = \frac{dt}{2}$

$\therefore \; I = \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \log |t| + C$

$= \frac{1}{2} \log |e^{2x} + e^{-2x}| + C$ [put $t = e^{2x} + e^{-2x}$] **(1)**

21. Let $I = \int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

Put $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} dx = dt \Rightarrow \frac{1}{\sqrt{x}} dx = 2dt$

$\therefore \; I = \int \frac{\cos \sqrt{x}}{\sqrt{x}} dx = 2\int \cos t \, dt$

$= 2\sin t + C = 2\sin \sqrt{x} + C$ [put $t = \sqrt{x}$] **(1)**

22. Let $I = \int \frac{2\cos x}{3\sin^2 x} dx$

Put $\sin x = t \Rightarrow \cos x \, dx = dt$

$\therefore \quad I = \int \frac{2}{3t^2} dt = \frac{2}{3} \int \frac{dt}{t^2} = \frac{2}{3} \frac{t^{-1}}{(-1)} + C$

$= \frac{-2}{3} (\sin x)^{-1} + C = \frac{-2}{3\sin x} + C$ **(1)**

[put $t = \sin x$]

Alternate Method

Let $I = \int \frac{2\cos x}{3\sin^2 x} dx = \frac{2}{3} \int \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} dx$

$= \frac{2}{3} \int \operatorname{cosec} x \cot x \, dx$

$= \frac{2}{3}(-\operatorname{cosec} x) + C = -\frac{2}{3\sin x} + C$ **(1)**

23. First, factorise numerator and cancel out common factor from numerator and denominator and then integrate.

Let $I = \int \frac{x^3 - x^2 + x - 1}{x - 1} dx$

$= \int \frac{x^2(x-1) + 1(x-1)}{x-1} dx$

$= \int \frac{(x^2+1)(x-1)}{x-1} dx = \int (x^2+1) \, dx$

$= \frac{x^3}{3} + x + C$ **(1)**

24. Let $I=\int\frac{1-\sin x}{\cos^2 x}dx=\int\left(\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}\right)dx$

$=\int\sec^2 x\,dx-\int\sec x\tan x\,dx$

$=\tan x-\sec x+C$ **(1)**

25. Do same as Q. No. 22. **[Ans.** $-2\text{cosec}\,x+C$]

26. Let $I=\int\frac{x^3-1}{x^2}dx=\int\left(\frac{x^3}{x^2}-\frac{1}{x^2}\right)dx$

$=\int x\,dx-\int\frac{1}{x^2}dx=\frac{x^2}{2}-\frac{x^{-1}}{-1}+C$

$=\frac{x^2}{2}+\frac{1}{x}+C$ **(1)**

27. Let $I=\int\sec^2(7-4x)\,dx$

Put $7-4x=t$

$\Rightarrow -4dx=dt \Rightarrow dx=\frac{-1}{4}dt$

$\therefore\quad I=\frac{-1}{4}\int\sec^2 t\,dt=\frac{-1}{4}\tan t+C$

$=-\frac{\tan(7-4x)}{4}+C$ **(1)**

28. Do same as Q. No. 16. $\left[\textbf{Ans.}\ \frac{(\log x)^2}{2}+C\right]$

29. Let $I=\int 2^x dx=\frac{2^x}{\log 2}+C$ $\left[\because\int a^x dx=\frac{a^x}{\log a}+C\right]$ **(1)**

30. $\int\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx$

Let $\tan x=t\Rightarrow \sec^2 x=\frac{dt}{dx}\Rightarrow dx=\frac{dt}{\sec^2 x}$ **(1)**

$\therefore\int\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx=\int\frac{\sec^2 x}{\sqrt{t^2+4}}\frac{dt}{\sec^2 x}$

$=\int\frac{dt}{\sqrt{t^2+2^2}}=\log|t+\sqrt{t^2+4}|+C$

$\left\{\because\int\frac{dx}{\sqrt{x^2+a^2}}=\log|x+\sqrt{x^2+a^2}|+C\right\}$

$=\log|\tan x+\sqrt{\tan^2 x+4}|+C$

[$\because t=\tan x$] **(1)**

31. $\int\sqrt{1-\sin 2x}\,dx$

$=\int\sqrt{\sin^2 x+\cos^2 x-2\sin x\cos x}\,dx$

[$\because\sin^2 x+\cos^2 x=1,\ 2\sin x\cos x=\sin 2x$] **(1)**

$=\int\sqrt{(\sin x-\cos x)^2}\,dx=\int(\sin x-\cos x)\,dx$

$\left[\because\text{In the interval } \frac{\pi}{4}<x<\frac{\pi}{2}, \sin x>\cos x\right]$

$=-\cos x-\sin x+C=-(\cos x+\sin x)+C$ **(1)**

32. Let $I=\int\sin^{-1}(2x)\,dx$

Let $2x=y\Rightarrow x=\frac{y}{2}\Rightarrow dx=\frac{dy}{2}$

$\therefore\ I=\frac{1}{2}\int\sin^{-1}(y)\,dy$

$=\frac{1}{2}\left[\sin^{-1}(y)\cdot y-\int\frac{1}{\sqrt{1-y^2}}\cdot y\,dy\right]$

[integrating by parts] **(1)**

$=\frac{1}{2}\left[y\sin^{-1}y+\frac{1}{2}\int-\frac{2y}{\sqrt{1-y^2}}dy\right]$

$=\frac{1}{2}[y\sin^{-1}y+\sqrt{1-y^2}+C]$

$\left[\because\int\frac{dy}{\sqrt{x}}=2\sqrt{x}+C\right]$

$=\frac{1}{2}[2x\sin^{-1}2x+\sqrt{1-4x^2}+C]$ [$\because y=2x$] **(1)**

33. $\int\frac{\tan^2 x\cdot\sec^2 x}{1-\tan^6 x}dx$

Let $\tan^3 x=t\Rightarrow 3\tan^2 x\sec^2 x\,dx=dt$

$\tan^2 x\sec^2 dx=\frac{1}{3}\cdot dt$ **(1)**

$\therefore\frac{1}{3}\int\frac{dt}{1-t^2}=\frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|+C\right]$

$\left[\because\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C\right]$

$=\frac{1}{6}\log\left|\frac{1+t}{1-t}\right|+C$

Now, put the value of t, we get

$\int\frac{\tan^2 x\sec^2 x}{1-\tan^6 x}dx=\frac{1}{6}\log\left|\frac{1+\tan^3 x}{1-\tan^3 x}\right|+C$ **(1)**

34. $\int\sin x\cdot\log\cos x\,dx$

Put $\cos x=t\Rightarrow-\sin x\,dx=dt$

$\therefore-\int\log t\,dt\Rightarrow-\int(\log t)\cdot 1\,dt$

$\Rightarrow-\left[\log t\int 1dt-\int\left\{\frac{d}{dt}(\log t)\int 1dt\right\}dt\right]$ **(1)**

$$\Rightarrow -\left[(\log t)\cdot t - \int \frac{1}{t}\cdot t\,dt\right] \Rightarrow -[t\cdot \log t - \int 1\,dt]$$

$$\Rightarrow -[t\cdot \log t - t] + C \Rightarrow -t\cdot \log t + t + C$$

$$\Rightarrow -\cos x \log \cos x + \cos x + C$$ (1)

35. Let $I = \int \sqrt{3-2x-x^2}\,dx$

$$= \int \sqrt{-(x^2+2x-3)}\,dx$$

$$= \int \sqrt{-(x^2+2x+1-4)}\,dx$$

$$= \int \sqrt{-((x+1)^2-2^2)}\,dx$$

$$= \int \sqrt{2^2-(x+1)^2}\,dx$$ (1)

Now, put $x+1=t \Rightarrow dx = dt$

$$\therefore I = \int \sqrt{2^2-t^2}\,dt = \frac{1}{2}\left[t\sqrt{2^2-t^2} + 2^2\sin^{-1}\left(\frac{t}{2}\right)\right] + C$$

$$\left[\because \int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right) + C\right]$$

$$= \frac{1}{2}\left[(x+1)\sqrt{3-2x-x^2} + 4\sin^{-1}\left(\frac{x+1}{2}\right)\right] + C$$

$[\because t = x+1]$ (1)

36. Let $I = \int \left(\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}\right)dx$

$$= \int \left(\frac{\sin^3 x}{\sin^2 x\cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x}\right)dx$$

$$= \int \left(\frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x}\right)dx$$ (1)

$$= \int [(\tan x\cdot \sec x) + (\cot x \cdot \operatorname{cosec} x)]\,dx$$

$$= \int \sec x\cdot \tan x\,dx + \int \cot x\cdot \operatorname{cosec} x\,dx$$

$$= \sec x + (-\operatorname{cosec} x) + C = \sec x - \operatorname{cosec} x + C$$ (1)

37. Let $I = \int \frac{(x-3)}{(x-1)^3}e^x\,dx = \int \frac{e^x(x-1-2)}{(x-1)^3}dx$

$$= \int e^x \left\{\frac{(x-1)}{(x-1)^3} - \frac{2}{(x-1)^3}\right\}dx$$

$$= \int e^x \left\{\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}\right\}dx$$ (1)

$$= \int e^x\cdot\{f(x)+f'(x)\}\,dx,$$

where $f(x) = \frac{1}{(x-1)^2}$ and $f'(x) = \frac{-2}{(x-1)^3}$

$$= e^x\cdot f(x) + C = e^x\cdot\frac{1}{(x-1)^2} + C = \frac{e^x}{(x-1)^2} + C$$ (1)

38. Let $I = \int \frac{(x-5)}{(x-3)^3}\cdot e^x\,dx = \int \frac{(x-3-2)}{(x-3)^3}\cdot e^x\,dx$

$$= \int e^x \left\{\frac{(x-3)}{(x-3)^3} - \frac{2}{(x-3)^3}\right\}dx$$

$$= \int e^x \left\{\frac{1}{(x-3)^2} - \frac{2}{(x-3)^3}\right\}dx$$ (1)

$$= \int e^x\,\{f(x)+f'(x)\}\,dx, \text{ where } f(x) = \frac{1}{(x-3)^2}$$

$$= e^x f(x) + C$$

$$[\because \int e^x[f(x)+f'(x)]\,dx = e^x f(x) + C]$$

$$= \frac{e^x}{(x-3)^2} + C \qquad \left[\because f(x) = \frac{1}{(x-3)^2}\right]$$ (1)

39. Let $I = \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx$

$$= \int \frac{1-2\sin^2 x + 2\sin^2 x}{\cos^2 x}dx$$

$[\because \cos 2A = 1 - 2\sin^2 A]$ (1)

$$= \int \frac{1}{\cos^2 x}dx = \int \sec^2 x\,dx = \tan x + C$$ (1)

40. Let $I = \int \frac{3-5\sin x}{\cos^2 x}dx$

$$= \int\left(\frac{3}{\cos^2 x} - \frac{5\sin x}{\cos^2 x}\right)dx$$ (1)

$$= 3\int \sec^2 x\,dx - 5\int \sec x \tan x\,dx$$

$$= 3\tan x - 5\sec x + C$$ (1)

41. $\int \frac{dx}{x^2+4x+8}$

Here, $x^2+4x+8 = x^2+4x+4+4$

$= (x+2)^2 + (2)^2$

Now, $\int \frac{dx}{x^2+4x+8} = \int \frac{dx}{(x+2)^2+(2)^2}$ (1)

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C\left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right]$$ (1)

42. Let $I = \int \frac{dx}{5-8x-x^2}$

$$= \int \frac{dx}{5-2\cdot 4\cdot x - x^2 - (4)^2 + (4)^2}$$

$$= \int \frac{dx}{5+16-[x^2+(4)^2+2\cdot 4\cdot x]}$$

$$= \int \frac{dx}{21-(x+4)^2}$$ (1)

$$= \int \frac{dx}{(\sqrt{21})^2 - (x+4)^2}$$

$$= \frac{1}{2\sqrt{21}} \log\left|\frac{\sqrt{21} + x + 4}{\sqrt{21} - x - 4}\right| + C$$

$$\left[\because \int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C\right]$$ (1)

43. Let $I = \int \frac{3x+5}{x^2+3x-18} dx$ ...(i)

Also, let $3x + 5 = A \frac{d}{dx}(x^2 + 3x - 18) + B$

$$3x + 5 = A(2x+3) + B$$ ...(ii) (1)

On comparing the coefficient of x, we get

$$2A = 3 \Rightarrow A = \frac{3}{2}$$

and on comparing the constant term, we get

$$B = 5 - 3A \Rightarrow B = 5 - 3\left(\frac{3}{2}\right) = \frac{1}{2}$$ (1)

From Eq. (ii), we get

$$3x + 5 = \frac{3}{2}(2x+3) + \frac{1}{2}$$ ...(iii)

From Eqs. (i) and (iii), we get

$$I = \int \frac{\frac{3}{2}(2x+3) + \frac{1}{2}}{x^2+3x-18} dx$$

$$= \frac{3}{2}\int \frac{2x+3}{x^2+3x-18} dx + \frac{1}{2}\int \frac{1}{x^2+3x-18} dx$$ (1)

$$= \frac{3}{2}\log|x^2+3x-18| + \frac{1}{2}\int \frac{1}{\left(x+\frac{3}{2}\right)^2 - \frac{81}{4}} dx$$

$$= \frac{3}{2}\log|x^2+3x-18| + \frac{1}{2}\int \frac{1}{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2} dx$$

$$= \frac{3}{2}\log|x^2+3x-18| + \frac{1}{2}\cdot\frac{1}{2\left(\frac{9}{2}\right)}\log\left|\frac{\left(x+\frac{3}{2}\right) - \frac{9}{2}}{\left(x+\frac{3}{2}\right) + \frac{9}{2}}\right| + C$$

$$\left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right]$$

$$= \frac{3}{2}\log|x^2+3x-18| + \frac{1}{18}\log\left|\frac{x-3}{x+6}\right| + C$$ (1)

44. Let $I = \int \frac{\cos x}{(1+\sin x)(2+\sin x)} dx$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$$\therefore I = \int \frac{dt}{(1+t)(2+t)} = \int \left[\frac{A}{1+t} + \frac{B}{2+t}\right] dt$$ ...(i) (1)

$$\therefore \frac{1}{(1+t)(2+t)} = \frac{A(2+t) + B(1+t)}{(1+t)(2+t)}$$

$$1 = 2A + tA + B + Bt$$

$$1 = 1(2A + B) + t(A + B)$$ (1)

On comparing the coefficients of t and constant term on both sides, we get

$$2A + B = 1 \text{ and } A + B = 0$$

$$\Rightarrow \quad A = 1 \text{ and } B = -1$$

$$\therefore I = \int \left(\frac{1}{1+t} - \frac{1}{2+t}\right) dt$$

$$= \int \frac{1}{(1+t)} dt - \int \frac{1}{(2+t)} dt$$ (1)

$$= \log|1+t| - \log|2+t| + C$$

$$= \log\left|\frac{1+t}{2+t}\right| + C$$

$$= \log\left|\frac{1+\sin x}{2+\sin x}\right| + C$$ [put $t = \sin x$] (1)

45. Let $I = \int \frac{x^2+x+1}{(x+2)(x^2+1)} dx$

By partial fraction,

$$\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{A}{(x+2)} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow \quad x^2 + x + 1 = A(x^2+1) + (Bx+C)(x+2)$$ (1)

Putting $x = -2$,

$$4 - 2 + 1 = A(5) + 0 \Rightarrow 5A = 3 \Rightarrow A = \frac{3}{5}$$

Putting $x = 0$,

$$0 + 0 + 1 = A(0+1) + (0+C)(0+2)$$

$$\Rightarrow 1 = A + 2C \Rightarrow 1 = \frac{3}{5} + 2C \Rightarrow 2C = \frac{2}{5} \Rightarrow C = \frac{1}{5}$$

and putting $x = 1$,

$$1 + 1 + 1 = 2A + (B + C)(3)$$

$$\Rightarrow \quad 3 = 2A + 3(B + C)$$

$$\Rightarrow \quad 3 = 2\left(\frac{3}{5}\right) + 3\left(B + \frac{1}{5}\right)$$ (1)

$$\Rightarrow \quad 3 - \frac{6}{5} = 3\left(B + \frac{1}{5}\right)$$

$$\Rightarrow \quad \frac{9}{5} = 3\left(B + \frac{1}{5}\right) \Rightarrow \frac{3}{5} - \frac{1}{5} = B \Rightarrow B = \frac{2}{5}$$

Thus, $\dfrac{x^2+x+1}{(x+2)(x^2+1)} = \dfrac{3}{5(x+2)} + \dfrac{\left(\frac{2}{5}x+\frac{1}{5}\right)}{(x^2+1)}$ **(1)**

Now, $\displaystyle\int \frac{x^2+x+1}{(x+2)(x^2+1)}dx$

$= \displaystyle\int \frac{3}{5(x+2)}dx + \frac{1}{5}\int \frac{2x}{x^2+1}dx + \frac{1}{5}\int \frac{dx}{x^2+1}$

$= \dfrac{3}{5}\log|x+2| + \dfrac{1}{5}\log|x^2+1| + \dfrac{1}{5}\tan^{-1}(x) + C$ **(1)**

46. Let $I = \displaystyle\int \frac{2\cos x}{(1-\sin x)(2-\cos^2 x)}dx$

$= \displaystyle\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$ **(1)**

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$\therefore \quad I = \displaystyle\int \frac{2}{(1-t)(1+t^2)}dt$

Now, let $\dfrac{2}{(1-t)(1+t^2)} = \dfrac{A}{1-t} + \dfrac{Bt+C}{1+t^2}$

$\Rightarrow \quad 2 = A(1+t^2) + (Bt+C)(1-t)$ ...(i) **(1)**

Putting $t=1$ in Eq. (i), we get

$2 = 2A \Rightarrow A = 1$

Putting $t=0$ in Eq. (i), we get

$2 = A + C \Rightarrow 2 = 1 + C \Rightarrow C = 1$

Putting $t=-1$ in Eq. (i), we get

$2 = 2A + (-B+C)(2)$

$\Rightarrow \quad 2 = 2 - 2B + 2$

$\Rightarrow \quad 2B = 2 \Rightarrow B = 1$ **(1)**

$\therefore \quad I = 1\displaystyle\int \frac{1}{1-t}dt + \int \frac{t+1}{1+t^2}dt$

$= \displaystyle\int \frac{1}{1-t}dt + \frac{1}{2}\int \frac{2t}{1+t^2}dt + \int \frac{1}{1+t^2}dt$

$= -\log|1-t| + \dfrac{1}{2}\log|1+t^2| + \tan^{-1}t + C$

$= -\log|1-\sin x| + \dfrac{1}{2}\log|1+\sin^2 x| + \tan^{-1}(\sin x) + C$

$= \tan^{-1}(\sin x) + \log\left|\dfrac{\sqrt{1+\sin^2 x}}{1-\sin x}\right| + C$

$\left[\because \log m - \log n = \log\left(\dfrac{m}{n}\right) \text{ and } n\log m = \log m^n\right]$ **(1)**

47. Let $\quad I = \displaystyle\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$

Put $\sin x = t$, then $\cos x\, dx = dt$

$\therefore \quad I = \displaystyle\int \frac{2dt}{(1-t)(1+t^2)}$ ...(i)

Now, let $\quad \dfrac{2}{(1-t)(1+t^2)} = \dfrac{A}{1-t} + \dfrac{Bt+C}{1+t^2}$

$\Rightarrow \quad 2 = (1+t^2)A + (1-t)(Bt+C)$

$\Rightarrow \quad 2 = (1+t^2)A + (Bt + C - Bt^2 - Ct)$

$\Rightarrow \quad 2 = t^2(A-B) + t(B-C) + (A+C)$ **(1)**

On comparing the coefficients of like powers of t, we get

$A - B = 0;\ B - C = 0$ and $A + C = 2$

$\Rightarrow \quad A = B;\ B = C$ and $A + C = 2$

$\Rightarrow \quad A = B = C = 1$

$\therefore \dfrac{2}{(1-t)(1+t^2)} = \dfrac{1}{1-t} + \dfrac{1+t}{1+t^2}$ **(1)**

Now, from Eq. (i), we get

$I = \displaystyle\int\left(\frac{1}{1-t} + \frac{1+t}{1+t^2}\right)dt$

$= \displaystyle\int \frac{dt}{1-t} + \int \frac{1}{1+t^2}dt + \frac{1}{2}\int \frac{2t}{1+t^2}dt$

$= \dfrac{\log|1-t|}{(-1)} + \tan^{-1}t + \dfrac{1}{2}\log|1+t^2| + C$ **(1)**

$= \dfrac{1}{2}\log|1+\sin^2 x| - \log|1-\sin x| + \tan^{-1}(\sin x) + C$

$[\because t = \sin x]$

$= \tan^{-1}(\sin x) + \log\left|\dfrac{\sqrt{1+\sin^2 x}}{1-\sin x}\right| + C$ **(1)**

$\left[\because \log m - \log n = \log\left(\dfrac{m}{n}\right) \text{ and } n\log m = \log m^n\right]$

48. Let $I = \displaystyle\int \frac{4}{(x-2)(x^2+4)}dx$

Again, let $\dfrac{4}{(x-2)(x^2+4)} = \dfrac{A}{(x-2)} + \dfrac{Bx+C}{x^2+4}$

$\Rightarrow \quad 4 = A(x^2+4) + (Bx+C)(x-2)$

$\Rightarrow \quad 4 = x^2(A+B) + x(-2B+C) + 4A - 2C$ **(1)**

On equating the coefficients of x^2, x and constant form both sides, we get

$A + B = 0$ ...(i)

$-2B + C = 0$ ...(ii)

and $\quad 4A - 2C = 4$ ...(iii) **(1)**

On solving Eqs. (i), (ii) and (iii), we get

$A = \dfrac{1}{2},\ B = -\dfrac{1}{2}$ and $C = 1$

$$\therefore \quad I = \int \frac{4}{(x-2)(x^2+4)}\,dx \quad (1)$$

$$= \int \frac{1/2}{(x-2)}\,dx + \int \frac{\frac{-1}{2}x + 1}{x^2+4}\,dx$$

$$= \frac{1}{2}\int \frac{dx}{(x-2)} - \int \frac{x-2}{2(x^2+4)}\,dx$$

$$= \frac{1}{2}\log|x-2| - \frac{1}{2}\log|x^2+4|$$

$$+ \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C \quad (1)$$

49. Let $I = \int \frac{2x}{(x^2+1)(x^2+2)^2}\,dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore \quad I = \int \frac{dt}{(t+1)(t+2)^2}$$

Let $\frac{1}{(t+1)(t+2)^2} = \frac{A}{t+1} + \frac{B}{t+2} + \frac{C}{(t+2)^2}$

$$\Rightarrow \frac{1}{(t+1)(t+2)^2} = \frac{A(t+2)^2 + B(t+1)(t+2) + C(t+1)}{(t+1)(t+2)^2} \quad (1)$$

$$\Rightarrow 1 = A(t+2)^2 + B(t+1)(t+2) + C(t+1)$$

$$\Rightarrow 1 = A(t^2 + 4 + 4t) + B(t^2 + 2t + t + 2) + C(t+1)$$

$$\Rightarrow 1 = A(t^2 + 4t + 4) + B(t^2 + 3t + 2) + C(t+1)$$

$$\Rightarrow 1 = t^2(A+B) + t(4A + 3B + C) + 4A + 2B + C \quad (1)$$

On comparing the coefficients of t^2, t and the constant term from both sides, we get

$$A + B = 0 \quad \ldots(i)$$

$$4A + 3B + C = 0 \quad \ldots(ii)$$

and $4A + 2B + C = 1 \quad \ldots(iii)$

From Eq. (i), $A = -B$

Put the value of A in Eqs. (ii) and (iii), we get

$$-4B + 3B + C = 0$$

$$\Rightarrow \quad -B + C = 0$$

$$\Rightarrow \quad B - C = 0 \quad \ldots(iv)$$

and $-4B + 2B + C = 1$

$$\Rightarrow \quad -2B + C = 1$$

$$\Rightarrow \quad 2B - C = -1 \quad \ldots(v)$$

Now, from Eqs. (iv) and (v), we get

$$-B = 1 \Rightarrow B = -1 \quad (1)$$

$\therefore \quad A = 1$ and $C = -1$

$$\therefore \quad I = \int \frac{1}{t+1}\,dt + \int \frac{(-1)}{t+2}\,dt + \int \frac{(-1)}{(t+2)^2}\,dt$$

$$\Rightarrow I = \log|t+1| - \log+|t+2| - \frac{(t+2)^{-1}}{-1} + C$$

$$\Rightarrow I = \log|t+1| - \log|t+2| + \frac{1}{(t+2)} + C$$

$$= \log|x^2+1| - \log|x^2+2| + \frac{1}{(x^2+2)} + C \quad (1)$$

50. Let $I = \int \frac{2x}{(x^2+1)(x^4+4)}\,dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore \quad I = \int \frac{dt}{(t+1)(t^2+4)}$$

Now, $\frac{1}{(t+1)(t^2+4)} = \frac{A}{t+1} + \frac{Bt + C}{t^2+4}$

$$\Rightarrow \quad 1 = A(t^2+4) + (Bt + C)(t+1)$$

$$\Rightarrow \quad 1 = A(t^2+4) + (Bt^2 + Bt + Ct + C)$$

$$\Rightarrow \quad 1 = t^2(A+B) + t(B+C) + (4A + C) \quad (1)$$

On comparing the coefficients of t^2, t and constant term from both sides, we get

$$A + B = 0 \quad \ldots(i)$$

$$B + C = 0 \quad \ldots(ii)$$

$$4A + C = 1 \quad \ldots(iii)$$

From Eqs. (i) and (ii), we get

$$A - C = 0 \quad \ldots(iv)$$

From Eqs. (iii) and (iv), we get

$$5A = 1$$

$$\Rightarrow \quad A = \frac{1}{5}$$

Then, $C = \frac{1}{5}, B = -\frac{1}{5}$ (1)

Now, $I = \int \frac{dt}{(t+1)(t^2+4)} = \int \frac{1}{5(t+1)}\,dt + \int \frac{(-1/5)t + (1/5)}{t^2+4}\,dt$

$$= \frac{1}{5}\int \frac{dt}{t+1} - \frac{1}{5}\int \frac{t-1}{t^2+4}\,dt \quad (1)$$

$$= \frac{1}{5}\log|t+1| - \frac{1}{5}\left[\int \frac{t}{t^2+4}\,dt - \int \frac{1}{t^2+4}\,dt\right]$$

$$= \frac{1}{5}\log|t+1| - \frac{1}{5}\left[\frac{1}{2}\log|t^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right] + C$$

$$= \frac{1}{5}\log|x^2+1| - \frac{1}{5}\left[\frac{1}{2}\log|x^4+4| - \frac{1}{2}\tan^{-1}\left(\frac{x^2}{2}\right)\right] + C$$ (1)

51. Let $I = \int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)]} d\theta$

$$= \int \frac{\cos\theta}{(4+\sin^2\theta)[5-4(1-\sin^2\theta)]} d\theta$$

$$= \int \frac{\cos\theta}{(4+\sin^2\theta)(5-4+4\sin^2\theta)} d\theta$$

$$= \int \frac{\cos\theta}{(4+\sin^2\theta)(1+4\sin^2\theta)} d\theta$$

Let $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$

Then, $I = \int \frac{dt}{(4+t^2)(1+4t^2)}$ ...(i)

Again, let $\frac{1}{(4+t^2)(1+4t^2)} = \frac{A}{4+t^2} + \frac{B}{1+4t^2}$ ...(ii)

[by partial fraction]

At $t = 0$, $\frac{A}{4} + \frac{B}{1} = \frac{1}{4\times 1} \Rightarrow A + 4B = 1$ ...(iii)

At $t = 1$, $\frac{A}{5} + \frac{B}{5} = \frac{1}{5\times 5} \Rightarrow 5A + 5B = 1$ ...(iv) (1)

On solving Eqs. (iii) and (iv), we get

$$A = -\frac{1}{15} \text{ and } B = \frac{4}{15}$$

On putting $A = -\frac{1}{15}$ and $B = \frac{4}{15}$ in Eq. (ii), we get

$$\frac{1}{(4+t^2)(1+4t^2)} = \frac{-\frac{1}{15}}{4+t^2} + \frac{\frac{4}{15}}{1+4t^2}$$ (1)

$$\Rightarrow \frac{1}{(4+t^2)(1+4t^2)} = \frac{-1}{15(4+t^2)} + \frac{4}{15(1+4t^2)}$$

On integrating both sides w.r.t. t, we get

$$\int \frac{1}{(4+t^2)(1+4t^2)} dt = \frac{-1}{15}\int \frac{1}{4+t^2} dt + \frac{4}{15}\int \frac{1}{1+4t^2} dt$$

$$= \frac{-1}{15}\int \frac{1}{2^2+t^2} + \frac{4}{15\times 4}\int \frac{1}{\left(\frac{1}{2}\right)^2 + t^2} dt$$ (1)

$$= \frac{-1}{15}\cdot\frac{1}{2}\tan^{-1}\frac{t}{2} + \frac{1}{15}\cdot\frac{1}{1/2}\tan^{-1}\frac{t}{1/2} + C$$

$$\left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right]$$

$$= \frac{-1}{30}\tan^{-1}\frac{\sin\theta}{2} + \frac{2}{15}\tan^{-1} 2\sin\theta + C \quad [\text{put } t = \sin\theta]$$ (1)

52. Let $I = \int \frac{(3\sin\theta - 2)\cos\theta}{5 - \cos^2\theta - 4\sin\theta} d\theta$

$$= \int \frac{(3\sin\theta - 2)\cos\theta}{5-(1-\sin^2\theta) - 4\sin\theta} d\theta$$

Put $\sin\theta = t \Rightarrow \cos\theta d\theta = dt$

$$\therefore \quad I = \int \frac{3t-2}{5-(1-t^2)-4t} dt$$ (1)

$$= \int \frac{3t-2}{4+t^2-4t} dt = \int \frac{3t-2}{(t-2)^2} dt$$

$$= \int \frac{3t-6+4}{(t-2)^2} dt = \int \frac{3(t-2)+4}{(t-2)^2} dt$$ (1)

$$= \int \frac{3}{(t-2)} dt + \int \frac{4}{(t-2)^2} dt$$

$$= 3\log|t-2| + \frac{4(t-2)^{-2+1}}{-2+1} + C$$ (1)

$$\left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1\right]$$

$$= 3\log|t-2| - \frac{4}{(t-2)} + C$$

$$= 3\log|\sin\theta - 2| - \frac{4}{(\sin\theta - 2)} + C$$ (1)

[put $t = \sin\theta$]

53. Let $I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx = \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}} dx$

Put $x^{3/2} = a^{3/2} t$

$$\Rightarrow \frac{3}{2}x^{1/2} dx = a^{3/2} dt \Rightarrow \sqrt{x}\, dx = \frac{2}{3}a^{3/2} dt$$ (1)

$$\therefore \quad I = \int \frac{\frac{2}{3}a^{3/2}}{\sqrt{(a^{3/2})^2 - (a^{3/2}t)^2}} dt$$

$$= \frac{2}{3}a^{3/2}\int \frac{dt}{a^{3/2}\sqrt{1-t^2}}$$ (1)

$$= \frac{2}{3}\int \frac{dt}{\sqrt{1-t^2}} = \frac{2}{3}\sin^{-1}\left(\frac{t}{1}\right) + C$$

$$\left[\because \int \frac{dx}{a^2 - x^2} = \sin^{-1}\left(\frac{x}{a}\right) + C\right]$$ (1)

$$= \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + C \quad \left[\text{put } t = \frac{x^{3/2}}{a^{3/2}}\right]$$

$$= \frac{2}{3}\sin^{-1}\left(\sqrt{\frac{x^3}{a^3}}\right) + C$$ (1)

54. First, use the method for integral of the form

$\int (px+q)\sqrt{ax^2+bx+c}\,dx$,

consider $(px+q) = A\frac{d}{dx}(ax^2+bx+c)+B$,

simplify and get the values of A and B.

Further, simplify the integrand and use the formula

$\int \sqrt{a^2-x^2}\,dx = \left[\frac{1}{2}x\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C\right]$.

Let $\quad I = \int (x+3)\sqrt{3-4x-x^2}\,dx$

Given integral is the form of

$$\int (px+q)\sqrt{ax^2+bx+c}\,dx$$

Let $(x+3) = A\frac{d}{dx}(3-4x-x^2)+B$

$\Rightarrow \quad x+3 = A(-4-2x)+B \quad \ldots(i)$

$\Rightarrow \quad x+3 = (-4A+B)-2Ax$

On comparing the coefficients of x and constant terms, we get

$$-2A = 1$$

$\Rightarrow \quad A = -\frac{1}{2}$

and $\;-4A+B = 3 \Rightarrow 2+B = 3 \Rightarrow B = 1 \quad$ **(1)**

Thus, $(x+3) = -\frac{1}{2}(-4-2x)+1 \quad$ [from Eq. (i)]

Now, given integral becomes

$$I = -\frac{1}{2}\int(-4-2x)\sqrt{3-4x-x^2}\,dx + \int\sqrt{3-4x-x^2}\,dx$$

$I = I_1 + I_2$ (say) $\quad \ldots(ii)$

Now, consider

$$I_1 = -\frac{1}{2}\int(-4-2x)\sqrt{3-4x-x^2}\,dx$$

Put $\quad 3-4x-x^2 = t$

$\Rightarrow \quad (-4-2x)dx = dt$

$\therefore \quad I_1 = -\frac{1}{2}\int\sqrt{t}\,dt = -\frac{1}{2}\times\frac{2}{3}(t)^{\frac{3}{2}}+C_1$

$= -\frac{1}{3}(3-4x-x^2)^{\frac{3}{2}}+C_1 \quad$ **(1)**

and $\quad I_2 = \int\sqrt{3-4x-x^2}\,dx$

$= \int\sqrt{-(x^2+4x-3)}\,dx$

$= \int\sqrt{-(x^2+2\times 2x+4-4-3)}\,dx$

$= \int\sqrt{-\{(x+2)^2-7\}}\,dx$

$= \int\sqrt{7-(x+2)^2}\,dx$

$= \int\sqrt{(\sqrt{7})^2-(x+2)^2}\,dx \quad$ **(1)**

$= \frac{1}{2}\left[(x+2)\sqrt{3-4x-x^2}+7\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right)\right]+C_2$

$\left[\because \int\sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)\right]+C\right]$

Now, from Eq. (ii), we have

$$I = -\frac{1}{3}(3-4x-x^2)^{\frac{3}{2}}+\frac{1}{2}(x+2)\sqrt{3-4x-x^2}+\frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right)+C$$

where, $C = C_1 + C_2$. $\quad$ **(1)**

55. First, use the partial fraction in the given integrand,

i.e. write $\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{A}{x+2}+\frac{Bx+C}{x^2+1}$.

Simplify it and get the values of constants A, B and C.

Further, integrate it to get the result.

Let $I = \int\frac{x^2+x+1}{(x^2+1)(x+2)}\,dx$

By using partial fraction method, we get

$$\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{A}{x+2}+\frac{Bx+C}{x^2+1} \quad \ldots(i)$$

$\Rightarrow \quad x^2+x+1 = A(x^2+1)+(Bx+C)(x+2)$

$\Rightarrow \quad x^2+x+1 = x^2(A+B)+x(2B+C)+(A+2C)$

On comparing the coefficients of x^2, x and constant terms both sides, we get

$A+B = 1 \quad \ldots(ii)$

$2B+C = 1 \quad \ldots(iii)$

and $\quad A+2C = 1 \quad \ldots(iv)$ **(1)**

On substituting the value of B from Eq. (ii) in Eq. (iii), we get

$2(1-A)+C = 1$

$\Rightarrow \quad 2-2A+C = 1$

$\Rightarrow \quad 2A-C = 1 \quad \ldots(v)$

Now, on solving Eqs. (iv) and (v), we get

$$C=\frac{1}{5} \text{ and } A=\frac{3}{5}$$

Now, from Eq. (ii), we get $B=1-\frac{3}{5}=\frac{2}{5}$ **(1)**

Thus, from Eq. (i), we have

$$\frac{x^2+x+1}{(x^2+1)(x+2)}=\frac{3}{5}\cdot\frac{1}{(x+2)}+\frac{1}{5}\frac{(2x+1)}{(x^2+1)}$$ **(1)**

On integrating both sides, we get

$$I=\frac{3}{5}\int\frac{dx}{x+2}+\frac{1}{5}\int\frac{(2x+1)}{x^2+1}dx$$

$$=\frac{3}{5}\int\frac{dx}{x+2}+\frac{1}{5}\int\frac{2x}{x^2+1}dx+\frac{1}{5}\int\frac{dx}{x^2+1}$$

$$=\frac{3}{5}\log|x+2|+\frac{1}{5}\log|x^2+1|+\frac{1}{5}\tan^{-1}x+C$$ **(1)**

$$\left[\because\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C \text{ and } \int\frac{f'(x)}{f(x)}dx=\log|f(x)|+C\right]$$

56. Let $I=\int\frac{(2x-5)e^{2x}}{(2x-3)^3}dx=\int\frac{(2x-3-2)e^{2x}}{(2x-3)^3}dx$

$$=\int\frac{e^{2x}}{(2x-3)^2}dx-2\int\frac{e^{2x}}{(2x-3)^3}dx$$

$$=\int \underset{\text{II}}{e^{2x}}\,\underset{\text{I}}{(2x-3)^{-2}}dx-2\int e^{2x}(2x-3)^{-3}dx$$ **(1)**

$$=\left[(2x-3)^{-2}\int e^{2x}dx-\int\left\{\frac{d}{dx}(2x-3)^{-2}\int e^{2x}dx\right\}dx\right]-2\int e^{2x}(2x-3)^{-3}dx$$

[using integration by parts] **(1)**

$$=(2x-3)^{-2}\frac{e^{2x}}{2}-\int -2(2x-3)^{-3}\times 2\times\frac{e^{2x}}{2}dx-2\int e^{2x}(2x-3)^{-3}dx$$ **(1)**

$$=\frac{e^{2x}(2x-3)^{-2}}{2}+2\int e^{2x}(2x-3)^{-3}dx-2\int e^{2x}(2x-3)^{-3}dx$$

$$=\frac{e^{2x}(2x-3)^{-2}}{2}+C$$ **(1)**

57. Let $I=\int(2x+5)\sqrt{10-4x-3x^2}dx$

Now, let us write $2x+5=A\frac{d}{dx}(10-4x-3x^2)+B$,

where A and B are constants. ...(i)

$\Rightarrow\quad 2x+5=A(-4-6x)+B$

$\Rightarrow\quad 2x+5=-6Ax+(B-4A)$ **(1/2)**

On comparing the coefficient of x and the constant term, we get

$$-6A=2 \text{ and } B-4A=5 \Rightarrow A=\frac{-1}{3}$$

and $\quad B=5+4A=5+4\left(\frac{-1}{3}\right)=\frac{11}{3}$

$\Rightarrow\quad A=\frac{-1}{3}$ and $B=\frac{11}{3}$

Thus, $(2x+5)=\frac{-1}{3}(-4-6x)+\frac{11}{3}$ [from Eq. (i)] **(1)**

Now, $\quad I=\frac{-1}{3}\int(-4-6x)\sqrt{10-4x-3x^2}dx+\frac{11}{3}\int\sqrt{10-4x-3x^2}dx$

$$=\frac{-1}{3}I_1+\frac{11}{3}I_2 \text{ (say)}$$ ...(ii) **(1/2)**

Consider $I_1=\int(-4-6x)\sqrt{10-4x-3x^2}dx$

Put $10-4x-3x^2=t\Rightarrow(-4-6x)dx=dt$

$\therefore\quad I_1=\int\sqrt{t}dt=\frac{2}{3}t^{3/2}+C_1$

$$=\frac{2}{3}(10-4x-3x^2)^{3/2}+C_1$$ ...(iii) **(1/2)**

Now, consider $I_2=\int\sqrt{10-4x-3x^2}dx$

$$=\sqrt{3}\int\sqrt{-\left(x^2+\frac{4}{3}x-\frac{10}{3}\right)}dx$$

$$=\sqrt{3}\int\sqrt{-\left(x^2+2\cdot\frac{2}{3}x+\left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^2-\frac{10}{3}\right)}dx$$ **(1/2)**

$$=\sqrt{3}\int\sqrt{\frac{34}{9}-\left(x+\frac{2}{3}\right)^2}dx$$

$$=\sqrt{3}\int\sqrt{\left(\frac{\sqrt{34}}{3}\right)^2-\left(x+\frac{2}{3}\right)^2}dx$$ **(1/2)**

$$=\frac{\sqrt{3}}{2}\left[\left(x+\frac{2}{3}\right)\sqrt{\left(\frac{\sqrt{34}}{3}\right)^2-\left(x+\frac{2}{3}\right)^2}+\frac{34}{9}\sin^{-1}\left(\frac{\left(x+\frac{2}{3}\right)}{\left(\frac{\sqrt{34}}{3}\right)}\right)\right]+C_2$$

$$\left[\because\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C\right]$$

$$= \frac{\sqrt{3}}{2}\left[\left(x+\frac{2}{3}\right)\sqrt{\frac{34}{9}-\left(x+\frac{2}{3}\right)^2}\right.$$

$$\left. + \frac{34}{9}\sin^{-1}\left(\frac{3x+2}{\sqrt{34}}\right) + C_2\right] \quad \text{...(iv)}$$

Now, from Eqs. (ii), (iii) and (iv), we get

$$I = \frac{-2}{9}(10-4x-3x^2)^{\frac{3}{2}} + \frac{11\sqrt{3}}{6}$$

$$\left[\left(x+\frac{2}{3}\right)\sqrt{\frac{34}{9}-\left(x+\frac{2}{3}\right)^2} + \frac{34}{9}\sin^{-1}\left(\frac{3x+2}{\sqrt{34}}\right)\right] + C$$

where, $C = \frac{-C_1}{3} + \frac{11}{3}C_2$ **(1/2)**

58. Let $I = \int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx$

Consider, $\frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}$

and put $x^2 = y$

Then, $\frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} = \frac{(y+1)(y+4)}{(y+3)(y-5)}$

$$= \frac{y^2+5y+4}{y^2-2y-15} = \frac{(y^2-2y-15)+(7y+19)}{y^2-2y-15}$$

$$= 1 + \frac{7y+19}{y^2-2y-15} = 1 + \frac{7y+19}{(y+3)(y-5)} \quad \text{...(i)} \quad \textbf{(1)}$$

Now, let us write $\frac{7y+19}{(y+3)(y-5)} = \frac{A}{y+3} + \frac{B}{y-5}$

$\Rightarrow$ $7y+19 = A(y-5)+B(y+3)$

On putting $y = 5$, we get

$$35+19 = 8B \Rightarrow B = \frac{54}{8} = \frac{27}{4}$$

and on putting $y = -3$, we get

$$-21+19 = -8A \Rightarrow A = \frac{(-2)}{(-8)} = \frac{1}{4}$$

Thus, $\frac{7y+19}{(y+3)(y-3)} = \frac{1}{4}\cdot\frac{1}{(y+3)} + \frac{27}{4}\cdot\frac{1}{(y-5)}$

...(ii) **(1)**

From Eqs. (i) and (ii), we get

$$\frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} = 1 + \frac{1}{4}\cdot\frac{1}{(x^2+3)} + \frac{27}{4}\frac{1}{(x^2-5)} \quad \textbf{(1/2)}$$

Now, $I = \int\left(1 + \frac{1}{4}\cdot\frac{1}{(x^2+3)} + \frac{27}{4}\frac{1}{(x^2-5)}\right)dx$ **(1/2)**

$$= \int dx + \frac{1}{4}\int\frac{dx}{x^2+3} + \frac{27}{4}\int\frac{dx}{x^2-5}$$

$$= x + \frac{1}{4}\int\frac{dx}{x^2+(\sqrt{3})^2} + \frac{27}{4}\int\frac{dx}{x^2-(\sqrt{5})^2}$$

$$= x + \frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{27}{4}\frac{1}{2\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + C$$

$$= x + \frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{27}{8\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + C \quad \textbf{(1)}$$

$$\left[\because \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \text{ and } \int\frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right]$$

59. First, put $x = \sin t$ and then use integration by parts and simplify it.

Let $I = \int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

Put $\sin^{-1}x = t$

$\Rightarrow$ $\frac{1}{\sqrt{1-x^2}}dx = dt$ **(1)**

$\therefore$ $I = \int \underset{\text{I}}{t}\ \underset{\text{II}}{\sin t}\, dt$

Using integration by parts, taking t as the first function and $\sin t$ as the second function, we get

$$I = t\int\sin t\,dt - \int\left[\frac{d}{dt}(t)\cdot\int\sin t\,dt\right]dt \quad \textbf{(1)}$$

$$= -t\cos t - \int(1\times -\cos t)\,dt$$

$$= -t\cos t + \int\cos t\,dt$$

$$= -t\cos t + \sin t + C \quad \textbf{(1)}$$

$$= -t\sqrt{1-\sin^2 t} + \sin t + C$$

$$[\because \cos^2 t = 1-\sin^2 t \Rightarrow \cos t = \sqrt{1-\sin^2 t}]$$

$\therefore$ $I = -\sin^{-1}x\sqrt{1-x^2} + x + C$ **(1)**

$[\text{put } t = \sin^{-1}x \Rightarrow x = \sin t]$

60. First, simplify the integrand in such a form that numerator is in sin form and denominator is in cos form. Substitute $\cos x = t$ and then convert the given integrand in the form of t.
Now, use partial fraction in the integrand and then integrate it. Further, substitute the value of t and get the required result.

Let $I = \int\frac{dx}{\sin x + \sin 2x} = \int\frac{dx}{\sin x + 2\sin x\cos x}$

$[\because \sin 2x = 2\sin x\cos x]$

$$= \int \frac{dx}{\sin x\,(1 + 2\cos x)} = \int \frac{\sin x}{\sin^2 x\,(1 + 2\cos x)} dx$$

[multiplying numerator and denominator by $\sin x$]

$$= \int \frac{\sin x}{(1 - \cos^2 x)\,(1 + 2\cos x)} dx \quad \textbf{(1)}$$

Put $\cos x = t$,

$\Rightarrow \quad -\sin x\, dx = dt$

$\Rightarrow \quad \sin x\, dx = -\,dt$

$$\therefore \quad I = \int \frac{-\,dt}{(1 - t^2)\,(1 + 2t)}$$

$$= \int \frac{-\,dt}{(1 - t)\,(1 + t)\,(1 + 2t)} \quad \ldots(\text{i})$$

Now, using partial fraction,

let $\dfrac{1}{(1 - t)\,(1 + t)\,(1 + 2t)}$

$$= \frac{A}{1 - t} + \frac{B}{1 + t} + \frac{C}{1 + 2t} \quad \ldots(\text{ii})$$

$$\Rightarrow \quad 1 = (1 + t)\,(1 + 2t)\,A + (1 - t)\,(1 + 2t)\,B + (1 - t)\,(1 + t)\,C \quad \ldots(\text{iii})\ \textbf{(1)}$$

On putting $t = -1$ in Eq. (iii), we get

$$1 = (2)\,(-1)\,B \Rightarrow B = -\frac{1}{2}$$

On putting $t = 1$ in Eq. (iii), we get

$$1 = 2 \cdot (3)\,A \Rightarrow A = \frac{1}{6}$$

On putting $t = -\frac{1}{2}$ in Eq. (iii), we get

$$1 = \left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{2}\right) C$$

$$\Rightarrow \quad 1 = \left(\frac{3}{2} \times \frac{1}{2}\right) C$$

$$\Rightarrow \quad C = \frac{4}{3} \quad \textbf{(1)}$$

$$\therefore I = -\left[\int \frac{A}{1 - t}\, dt + \int \frac{B}{1 + t}\, dt + \int \frac{C}{1 + 2t}\, dt\right]$$

[using Eqs. (i) and (ii)]

$$= -\left[\frac{1}{6}\int \frac{dt}{1 - t} + \left(-\frac{1}{2}\right)\int \frac{dt}{1 + t} + \frac{4}{3}\int \frac{dt}{1 + 2t}\right]$$

$$= -\left[\frac{1}{6}\,\frac{\log|1 - t|}{-1} - \frac{1}{2}\log|1 + t| + \frac{4}{3}\,\frac{\log|1 + 2t|}{2}\right] + C$$

$$= \frac{1}{6}\log|1 - t| + \frac{1}{2}\log|1 + t| - \frac{2}{3}\log|1 + 2t| + C$$

$$= \frac{1}{6}\log|1 - \cos x| + \frac{1}{2}\log|1 + \cos x| - \frac{2}{3}\log|1 + 2\cos x| + C$$

[put $t = \cos x$] **(1)**

61. Let $I = \int \frac{x^2 - 3x + 1}{\sqrt{1 - x^2}} = (-1)\int \frac{-x^2 + 3x - 1}{\sqrt{1 - x^2}} dx$

$$= (-1) \int \frac{1 - x^2 + 3x - 2}{\sqrt{1 - x^2}} dx$$

$$= (-1) \int \left[\frac{1 - x^2}{\sqrt{1 - x^2}} + \frac{3x - 2}{\sqrt{1 - x^2}}\right] dx$$

$$= (-1) \int \left[\sqrt{1 - x^2} + \int \frac{3x - 2}{\sqrt{1 - x^2}}\right] dx$$

$$= (-1) \left[\int \sqrt{1 - x^2}\, dx + \int \frac{3x - 2}{\sqrt{1 - x^2}}\, dx\right]$$

$$= (-1)\,[I_1 + I_2]\ (\text{say}) \quad \ldots(\text{i})\ \textbf{(1)}$$

Now, $I_1 = \int \sqrt{1 - x^2} dx$

$$= \frac{1}{2}[x\sqrt{1 - x^2} + \sin^{-1}(x)] + C_1 \quad \ldots(\text{ii})\ \textbf{(1)}$$

$$\left[\because \int \sqrt{a^2 - x^2} dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right] + C\right]$$

and $I_2 = \int \frac{3x - 2}{\sqrt{1 - x^2}} dx$

$$= \int \frac{3x}{\sqrt{1 - x^2}}\, dx - 2\int \frac{dx}{\sqrt{1 - x^2}}$$

$$= -\frac{3}{2}\int \frac{-2x}{\sqrt{1 - x^2}} dx - 2\int \frac{dx}{\sqrt{1 - x^2}}$$

$$= -\frac{3}{2} \times 2\sqrt{1 - x^2} - 2\sin^{-1}(x) + C_2$$

$$\left[\because \int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C\right] \textbf{(1)}$$

$$= -3\sqrt{1 - x^2} - 2\sin^{-1}(x) + C_2 \quad \ldots(\text{iii})$$

From Eqs. (i), (ii) and (iii), we have

$$I = (-1)\left[\frac{x}{2}\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1}(x) - 2\sin^{-1}(x) - 3\sqrt{1 - x^2} + C_1 + C_2\right]$$

$$= \frac{3}{2}\sin^{-1}(x) - \frac{x}{2}\sqrt{1 - x^2} + 3\sqrt{1 - x^2} + C \quad \textbf{(1)}$$

where, $C = -C_1 - C_2$

62. Do same as Q. No. 54.

$$\left[\textbf{Ans. } \frac{2}{3}(2 + x - x^2)^{3/2} + \frac{(2x - 1)}{2}\sqrt{2 + x - x^2} + \frac{9}{4}\sin^{-1}\left(\frac{2x - 1}{3}\right) + C\right]$$

63. Let $I = \int \frac{\log|x|}{(x+1)^2}dx = \int \underset{\text{I}}{\log|x|} \cdot \underset{\text{II}}{\frac{1}{(x+1)^2}}dx$

On applying integration by parts, we get

$$I = \log|x| \cdot \int \frac{dx}{(x+1)^2} - \int \frac{d}{dx}(\log|x|) \cdot \left(\int \frac{dx}{(x+1)^2}\right)dx$$ **(1)**

$$\left[\because \int \underset{\text{I}}{u}\,\underset{\text{II}}{v}\,dx = u\int v\,dx - \int\left(\frac{d}{dx}(u)\int v\,dx\right)dx\right]$$

$$= \log|x| \cdot \frac{(-1)}{x+1} + \int \frac{1}{x(x+1)}dx$$

$$= \frac{-\log|x|}{x+1} + I_1 \text{ (say)} \quad \text{...(i)}$$ **(1)**

Consider, $I_1 = \int \frac{dx}{x(x+1)}$

Now, by using partial fraction method,

Let $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

$\Rightarrow \quad 1 = A(x+1) + Bx$

On putting $x = 0$, we get $A = 1$

and again putting $x = -1$, we get $B = -1$ **(1)**

$$\therefore \quad I_1 = \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \int \frac{1}{x}dx - \int \frac{dx}{x+1}$$

$$= \log|x| - \log|x+1| + C \quad \text{...(ii)}$$

Now, from Eqs. (i) and (ii), we get

$$I = \frac{-\log|x|}{x+1} + \log|x| - \log|x+1| + C$$

$$= \frac{-\log|x|}{x+1} + \log\left|\frac{x}{x+1}\right| + C$$

$$\left[\because \log m - \log n = \log\frac{m}{n}\right]$$ **(1)**

64. Let $I = \int \frac{\sin(x-a)}{\sin(x+a)}dx$

Put $x + a = t \quad \Rightarrow \quad dx = dt$ **(1)**

$$\therefore I = \int \frac{\sin(t-a-a)}{\sin t}dt = \int \frac{\sin(t-2a)}{\sin t}dt$$ **(1)**

$$= \int \frac{\sin t \cos 2a - \cos t \sin 2a}{\sin t}dt$$

$$[\because \sin(A-B) = \sin A \cos B - \cos A \sin B]$$

$$= \int \cos 2a\, dt - \int \sin 2a \cdot \cot t\, dt$$

$$= \cos 2a\,[t] - \sin 2a\,[\log|\sin t|] + C_1$$

$$= (x+a)\cos 2a - \sin 2a \log|\sin(x+a)| + C_1$$

[put $t = x + a$] **(1)**

$$= x\cos 2a - \sin 2a \log|\sin(x+a)| + C,$$

where, $C = a\cos 2a + C_1$ **(1)**

65. Let $I = \int \underset{\text{II}}{e^{2x}} \underset{\text{I}}{\sin(3x+1)}dx \quad \text{...(i)}$

$$= \sin(3x+1)\int e^{2x}dx - \int\left\{\frac{d}{dx}\sin(3x+1)\int e^{2x}dx\right\}dx$$

[by using integration by parts]

$$= \frac{\sin(3x+1)\cdot e^{2x}}{2} - \int 3\cos(3x+1)\cdot\frac{e^{2x}}{2}dx$$ **(1)**

$$= \frac{e^{2x}\sin(3x+1)}{2} - \frac{3}{2}\int \underset{\text{II}}{e^{2x}} \underset{\text{I}}{\cos(3x+1)}dx$$

$$= \frac{e^{2x}\sin(3x+1)}{2} - \frac{3}{2}\left[\cos(3x+1)\int e^{2x}dx - \int\left\{\frac{d}{dx}\cos(3x+1)\int e^{2x}dx\right\}dx\right]$$ **(1)**

[again by using integration by parts]

$$= \frac{e^{2x}\sin(3x+1)}{2} - \frac{3}{2}\left[\cos(3x+1)\cdot\frac{e^{2x}}{2} - \int -3\sin(3x+1)\cdot\frac{e^{2x}}{2}dx\right] + C_1$$ **(1)**

$$\Rightarrow I = \frac{e^{2x}\sin(3x+1)}{2} - \frac{3}{4}e^{2x}\cos(3x+1) - \frac{9}{4}\int e^{2x}\sin(3x+1)dx + C_1$$

$$\Rightarrow I = \frac{e^{2x}\sin(3x+1)}{2} - \frac{3}{4}e^{2x}\cos(3x+1) - \frac{9}{4}I + C_1$$

[from Eq. (i)]

$$\Rightarrow \frac{13}{4}I = \frac{e^{2x}\sin(3x+1)}{2} - \frac{3e^{2x}\cos(3x+1)}{4} + C_1$$

$$\therefore \quad I = \frac{2e^{2x}\sin(3x+1)}{13} - \frac{3e^{2x}\cos(3x+1)}{13} + C$$

where, $C = \frac{4C_1}{13}$. **(1)**

66. Let $I = \int \frac{x^2}{(x^2+4)(x^2+9)}dx$

Consider, $\frac{x^2}{(x^2+4)(x^2+9)}$ and put $x^2 = t$

Then, $\frac{x^2}{(x^2+4)(x^2+9)} = \frac{t}{(t+4)(t+9)}$

Now, let $\frac{t}{(t+4)(t+9)} = \frac{A}{(t+4)} + \frac{B}{(t+9)}$ **(1)**

$\Rightarrow t = A(t+9) + B(t+4)$

On putting $t = -9$, we get

$-9 = -5B$

$\Rightarrow \quad B = \frac{9}{5}$

On putting $t = -4$, we get

$$-4 = 5A \Rightarrow A = \frac{-4}{5}$$ **(1)**

Thus, $\frac{t}{(t+4)(t+9)} = \frac{-4}{5(t+4)} + \frac{9}{5(t+9)}$

$$\therefore \quad I = \int \frac{x^2}{(x^2+4)(x^2+9)} dx$$

$$= \int \frac{-4}{5(x^2+4)} dx + \int \frac{9}{5(x^2+9)} dx$$ **(1)**

$$= \frac{-4}{5}\int \frac{dx}{x^2+(2)^2} + \frac{9}{5}\int \frac{dx}{x^2+(3)^2}$$

$$= \frac{-4}{5}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{9}{5}\cdot\frac{1}{3}\cdot\tan^{-1}\left(\frac{x}{3}\right) + C$$

$$\left[\because \int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right]$$

$$= \frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right) - \frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + C$$ **(1)**

67. Let $I = \int e^x \frac{(x^2+1)}{(x+1)^2} dx$

$$= \int e^x \frac{(x^2+1+2x-2x)}{(x+1)^2} dx$$ **(1/2)**

$$= \int e^x \left(\frac{(x+1)^2-2x}{(x+1)^2}\right) dx$$

$$= \int e^x \left(1 - \frac{2x}{(x+1)^2}\right) dx$$ **(1/2)**

$$= \int e^x\, dx - 2\int e^x \cdot \frac{x}{(x+1)^2} dx$$ **(1/2)**

$$= e^x - 2\int e^x \left(\frac{x+1-1}{(x+1)^2}\right) dx$$ **(1)**

$$= e^x - 2\int e^x \left(\frac{1}{(x+1)} + \frac{(-1)}{(x+1)^2}\right) dx$$

Now, consider $f(x) = \frac{1}{x+1}$, then $f'(x) = \frac{(-1)}{(x+1)^2}$

Thus, the above integrand is of the form $e^x[f(x) + f'(x)]$.

$$\therefore \quad I = e^x - 2e^x \frac{1}{(x+1)} + C$$ **(1)**

$$[\because \int e^x [f(x) + f'(x)] dx = e^x f(x) + C]$$

$$\Rightarrow \quad I = e^x\left(\frac{x+1-2}{x+1}\right) + C \Rightarrow I = e^x\left(\frac{x-1}{x+1}\right) + C$$ **(1/2)**

68. Here, integrand is of the form $(px-q)\sqrt{ax^2+bx+c}$, so firstly write $x-3$ as $x - 3 = A\frac{d}{dx}(x^2+3x-18) + B$ and find A and B. Then, integrate by using suitable method.

Let $I = \int (x-3)\sqrt{x^2+3x-18}\, dx$

Now, let us write $(x-3)$ as

$$x - 3 = A\frac{d}{dx}(x^2+3x-18) + B$$

$$\Rightarrow \quad x - 3 = A(2x+3) + B$$

On equating the coefficients of x and constant terms from both sides, we get

$$2A = 1$$

and $$3A + B = -3$$

$$\Rightarrow \quad A = \frac{1}{2} \text{ and } 3\times\frac{1}{2} + B = -3$$

$$\Rightarrow \quad A = \frac{1}{2} \text{ and } B = -\frac{3}{2} - 3$$

$$\Rightarrow \quad A = \frac{1}{2} \text{ and } B = -\frac{9}{2}$$ **(1)**

Thus, the given integral reduces in the following form

$$I = \int\left\{\frac{1}{2}(2x+3) - \frac{9}{2}\right\}\sqrt{x^2+3x-18}\, dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int (2x+3)\sqrt{x^2+3x-18}\, dx - \frac{9}{2}\int\sqrt{x^2+3x-18}\, dx$$

$$= \frac{1}{2}I_1 - \frac{9}{2}I_2, \text{ (say)} \qquad \text{...(i)}$$

Consider $I_1 = \int (2x+3)\sqrt{x^2+3x-18}\, dx$.

Put $x^2 + 3x - 18 = t$

$$\Rightarrow \quad (2x+3)\, dx = dt$$

$$\therefore \quad I_1 = \int t^{1/2}\, dt = \frac{2}{3}t^{3/2} + C_1$$

$$= \frac{2}{3}(x^2+3x-18)^{3/2} + C_1$$ **(1)**

$[\text{put } t = x^2 + 3x - 18]$

and $I_2 = \int\sqrt{x^2+3x-18}\, dx$

$$= \int\sqrt{\left(x+\frac{3}{2}\right)^2 - 18 - \frac{9}{4}}\, dx$$

$$= \int\sqrt{\left(x+\frac{3}{2}\right)^2 - \frac{81}{4}}\, dx$$

$$= \int \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\, dx$$

$$= \frac{\left(x + \frac{3}{2}\right)}{2} \sqrt{x^2 + 3x - 18}$$

$$- \frac{81}{8} \log\left|\left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x - 18}\right| + C_2$$

$$\left[\because \int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log|x + \sqrt{x^2 - a^2}| + C\right]$$

$$= \frac{2x + 3}{4} \sqrt{x^2 + 3x - 18}$$

$$- \frac{81}{8} \log\left|\frac{2x + 3}{2} + \sqrt{x^2 + 3x - 18}\right| + C_2$$

(1)

On putting the values of I_1 and I_2 in Eq. (i), we get

$$I = \frac{1}{2}\left[\frac{2}{3}(x^2 + 3x - 18)^{3/2} + C_1\right]$$

$$- \frac{9}{2}\left[\frac{2x + 3}{4}\sqrt{x^2 + 3x - 18}\right.$$

$$\left.- \frac{81}{8} \log\left|\frac{2x + 3}{2} + \sqrt{x^2 + 3x - 18}\right| + C_2\right]$$

$$\Rightarrow \quad I = \frac{1}{3}(x^2 + 3x - 18)^{3/2}$$

$$- \frac{9}{8}(2x + 3)\sqrt{x^2 + 3x - 18}$$

$$+ \frac{729}{16} \log\left|\frac{2x + 3}{2} + \sqrt{x^2 + 3x - 18}\right| + C$$

where, $C = \frac{C_1}{2} - \frac{9C_2}{2}$. (1)

69. Let $I = \int \frac{x + 2}{\sqrt{x^2 + 5x + 6}}\, dx$

Now, let us write, $(x + 2)$ as

$$x + 2 = A \frac{d}{dx}(x^2 + 5x + 6) + B$$

$$\Rightarrow \quad x + 2 = A(2x + 5) + B$$

On equating the coefficients of x and constant terms from both sides, we get

$$2A = 1 \text{ and } 5A + B = 2$$

$$\Rightarrow \quad A = \frac{1}{2} \text{ and } B = -\frac{1}{2}$$ (1)

$$\therefore \quad I = \int \frac{\left\{\frac{1}{2}(2x + 5) - \frac{1}{2}\right\}}{\sqrt{x^2 + 5x + 6}}\, dx$$

$$= \frac{1}{2}\int \frac{2x + 5}{\sqrt{x^2 + 5x + 6}}\, dx - \frac{1}{2}\int \frac{1}{\sqrt{x^2 + 5x + 6}}\, dx$$

$$= \frac{1}{2} I_1 - \frac{1}{2} I_2 \text{ (say)} \quad \text{...(i)}$$

Consider, $I_1 = \int \frac{2x + 5}{\sqrt{x^2 + 5x + 6}}\, dx$

Put $x^2 + 5x + 6 = t \Rightarrow (2x + 5)\, dx = dt$

$$\therefore \quad I_1 = \int \frac{1}{\sqrt{t}}\, dt = 2\sqrt{t} + C_1$$

$$= 2\sqrt{x^2 + 5x + 6} + C_1 \quad \text{...(ii)}$$ (1)

[put $t = x^2 + 5x + 6$]

and $I_2 = \int \frac{1}{\sqrt{x^2 + 5x + 6}}\, dx$

$$= \int \frac{1}{\sqrt{x^2 + 2 \times \frac{5}{2} \times x + 6 + \frac{25}{4} - \frac{25}{4}}}\, dx$$

$$= \int \frac{1}{\sqrt{\left(x + \frac{5}{2}\right)^2 + 6 - \frac{25}{4}}}\, dx$$

$$= \int \frac{1}{\sqrt{\left(x + \frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\, dx$$

$$= \log\left|\left(x + \frac{5}{2}\right) + \sqrt{\left(x + \frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + C_2$$

$$\left[\because \int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C\right]$$

$$\Rightarrow \quad I_2 = \log\left|x + \frac{5}{2} + \sqrt{x^2 + 5x + 6}\right| + C_2 \quad \text{...(iii)}$$ (1)

On putting the values of I_1 and I_2 from Eqs. (ii) and (iii) in Eq. (i), we get

$$I = \frac{1}{2}[2\sqrt{x^2 + 5x + 6} + C_1]$$

$$- \frac{1}{2}\left[\log\left|x + \frac{5}{2} + \sqrt{x^2 + 5x + 6}\right| + C_2\right]$$

$$= \sqrt{x^2 + 5x + 6} + \frac{C_1}{2}$$

$$- \frac{1}{2}\log\left|x + \frac{5}{2} + \sqrt{x^2 + 5x + 6}\right| - \frac{C_2}{2}$$

$$\Rightarrow \quad I = \sqrt{x^2 + 5x + 6}$$

$$- \frac{1}{2}\log\left|x + \frac{5}{2} + \sqrt{x^2 + 5x + 6}\right| + C$$

where, $C = \frac{C_1}{2} - \frac{C_2}{2}$. (1)

70. Do same as Q. 68.

$$\left[\text{Ans. } I = (x^2 + x + 1)^{3/2} - \frac{7}{8}(2x+1)\sqrt{x^2+x+1} - \frac{21}{16}\log\left|\frac{(2x+1)}{2} + \sqrt{x^2+x+1}\right| + C\right]$$

71. Let $I = \int \frac{5x-2}{1+2x+3x^2}dx$ ...(i)

Here, $(5x-2)$ can be written as

$$5x - 2 = A\frac{d}{dx}(1+2x+3x^2) + B$$

$$\Rightarrow \quad 5x - 2 = A(2+6x) + B$$

On comparing the coefficients of x and constant terms, we get

$$5 = 6A \Rightarrow A = \frac{5}{6}$$

and $-2 = 2A + B \Rightarrow B = -2A - 2$

$$= -\frac{5}{3} - 2 = -\frac{11}{3} \quad \left[\because A = \frac{5}{6}\right]$$

Then, from Eq. (i), we get

$$I = \int \frac{\frac{5}{6}(2+6x) - \frac{11}{3}}{1+2x+3x^2}dx$$

$$\Rightarrow I = \int \frac{\frac{5}{6}(2+6x)}{1+2x+3x^2}dx - \int \frac{\left(\frac{11}{3}\right)}{1+2x+3x^2}dx \quad \textbf{(1)}$$

$$\Rightarrow I = I_1 - I_2 \quad \text{...(ii)}$$

where, $I_1 = \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}dx$

Put $1+2x+3x^2 = t \Rightarrow (2+6x)dx = dt$

$$\therefore \quad I_1 = \frac{5}{6}\int \frac{dt}{t} = \frac{5}{6}\log|t| + C_1$$

$$= \frac{5}{6}\log|1+2x+3x^2| + C_1$$

[put $t = 1 + 2x + 3x^2$] **(1)**

and $I_2 = \frac{11}{3}\int \frac{dx}{3x^2+2x+1}$

$$= \frac{11}{9}\int \frac{dx}{\left[x^2 + \frac{2x}{3} + \frac{1}{3}\right]}$$

$$= \frac{11}{9}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2 + \frac{2}{9}}$$

$$= \frac{11}{9}\cdot\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right) + C_2$$

$$\left[\because \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C\right]$$

$$= \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C_2 \quad \textbf{(1)}$$

On putting the values of I_1 and I_2 in Eq. (ii), we get

$$I = \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C,$$

where $C = C_1 - C_2$ **(1)**

72. First, put $x^2 = t$ and use partial fraction to write integrand in simplest form, then integrate by using suitable formula.

Let $I = \int \frac{x^3}{x^4+3x^2+2}dx$

Put $x^2 = t \Rightarrow 2x = \frac{dt}{dx}$

$$\Rightarrow \quad x\,dx = \frac{dt}{2} \quad \textbf{(1)}$$

$$\therefore \quad I = \frac{1}{2}\int \frac{t}{t^2+3t+2}dt = \frac{1}{2}\int \frac{t}{(t+2)(t+1)}dt$$

$$= \frac{1}{2}\left[\int \frac{2}{t+2}dt - \int \frac{1}{t+1}dt\right] \quad \textbf{(1½)}$$

[by using partial fraction]

$$= \frac{1}{2}[2\log|t+2| - \log|t+1|] + C$$

$$= \log|t+2| - \frac{1}{2}\log|t+1| + C$$

$$= \log|t+2| - \log\sqrt{t+1} + C$$

$$= \log\left|\frac{t+2}{\sqrt{t+1}}\right| + C = \log\left|\frac{x^2+2}{\sqrt{x^2+1}}\right| + C \quad \textbf{(1½)}$$

[put $t = x^2$]

73. Do same as Q. No. 59.

$$[\text{Ans. } -\sqrt{1-x^2}\cos^{-1}x - x + C]$$

74. First, use $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$ to write numerator of integrand in simplest form and then integrate by using suitable method.

Let $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x}dx$

$$\Rightarrow I = \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\left[\begin{array}{l}(\sin^2 x + \cos^2 x)^3 \\ -3\sin^2 x \cos^2 x (\sin^2 x + \cos^2 x)\end{array}\right]}{\sin^2 x \cos^2 x} dx \quad \text{(1)}$$

$[\because a^3 + b^3 = (a+b)^3 - 3ab(a+b)]$

$$= \int \frac{(1)^3 - 3\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx$$

$[\because \sin^2 x + \cos^2 x = 1]$ (1½)

$$= \int \frac{1}{\sin^2 x \cos^2 x} dx - 3\int \frac{\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx - 3\int 1\, dx$$

$$= \int \left[\frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x}\right] dx - 3\int 1\, dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x)\, dx - 3\int 1\, dx$$

$$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx - 3\int 1\, dx$$

$$= \tan x - \cot x - 3x + C \quad \text{(1½)}$$

75. First, use trigonometric formulae $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$ to write integrand in simplest form and then apply integration by parts to integrate.

Let $I = \int e^{2x}\left(\frac{1 - \sin 2x}{1 - \cos 2x}\right) dx$

$$= \int e^{2x}\left(\frac{1 - 2\sin x \cos x}{2\sin^2 x}\right) dx \quad \text{(1)}$$

$\left[\begin{array}{l}\because 1 - \cos 2x = 2\sin^2 x \\ \text{and } \sin 2x = 2\sin x \cos x\end{array}\right]$

$$= \frac{1}{2}\int e^{2x}(\operatorname{cosec}^2 x - 2\cot x)\, dx \quad \text{(1½)}$$

$$= \frac{1}{2}\int \underset{\text{I}}{e^{2x}} \underset{\text{II}}{\operatorname{cosec}^2 x}\, dx - \int e^{2x} \cot x\, dx$$

$$= \frac{1}{2}\left[e^{2x}\int \operatorname{cosec}^2 x\, dx - \int\left\{\frac{d}{dx}(e^{2x})\int \operatorname{cosec}^2 x\, dx\right\} dx\right] - \int e^{2x} \cot x\, dx$$

[by using integration by parts]

$$= \frac{1}{2}[-e^{2x}\cot x + \int 2e^{2x}\cot x\, dx] + C - \int e^{2x}\cot x\, dx$$

$$= -\frac{e^{2x}}{2}\cot x + \int e^{2x}\cot x\, dx - \int e^{2x}\cot x\, dx + C$$

$$= -\frac{e^{2x}}{2}\cot x + C \quad \text{(1½)}$$

76. Let $I = \int \frac{3x+1}{(x+1)^2(x+3)} dx$

Again, let $\frac{3x+1}{(x+1)^2(x+3)}$

$$= \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+3)} \quad \ldots\text{(i)}$$

$$\Rightarrow 3x + 1 = A(x+1)(x+3) + B(x+3) + C(x+1)^2$$

$$\Rightarrow 3x + 1 = A(x^2 + 4x + 3) + B(x+3) + C(x^2 + 1 + 2x)$$

$$\Rightarrow 3x + 1 = (A + C)x^2 + (4A + B + 2C)x + 3A + 3B + C \quad \text{(1)}$$

On comparing like powers of x from both sides, we get

$$A + C = 0$$
$$4A + B + 2C = 3$$

and $3A + 3B + C = 1$

On solving, we get $A = 2, B = -1$

and $C = -2$ (1)

$\therefore$ Eq. (i) becomes

$$\frac{3x+1}{(x+1)^2(x+3)} = \frac{2}{(x+1)} - \frac{1}{(x+1)^2} - \frac{2}{(x+3)}$$

On integrating both sides, we get

$$\int \frac{3x+1}{(x+1)^2(x+3)} dx = \int \frac{2}{(x+1)} dx - \int \frac{1}{(x+1)^2} dx - \int \frac{2}{(x+3)} dx \quad \text{(1)}$$

$$\Rightarrow I = 2\log|x+1| - \frac{(x+1)^{-2+1}}{(-2+1)} - 2\log|x+3| + C$$

$$= 2\log\left|\frac{x+1}{x+3}\right| + \frac{1}{(x+1)} + C \quad \text{(1)}$$

$\left[\because \log m - \log n = \log\frac{m}{n}\right]$

77. Let $I = \int \frac{2x^2+1}{x^2(x^2+4)} dx$

Consider $\frac{2x^2+1}{x^2(x^2+4)}$ and put $x^2 = t$ and then $\frac{2x^2+1}{x^2(x^2+4)} = \frac{2t+1}{t(t+1)}$ and by using partial fraction, we get

$$\frac{2t+1}{t(t+4)} = \frac{A}{t} + \frac{B}{t+4} \Rightarrow 2t + 1 = A(t+4) + Bt \quad \text{(1/2)}$$

On comparing the coefficients of t and constant terms, we get

$$2 = A + B \text{ and } 1 = 4A \Rightarrow A = \frac{1}{4}$$

$\therefore \quad B = 2 - A = 2 - \frac{1}{4} = \frac{7}{4}$ **(1)**

Thus, $\frac{2x^2+1}{x^2(x^2+4)} = \frac{1}{4x^2} + \frac{7}{4(x^2+4)}$

$\therefore \quad I = \frac{1}{4}\int \frac{dx}{x^2} + \frac{7}{4}\int \frac{dx}{x^2+4}$ **(1)**

$= -\frac{1}{4x} + \frac{7}{4} \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

$\left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C\right]$

$= -\frac{1}{4x} + \frac{7}{8}\tan^{-1}\left(\frac{x}{2}\right) + C$ **(1½)**

78. Do same as Q. No. 77.

$\left[\textbf{Ans.} -\frac{1}{14}\tan^{-1}\left(\frac{x}{2}\right) + \frac{8}{35}\tan^{-1}\left(\frac{x}{5}\right) + C\right]$

79. Let $\quad I = \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}dx$

$= \int \frac{(2\cos^2 x - 1) - (2\cos^2\alpha - 1)}{\cos x - \cos\alpha}dx$

$[\because \cos 2\theta = 2\cos^2\theta - 1]$ **(1)**

$= \int \frac{2(\cos^2 x - \cos^2\alpha)}{(\cos x - \cos\alpha)}dx$

$= \int \frac{2(\cos x - \cos\alpha)(\cos x + \cos\alpha)}{(\cos x - \cos\alpha)}dx$

$[\because a^2 - b^2 = (a+b)(a-b)]$ **(1)**

$= \int 2(\cos x + \cos\alpha)\,dx$

$= 2\left[\int \cos x\, dx + \cos\alpha \int dx\right]$

$\therefore \quad I = 2(\sin x + x\cos\alpha) + C$ **(2)**

80. Do same as Q. No. 69.

$\left[\textbf{Ans.} = \sqrt{x^2+2x+3} + \log\left|(x+1) + \sqrt{x^2+2x+3}\right| + C\right]$

81. Let $\quad I = \int \frac{dx}{x(x^5+3)} = \int \frac{x^4}{x^5(x^5+3)}dx$

[$\because$ multiplying numerator and denominator by x^4]

Put $\quad t = x^5 \Rightarrow dt = 5x^4\, dx$

$\therefore \quad I = \int \frac{dt}{5t(t+3)}$ **(1)**

$= \frac{1}{5}\int \frac{1}{3}\left[\frac{1}{t} - \frac{1}{t+3}\right]dt$

$= \frac{1}{15}[\log|t| - \log|t+3|] + C$ **(1)**

$= \frac{1}{15}\log\left|\frac{t}{t+3}\right| + C$ **(1)**

$= \frac{1}{15}\log\left|\frac{x^5}{x^5+3}\right| + C \quad [\text{put } t = x^5]$ **(1)**

82. Do same as Q. No. 81. $\left[\textbf{Ans.} \frac{1}{3}\log\left|\frac{x^3}{x^3+1}\right| + C\right]$

83. Do same as Q. No. 81. $\left[\textbf{Ans.} \frac{1}{8}\log\left|\frac{x}{(x^3+8)^{1/3}}\right| + C\right]$

84. Let $I = \int \frac{\sqrt{1-\sin x}}{1+\cos x} \cdot e^{\frac{-x}{2}}dx$

Put $\frac{-x}{2} = t \Rightarrow dx = -2dt$

$\Rightarrow \quad I = \int \frac{\sqrt{1-\sin(-2t)}}{1+\cos(-2t)} \cdot e^t(-2dt) \quad [\because x = -2t]$

$= -2\int e^t \frac{\sqrt{1+\sin 2t}}{1+\cos 2t}dt$

$[\because \sin(-\theta) = -\sin\theta \text{ and } \cos(-\theta) = \cos\theta]$ **(1)**

$= -2\int e^t\left(\frac{\sqrt{(\cos t + \sin t)^2}}{2\cos^2 t}\right)dt$

$= -2\int e^t\left(\frac{\cos t + \sin t}{2\cos^2 t}\right)dt$ **(1)**

$= -\int e^t(\sec t + \tan t \sec t)\, dt$ **(1/2)**

Now, consider $f(t) = \sec t$, then $f'(t) = \sec t \tan t$

Thus, the above integrand is of the form

$\int e^t[f(t) + f'(t)]dt$ **(1)**

$\therefore \quad I = -e^t \sec t + C$

$[\because \int e^t[(f(t) + f'(t)]dt = e^t f(t) + C]$

$= -e^{-x/2}\sec\frac{x}{2} + C$

$\left[\because t = \frac{-x}{2} \text{ and } \sec(-\theta) = \sec\theta\right]$ **(1/2)**

85. Let $I = \int \frac{3x+5}{x^3 - x^2 - x + 1}dx$

$= \int \frac{3x+5}{x^2(x-1) - 1(x-1)}dx$

$= \int \frac{3x+5}{(x-1)(x^2-1)}dx$

$= \int \frac{3x+5}{(x-1)(x-1)(x+1)}dx = \int \frac{3x+5}{(x-1)^2(x+1)}dx$

Now, do same as Q. No. 67.

$$\left[\textbf{Ans. } \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{x-1} + C\right]$$

86. It is a product of three trigonometric functions. So, firstly we take two functions at a time and use the relation $2\sin A \sin B = \cos(A-B) - \cos(A+B)$ and then integrate it.

Let $I = \int \sin x \sin 2x \sin 3x \, dx$

$= \frac{1}{2}\int \sin x \,(2\sin 2x \sin 3x)\, dx$

[multiplying numerator and denominator by 2]

$= \frac{1}{2}\int \sin x [\cos(2x - 3x) - \cos(2x + 3x)] dx$

[$\because 2\sin A \sin B = \cos(A - B) - \cos(A + B)$] (1)

$= \frac{1}{2}\int \sin x [\cos(-x) - \cos 5x] \, dx$

$= \frac{1}{2}\int \sin x (\cos x - \cos 5x) dx$ [$\because \cos(-x) = \cos x$]

$= \frac{1}{2}\int \sin x \cos x \, dx - \frac{1}{2}\int \sin x \cos 5x \, dx$ (1)

$= \frac{1}{4}\int 2\sin x \cos x \, dx - \frac{1}{4}\int (2\sin x \cos 5x)\, dx$

[multiplying numerator and denominator by 2]

$= \frac{1}{4}\int \sin 2x \, dx - \frac{1}{4}\int \{\sin(x + 5x) + \sin(x - 5x)\} dx$

$\left[\because 2\sin x \cos x = \sin 2x \text{ and } 2\sin A \cos B = \sin(A + B) + \sin(A - B)\right]$

$= \frac{1}{4}\int \sin 2x \, dx - \frac{1}{4}\int [\sin 6x + \sin(-4x)] dx$ (1)

$= \frac{1}{4}\int \sin 2x \, dx - \frac{1}{4}\int (\sin 6x - \sin 4x) dx$

[$\because \sin(-\theta) = -\sin\theta$]

$= \frac{-1}{4}\cdot\frac{\cos 2x}{2} - \frac{1}{4}\left[\frac{-\cos 6x}{6} + \frac{\cos 4x}{4}\right] + C$

$\left[\because \int \sin ax \, dx = \frac{-\cos ax}{a}\right]$

$= \frac{-\cos 2x}{8} + \frac{\cos 6x}{24} - \frac{\cos 4x}{16} + C$ (1)

87. Here, denominator is a product of two algebraic functions. So, firstly we use partial fraction method and then integrate it.

Let $I = \int \frac{2}{(1 - x)(1 + x^2)} dx$

By using partial fraction,

let $\frac{2}{(1 - x)(1 + x^2)} = \frac{A}{1 - x} + \frac{Bx + C}{1 + x^2}$ ...(i)

$\Rightarrow \frac{2}{(1 - x)(1 + x^2)} = \frac{A(1 + x^2) + (Bx + C)(1 - x)}{(1 - x)(1 + x^2)}$

$\Rightarrow 2 = A(1 + x^2) + (Bx + C)(1 - x)$

$\Rightarrow 2 = A + Ax^2 + Bx + C - Bx^2 - Cx$

$\Rightarrow 2 = (A - B)x^2 + (B - C)x + (A + C)$

On comparing coefficients of x^2, x and constant terms from both sides, we get

$A - B = 0$ ...(ii)

$B - C = 0$ ...(iii)

and $A + C = 2$ ...(iv)

On solving Eqs. (ii), (iii) and (iv), we get

$A = 1, B = 1$ and $C = 1$

Now, Eq. (i) becomes

$\frac{2}{(1 - x)(1 + x^2)} = \frac{1}{1 - x} + \frac{x + 1}{1 + x^2}$

On integrating both sides w.r.t. x, we get

$\int \frac{2}{(1 - x)(1 + x^2)} dx = \int \frac{1}{1 - x} dx + \int \frac{x + 1}{1 + x^2} dx$

$= -\log|1 - x| + \int \frac{x}{1 + x^2} dx + \int \frac{1}{1 + x^2} dx$

$= -\log|1 - x| + \frac{1}{2}\int \frac{2x}{1 + x^2} dx + \int \frac{1}{1 + x^2} dx$

$= -\log|1 - x| + \frac{1}{2}\log|1 + x^2| + \tan^{-1} x + C$ (1)

$\left[\because \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C\right]$

88. Let $I = \int \left(\frac{1 + \sin x}{1 + \cos x}\right) e^x \, dx$

$= \int \frac{1 + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} \cdot e^x \, dx$ (1)

$\left[\because \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \text{ and } 1 + \cos x = 2\cos^2\frac{x}{2}\right]$

$= \int \left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right) e^x \, dx$

$= \int e^x \left(\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2}\right) dx$

We know that,

$\int e^x [f(x) + f'(x)] dx = e^x f(x) + C,$ (1)

Here, $f(x) = \tan\frac{x}{2}$

$\Rightarrow f'(x) = \frac{1}{2}\sec^2\frac{x}{2}$ (1)

$\therefore I = e^x \tan\frac{x}{2} + C$ (1)

89. Let $I = \int \frac{x^2}{(x\sin x + \cos x)^2} dx$

$\Rightarrow \quad I = \int \frac{x\cos x}{(x\sin x + \cos x)^2} \cdot x\sec x \, dx \quad \ldots(i)$ **(1)**

Put $\quad x\sin x + \cos x = t$

$\Rightarrow \quad (x\cos x + \sin x - \sin x)\, dx = dt$

$\Rightarrow \quad x\cos x \, dx = dt$

$\therefore \quad I_1 = \int \frac{x\cos x}{(x\sin x + \cos x)^2} dx$ (say)

$= \int \frac{dt}{t^2} = \frac{-1}{t} = \frac{-1}{x\sin x + \cos x}$

[put $t = x\sin x + \cos x$] **(1)**

Now, integrating Eq. (i) by parts, we get

$I = \int \underset{\text{I}}{x\sec x} \cdot \underset{\text{II}}{\frac{x\cos x}{(x\sin x + \cos x)^2}} dx$

$= x\sec x \cdot \frac{(-1)}{x\sin x + \cos x}$

$- \int (1 \cdot \sec x + x\sec x \tan x) \cdot \frac{-dx}{x\sin x + \cos x}$ **(1)**

$= \frac{-x\sec x}{x\sin x + \cos x} + \int \sec x \left(1 + \frac{x\sin x}{\cos x}\right) \frac{dx}{x\sin x + \cos x}$

$= \frac{-x\sec x}{x\sin x + \cos x} + \int \sec^2 x \, dx$

$= \frac{-x\sec x}{x\sin x + \cos x} + \tan x + C$ **(1)**

90. Do same as Q. No. 65.

$\left[\textbf{Ans. } \frac{1}{5} e^{2x} (2\sin x - \cos x) + C\right]$

91. Do same as Q. No. 69.

$\left[\textbf{Ans. } 3\sqrt{x^2 - 8x + 7} + 17 \log |(x-4) + \sqrt{(x-4)^2 - 9}| + C\right]$

92. First, divide numerator and denominator by x^2 and reduce the integrand in standard form.

Let $\quad I = \int \frac{x^2 + 4}{x^4 + 16} dx$

On dividing numerator and denominator by x^2, we get

$I = \int \frac{\left(1 + \frac{4}{x^2}\right)}{\left(x^2 + \frac{16}{x^2}\right)} dx = \int \frac{\left(1 + \frac{4}{x^2}\right)}{x^2 + \left(\frac{4}{x}\right)^2 - 8 + 8} dx$

$= \int \frac{\left(1 + \frac{4}{x^2}\right)}{\left(x - \frac{4}{x}\right)^2 + 8} dx$ **(1)**

Put $\quad x - \frac{4}{x} = t$

$\Rightarrow \quad \left(1 + \frac{4}{x^2}\right) dx = dt$

$\therefore \quad I = \int \frac{dt}{t^2 + 8} = \int \frac{dt}{t^2 + (2\sqrt{2})^2}$ **(1)**

$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{t}{2\sqrt{2}}\right) + C$

$\left[\because \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C\right]$

$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x - \frac{4}{x}}{2\sqrt{2}}\right) + C \quad \left[\text{put } t = x - \frac{4}{x}\right]$

$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 4}{2\sqrt{2}x}\right) + C$ **(2)**

93. Do same as Q. No. 92.

$\left[\textbf{Ans. } \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{x\sqrt{2}}\right) + C\right]$

94. Let $I = \int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx$

$= \int \frac{\sin x - \cos x}{\sqrt{1 + \sin 2x - 1}} dx$

$= \int \frac{\sin x - \cos x}{\sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x - 1}} dx$

$[\because \sin^2\theta + \cos^2\theta = 1]$

$= \int \frac{\sin x - \cos x}{\sqrt{(\sin x + \cos x)^2 - 1}} dx$ **(1)**

Put $\quad \sin x + \cos x = t$

$\Rightarrow \quad (\cos x - \sin x)\, dx = dt$

$\therefore \quad I = \int \frac{-dt}{\sqrt{t^2 - 1}} = -\log \left| t + \sqrt{t^2 - 1} \right| + C$ **(1)**

$\left[\because \int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C\right]$

$\Rightarrow \quad I = -\log |(\sin x + \cos x) + \sqrt{(\sin x + \cos x)^2 - 1}| + C$

[put $t = \sin x + \cos x$] **(1)**

$= -\log |(\sin x + \cos x) + \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x - 1}| + C$

$= -\log |(\sin x + \cos x) + \sqrt{\sin 2x}| + C$ **(1)**

95. Let $I = \int \frac{2x}{(1+x^2)(x^2+3)} dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt$ **(1)**

$\therefore \quad I = \int \frac{dt}{(t+1)(3+t)}$

By using partial fraction,

let $\frac{1}{(t+1)(3+t)} = \frac{A}{(1+t)} + \frac{B}{(3+t)}$ ...(i) **(1)**

$\Rightarrow \quad 1 = A(3+t) + B(1+t)$

On putting $t = -3$, we get

$1 = -2B \Rightarrow B = -\frac{1}{2}$

Now, on putting $t = -1$, we get

$1 = 2A \Rightarrow A = 1/2$

On putting $A = \frac{1}{2}$ and $B = -\frac{1}{2}$ in Eq. (i), we get

$\frac{1}{(1+t)(3+t)} = \frac{1/2}{1+t} + \frac{-1/2}{3+t}$ **(1/2)**

On integrating both sides, we get

$\int \frac{1}{(1+t)(3+t)} dt = \frac{1}{2}\int \frac{1}{1+t} dt - \frac{1}{2}\int \frac{1}{3+t} dt$

$= \frac{1}{2}\log|1+t| - \frac{1}{2}\log|3+t|$

$= \frac{1}{2}\log|1+x^2| - \frac{1}{2}\log|3+x^2| + C$ [put $t = x^2$]

$\therefore \quad I = \frac{1}{2}\log\left|\frac{1+x^2}{3+x^2}\right| + C$

$\left[\because \log m - \log n = \log \frac{m}{n}\right]$ **(1½)**

96. Do same as Q. No. 69.

$\left[\textbf{Ans. } 5\sqrt{x^2+4x+10} - 7\log|x+2+\sqrt{x^2+4x+10}| + C\right]$

97. Do same as Q. No. 75. $\left[\textbf{Ans. } \frac{1}{2}e^{2x}\tan x + C\right]$

98. Do same as Q. No. 77.

$\left[\textbf{Ans. } \tan^{-1}x - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C\right]$

99. Use integration by parts, i.e.

$\int \underset{\text{I}}{u} \cdot \underset{\text{II}}{v}\,dx = \left[u\int v\,dx - \int\left\{\frac{d}{dx}u \cdot \int v\,dx\right\}dx\right]$ and choose Ist function with the help of ILATE procedure.

Let $I = \int \left[\log(\log x) + \frac{1}{(\log x)^2}\right] dx$

$= \int \underset{\text{I}}{\log(\log x)} \cdot \underset{\text{II}}{1}\,dx + \int \frac{1}{(\log x)^2} dx$ **(1/2)**

Using integration by parts in first integral, we get

$I = \log(\log x)\int 1\,dx - \int\left[\frac{d}{dx}\log(\log x)\int 1\,dx\right]dx + \int \frac{1}{(\log x)^2} dx$ **(1/2)**

$= \log(\log x)\cdot x - \int \frac{1}{\log x}\cdot\frac{1}{x}\cdot x\,dx + \int \frac{1}{(\log x)^2} dx$

$= x\log(\log x) - \int \underset{\text{I}}{(\log x)^{-1}}\,\underset{\text{II}}{1}\,dx + \int \frac{1}{(\log x)^2} dx$ **(1)**

Again, applying integration by parts in the middle integral, we get

$I = x\log(\log x) - \left[(\log x)^{-1}\int 1\,dx - \int\left\{\frac{d}{dx}(\log x)^{-1}\int 1\,dx\right\}dx\right] + \int \frac{1}{(\log x)^2} dx + C$ **(1)**

$= x\log(\log x) - \left[\frac{x}{\log x} - \int -(\log x)^{-2}\cdot\frac{1}{x}\cdot x\,dx\right] + \int \frac{1}{(\log x)^2} dx + C$

$= x\log(\log x) - \frac{x}{\log x} - \int \frac{1}{(\log x)^2} dx + \int \frac{1}{(\log x)^2} dx + C$

$= x\log(\log x) - \frac{x}{\log x} + C$ **(1)**

100. Do same as Q. No. 69.

$\left[\textbf{Ans. } \sqrt{x^2-5x+6} + \frac{9}{2}\log\left|\left(x-\frac{5}{2}\right) + \sqrt{(x-2)(x-3)}\right| + C\right]$

101. Let $I = \int \frac{1-x^2}{x(1-2x)} dx$

$= \int \frac{1-x^2}{x-2x^2} dx$

Given integral can be rewritten as

$I = \int \left[\frac{1}{2} + \frac{1-\frac{1}{2}x}{x(1-2x)}\right] dx$

$\Rightarrow \quad I = \frac{1}{2}\int dx + \int \frac{1-\frac{1}{2}x}{x(1-2x)} dx$ ...(i) **(1/2)**

By using partial fraction,

let $$\frac{\left(1-\frac{1}{2}x\right)}{x(1-2x)} = \frac{A}{x} + \frac{B}{1-2x} \quad \dots(ii)$$

$$\Rightarrow \quad 1-\frac{1}{2}x = A(1-2x) + Bx \quad \dots(iii) \quad \textbf{(1/2)}$$

On putting $x = 0$ and $x = \frac{1}{2}$ in Eq. (iii), we get

$$1 - 0 = A(1-0) + 0 \Rightarrow A = 1$$

and $$1-\frac{1}{4} = A(1-1) + \frac{1}{2}B$$

$$\Rightarrow \quad \frac{3}{4} = \frac{1}{2}B$$

$$\Rightarrow \quad B = \frac{3}{2} \quad \textbf{(1)}$$

On putting the values of A and B in Eq. (ii), we get

$$\frac{1-\frac{1}{2}x}{x(1-2x)} = \frac{1}{x} + \frac{3/2}{1-2x} \quad \textbf{(1/2)}$$

Then, from Eq. (i), we get

$$I = \frac{1}{2}\int dx + \int \frac{1}{x}dx + \int \frac{3/2}{1-2x}dx$$

$$= \frac{1}{2}x + \log|x| + \frac{3}{2}\frac{\log|1-2x|}{-2} + C \quad \textbf{(1/2)}$$

$$\left[\because \int \frac{1}{1-ax}dx = -\frac{1}{a}\log|1-ax| + C\right]$$

$$= \frac{1}{2}x + \log|x| - \frac{3}{4}\log|1-2x| + C \quad \textbf{(1)}$$

102. Let $$I = \int e^x \left(\frac{\sin 4x - 4}{1-\cos 4x}\right)dx$$

$$= \int e^x \left(\frac{2\sin 2x \cos 2x - 4}{2\sin^2 2x}\right)dx \quad \textbf{(1)}$$

$$\left[\because \sin 2x = 2\sin x \cos x \text{ and } \cos 2x = 1 - 2\sin^2 x\right]$$

$$= \int e^x \left(\frac{2\sin 2x \cos 2x}{2\sin^2 2x} - \frac{4}{2\sin^2 2x}\right)dx$$

$$= \int e^x (\cot 2x - 2\,\text{cosec}^2\, 2x)\,dx \quad \textbf{(1)}$$

We know that,

$$\int e^x [f(x) + f'(x)]\,dx = e^x f(x) + C$$

Here, $$f(x) = \cot 2x$$

$$\Rightarrow \quad f'(x) = -2\,\text{cosec}^2\, 2x$$

$$\therefore \quad I = e^x \cot 2x + C \quad \textbf{(2)}$$

103. First, divide numerator and denominator by $\cos^4 x$ to convert integrand in terms of $\tan x$ and then put $\tan x = t$ and convert integrand into standard form which can integrate easily.

Let $$I = \int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x}dx$$

On dividing numerator and denominator by $\cos^4 x$ in RHS, we get

$$I = \int \frac{\sec^4 x}{\tan^4 x + \tan^2 x + 1}dx$$

$$= \int \frac{(\sec^2 x)(\sec^2 x)}{\tan^4 x + \tan^2 x + 1}dx \quad \textbf{(1)}$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$ and

$$\sec^2 x = 1 + \tan^2 x = 1 + t^2 \quad \textbf{(1)}$$

$$\therefore I = \int \frac{1+t^2}{t^4+t^2+1}dt = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1}dt \quad \textbf{(1)}$$

[$\because$ divide numerator and denominator by t^2]

$$= \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}-2+2+1}dt = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+3}dt \quad \textbf{(1)}$$

Again, put $u = t - \frac{1}{t}$

$$\Rightarrow \quad \left(1+\frac{1}{t^2}\right)dt = du$$

$$\therefore \quad I = \int \frac{du}{u^2 + (\sqrt{3})^2}$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C$$

$$\left[\because \int \frac{dx}{\sqrt{x^2+a^2}} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C\right] \quad \textbf{(1)}$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{3}}\right) + C \quad \left[\because u = t - \frac{1}{t}\right]$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{3}t}\right) + C$$

$$\therefore \quad I = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{3}\tan x}\right) + C \quad \textbf{(1)}$$

[put $t = \tan x$]

104. Let $I = \int [\sqrt{\cot x} + \sqrt{\tan x}]\, dx$

$$= \int \sqrt{\tan x}\,(1 + \cot x)\, dx$$

Put $\tan x = t^2 \Rightarrow \sec^2 x\, dx = 2t\, dt$

$$\Rightarrow \quad dx = \frac{2t}{1 + t^4} \quad \text{(1)}$$

$$[\because 1 + \tan^2 x = \sec^2 x \Rightarrow 1 + t^4 = \sec^2 x]$$

$$\therefore \quad I = \int t\left(1 + \frac{1}{t^2}\right)\frac{2t}{(1 + t^4)}\, dt$$

$$\Rightarrow \quad I = 2\int \frac{t^2 + 1}{t^4 + 1}\, dt \quad \text{(1)}$$

On dividing numerator and denominator by t^2 in RHS, we get

$$I = 2\int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t^2 + \frac{1}{t^2}\right)}\, dt = 2\int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2} - 2 + 2}\, dt$$

$$= 2\int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2}\, dt \quad \text{(1)}$$

Again, put $t - \frac{1}{t} = y \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = dy$

$$\therefore \; I = 2\int \frac{dy}{y^2 + (\sqrt{2})^2} \Rightarrow I = \frac{2}{\sqrt{2}}\tan^{-1}\frac{y}{\sqrt{2}} + C$$

$$\left[\because \int \frac{dx}{\sqrt{x^2 + a^2}} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C\right] \quad \text{(1)}$$

$$= \sqrt{2}\tan^{-1}\frac{\left(t - \frac{1}{t}\right)}{\sqrt{2}} + C \quad \left[\text{put } y = t - \frac{1}{t}\right] \quad \text{(1)}$$

$$= \sqrt{2}\tan^{-1}\left(\frac{t^2 - 1}{\sqrt{2}t}\right) + C$$

$$= \sqrt{2}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + C \quad \text{(1)}$$

$$[\text{put } t^2 = \tan x]$$

105. Let $I = \int \frac{1}{\cos^4 x + \sin^4 x}\, dx$

On dividing numerator and denominator by $\cos^4 x$ in RHS, we get

$$I = \int \frac{\sec^4 x}{1 + \tan^4 x}\, dx$$

$$\Rightarrow \quad I = \int \frac{(\sec^2 x)(\sec^2 x)}{1 + (\tan^2 x)^2}\, dx$$

$$\Rightarrow \quad I = \int \frac{\sec^2 x\,(1 + \tan^2 x)}{1 + (\tan^2 x)^2}\, dx \quad \text{(1)}$$

Put $\tan x = t$

$\Rightarrow \sec^2 x\, dx = dt$

$$\therefore \quad I = \int \frac{1 + t^2}{1 + t^4}\, dt \quad \text{(1)}$$

Again, dividing numerator and denominator by t^2 in RHS, we get

$$I = \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2} + 2 - 2}\, dt = \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 2}\, dt \quad \text{(1)}$$

Put $t - \frac{1}{t} = u$

$$\Rightarrow \quad \left(1 + \frac{1}{t^2}\right) dt = du \quad \text{(1)}$$

Then, $I = \int \frac{du}{u^2 + (\sqrt{2})^2}$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$\left[\because \int \frac{dx}{\sqrt{x^2 + a^2}} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C\right]$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t - \frac{1}{t}}{\sqrt{2}}\right) + C \quad \text{(1)}$$

$$\left[\text{put } u = t - \frac{1}{t}\right]$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2 - 1}{\sqrt{2}t}\right) + C$$

$$\therefore \quad I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{2}\tan x}\right) + C \quad \text{(1)}$$

$$[\text{put } t = \tan x]$$

106. Do same as Q. No. 66.

$$\left[\textbf{Ans. } -\frac{1}{3}\tan^{-1} x + \frac{2}{3}\tan^{-1}\frac{x}{2} + C\right]$$

107. First, use the identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ to convert integrand in terms of $\sin^{-1}$ only. Then, integrate by using substitution.

Let $I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\, dx$

We know that, $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \pi/2$

$$\Rightarrow \quad \cos^{-1}\sqrt{x} = \frac{\pi}{2} - \sin^{-1}\sqrt{x}$$

$$\therefore I=\int \frac{\sin^{-1}\sqrt{x}-\left(\frac{\pi}{2}-\sin^{-1}\sqrt{x}\right)}{\pi/2}dx$$

$$=\int \frac{2\sin^{-1}\sqrt{x}-\frac{\pi}{2}}{\frac{\pi}{2}}dx=\frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x}-\frac{\pi}{2}\right)dx$$

$$=\frac{4}{\pi}\int \sin^{-1}\sqrt{x}\,dx-\int 1\,dx=\frac{4}{\pi}\int \sin^{-1}\sqrt{x}\,dx-x$$

$$\Rightarrow I=\frac{4}{\pi}I_1-x \quad \text{...(i) (1)}$$

where, $I_1=\int \sin^{-1}\sqrt{x}\,dx$

Put $\sqrt{x}=t$

$\Rightarrow \quad x=t^2$ and $dx=2t\,dt$

$$\therefore \quad I_1=\int \sin^{-1}t\,2t\,dt=2\int \underset{\text{I}}{\sin^{-1}t}\cdot \underset{\text{II}}{t}\,dt$$

$$=2\left[\sin^{-1}t\int t\,dt-\int\left\{\frac{d}{dt}(\sin^{-1}t)\int t\,dt\right\}dt\right]$$

[using integration by parts]

$$=2\left[\sin^{-1}t\cdot\frac{t^2}{2}-\int\frac{1}{\sqrt{1-t^2}}\cdot\frac{t^2}{2}dt\right] \quad \text{(1)}$$

$$=t^2\sin^{-1}t+\int\frac{(1-t^2)-1}{\sqrt{1-t^2}}dt$$

$$=t^2\sin^{-1}t+\int\sqrt{1-t^2}\,dt-\int\frac{1}{\sqrt{1-t^2}}dt \quad \text{(1)}$$

$$=t^2\sin^{-1}t+\frac{t\sqrt{1-t^2}}{2}+\frac{1}{2}\sin^{-1}t-\sin^{-1}t+C_1$$

$$=\left(t^2-\frac{1}{2}\right)\sin^{-1}t+\frac{1}{2}t\sqrt{1-t^2}+C_1 \quad \text{(1)}$$

$$=\frac{1}{2}[(2x-1)\sin^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}]+C_1$$

[put $t=\sqrt{x}$]

$$=\frac{1}{2}[(2x-1)\sin^{-1}\sqrt{x}+\sqrt{x-x^2}]+C_1 \quad \text{(1)}$$

On putting the value of I_1 in Eq. (i), we get

$$I=\frac{2}{\pi}[(2x-1)\sin^{-1}\sqrt{x}+\sqrt{x-x^2}]-x+C,$$

where $C=\frac{4}{\pi}C_1$ **(1)**

108. We have, $\int\frac{x^2+x+1}{(x+1)^2(x+2)}dx$

The integrand $\frac{x^2+x+1}{(x+1)^2(x+2)}$ is a proper rational function.

Now, by using partial fraction,

let $\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+2}$ ...(i) **(1)**

$$\Rightarrow x^2+x+1=A(x+1)(x+2)+B(x+2)+C(x+1)^2$$

$$\Rightarrow x^2+x+1=A(x^2+3x+2)+B(x+2)+C(x^2+2x+1)$$

$$\Rightarrow x^2+x+1=(A+C)x^2+(3A+B+2C)x+(2A+2B+C) \quad \text{(1)}$$

On comparing the coefficients of like powers from both sides, we get **(1)**

$$A+C=1,$$
$$3A+B+2C=1$$

and $\quad 2A+2B+C=1$

On solving these equations, we get **(1)**

$$A=-2, B=1$$

and $\quad C=3$

From Eq. (i), we get

$$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{-2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x+2} \quad \text{(1)}$$

$$\therefore \int\frac{x^2+x+1}{(x+1)^2(x+2)}dx=-2\int\frac{1}{x+1}dx+\int\frac{dx}{(x+1)^2}+3\int\frac{dx}{(x+2)}$$

$$=-2\log|x+1|-\frac{1}{x+1}+3\log|x+2|+C \quad \text{(1)}$$

109. Let $I=\int\frac{\sqrt{x^2+1}\,[\log|x^2+1|-2\log|x|]}{x^4}dx$

$$=\int\frac{\sqrt{x^2+1}\,\log\left|\frac{x^2+1}{x^2}\right|}{x^4}dx$$

$$\left[\because \log m-a\log n=\log\frac{m}{n^a}\right] \quad \text{(1)}$$

$$=\int\frac{x\sqrt{1+\frac{1}{x^2}}\log\left|1+\frac{1}{x^2}\right|}{x^4}dx \quad \text{(1)}$$

$$=\int\frac{\sqrt{1+\frac{1}{x^2}}\log\left|1+\frac{1}{x^2}\right|}{x^3}dx$$

Put $1+\frac{1}{x^2}=t$

$$\Rightarrow \frac{-2}{x^3}dx=dt\Rightarrow\frac{dx}{x^3}=-\frac{dt}{2} \quad \text{(1)}$$

$$\therefore \quad I=-\frac{1}{2}\int\sqrt{t}\log|t|dt$$

$$= -\frac{1}{2}\left[\log|t|\int t^{1/2}dt - \int\left\{\frac{d}{dt}(\log|t|)\int t^{1/2}dt\right\}dt\right]$$

[using integration by parts] **(1)**

$$= -\frac{1}{2}\left[\log|t| \times \frac{t^{3/2}}{3/2} - \int \frac{t^{3/2}}{3/2} \times \frac{1}{t}dt\right]$$

$$= -\frac{1}{3}[t^{3/2}\log|t| - \int \sqrt{t}\, dt]$$

$$= -\frac{1}{3}\left[t^{3/2}\log|t| - \frac{t^{3/2}}{3/2}\right] + C \quad \textbf{(1)}$$

$$= -\frac{1}{3}t^{3/2}\left[\log|t| - \frac{2}{3}\right] + C$$

$$= -\frac{1}{3}\left(1 + \frac{1}{x^2}\right)^{3/2}\left[\log\left|1 + \frac{1}{x^2}\right| - \frac{2}{3}\right] + C$$

(1)

$\left[\text{put } t = 1 + \frac{1}{x^2}\right]$

110. Do same as Q. No. 108.

$$\left[\textbf{Ans. } \frac{3}{8}\log|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\log|x+3| + C\right]$$

111. Do same as Q. No. 69.

$$\left[\begin{aligned}\textbf{Ans. } I &= 6\sqrt{x^2 - 9x + 20} \\ &+ 34\log\left|x - \frac{9}{2} + \sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}}\right| + C\end{aligned}\right]$$