Chapter 5

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

5.1 Overview

We know that the square of a real number is always non-negative e.g. $(4)^2 = 16$ and $(-4)^2 = 16$. Therefore, square root of 16 is ± 4. What about the square root of a negative number? It is clear that a negative number can not have a real square root. So we need to extend the system of real numbers to a system in which we can find out the square roots of negative numbers. Euler (1707 - 1783) was the first mathematician to introduce the symbol i (iota) for positive square root of -1 i.e., $i = \sqrt{-1}$.

5.1.1 *Imaginary numbers*

Square root of a negative number is called an imaginary number., for example,

$$\sqrt{-9} = \sqrt{-1}\sqrt{9} = i3, \quad \sqrt{-7} = \sqrt{-1}\sqrt{7} = i\sqrt{7}$$

5.1.2 *Integral powers of i*

$i = \sqrt{-1}$, $i^2 = -1$, $i^3 = i^2 i = -i$, $i^4 = (i^2)^2 = (-1)^2 = 1$.

To compute i^n for $n > 4$, we divide n by 4 and write it in the form $n = 4m + r$, where m is quotient and r is remainder $(0 \le r \le 4)$

Hence $\qquad i^n = i^{4m+r} = (i^4)^m \,.\, (i)^r = (1)^m \,(i)^r = i^r$

For example, $\qquad (i)^{39} = i^{4 \times 9 + 3} = (i^4)^9 \,.\, (i)^3 = i^3 = -i$

and $\qquad (i)^{-435} = i^{-(4 \times 108 + 3)} = (i)^{-(4 \times 108)} \,.\, (i)^{-3}$

$$= \frac{1}{(i^4)^{108}} \cdot \frac{1}{(i)^3} = \frac{i}{(i)^4} = i$$

(i) If a and b are positive real numbers, then

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{-1}\sqrt{a} \times \sqrt{-1}\sqrt{b} = i\sqrt{a} \times i\sqrt{b} = -\sqrt{ab}$$

(ii) $\sqrt{a} \,.\, \sqrt{b} = \sqrt{ab}$ if a and b are positive or at least one of them is negative or zero. However, $\sqrt{a}\sqrt{b} \ne \sqrt{ab}$ if a and b, both are negative.

5.1.3 *Complex numbers*

(a) A number which can be written in the form $a + ib$, where a, b are real numbers and $i = \sqrt{-1}$ is called a complex number.

(b) If $z = a + ib$ is the complex number, then a and b are called real and imaginary parts, respectively, of the complex number and written as Re $(z) = a$, Im $(z) = b$.

(c) Order relations "greater than" and "less than" are not defined for complex numbers.

(d) If the imaginary part of a complex number is zero, then the complex number is known as purely real number and if real part is zero, then it is called purely imaginary number, for example, 2 is a purely real number because its imaginary part is zero and $3i$ is a purely imaginary number because its real part is zero.

5.1.4 *Algebra of complex numbers*

(a) Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are said to be equal if $a = c$ and $b = d$.

(b) Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers then $z_1 + z_2 = (a + c) + i\,(b + d)$.

5.1.5 *Addition of complex numbers satisfies the following properties*

1. As the sum of two complex numbers is again a complex number, the set of complex numbers is closed with respect to addition.
2. Addition of complex numbers is commutative, i.e., $z_1 + z_2 = z_2 + z_1$
3. Addition of complex numbers is associative, i.e., $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
4. For any complex number $z = x + i\,y$, there exist 0, i.e., $(0 + 0i)$ complex number such that $z + 0 = 0 + z = z$, known as identity element for addition.
5. For any complex number $z = x + iy$, there always exists a number $-z = -a - ib$ such that $z + (-z) = (-z) + z = 0$ and is known as the additive inverse of z.

5.1.6 *Multiplication of complex numbers*

Let $z_1 = a + ib$ and $z_2 = c + id$, be two complex numbers. Then

$z_1 \,.\, z_2 = (a + ib)\,(c + id) = (ac - bd) + i\,(ad + bc)$

1. As the product of two complex numbers is a complex number, the set of complex numbers is closed with respect to multiplication.
2. Multiplication of complex numbers is commutative, i.e., $z_1.z_2 = z_2.z_1$
3. Multiplication of complex numbers is associative, i.e., $(z_1.z_2) \,.\, z_3 = z_1 \,.\, (z_2.z_3)$

4. For any complex number $z = x + iy$, there exists a complex number 1, i.e., $(1 + 0i)$ such that
   $z \,.\, 1 = 1 \,.\, z = z$, known as identity element for multiplication.
5. For any non zero complex number $z = x + i\,y$, there exists a complex number $\frac{1}{z}$ such that $z \cdot \frac{1}{z} = \frac{1}{z} \cdot z = 1$, i.e., multiplicative inverse of $a + ib = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$.
6. For any three complex numbers z_1, z_2 and z_3,
   $$z_1 \,.\, (z_2 + z_3) = z_1 \,.\, z_2 + z_1 \,.\, z_3$$
   and
   $$(z_1 + z_2) \,.\, z_3 = z_1 \,.\, z_3 + z_2 \,.\, z_3$$
   i.e., for complex numbers multiplication is distributive over addition.

5.1.7 Let $z_1 = a + ib$ and $z_2 (\neq 0) = c + id$. Then

$$z_1 \div z_2 = \frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{(ac+bd)}{c^2+d^2} + i\,\frac{(bc-ad)}{c^2+d^2}$$

5.1.8 *Conjugate of a complex number*

Let $z = a + ib$ be a complex number. Then a complex number obtained by changing the sign of imaginary part of the complex number is called the conjugate of z and it is denoted by $\bar{z}$, i.e., $\bar{z} = a - ib$.

Note that additive inverse of z is $-a - ib$ but conjugate of z is $a - ib$.

We have :

1. $\overline{(\bar{z})} = z$
2. $z + \bar{z} = 2\,\text{Re}\,(z)$, $z - \bar{z} = 2\,i\,\text{Im}(z)$
3. $z = \bar{z}$, if z is purely real.
4. $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary
5. $z \,.\, \bar{z} = \{\text{Re}\,(z)\}^2 + \{\text{Im}\,(z)\}^2$.
6. $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$, $\overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2$
7. $\overline{(z_1 \,.\, z_2)} = (\bar{z}_1)\,(\bar{z}_2)$, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{(\bar{z}_1)}{(\bar{z}_2)}$ $(\bar{z}_2 \neq 0)$

5.1.9 *Modulus of a complex number*

Let $z = a + ib$ be a complex number. Then the positive square root of the sum of square of real part and square of imaginary part is called modulus (absolute value) of z and it is denoted by $|z|$ i.e., $|z| = \sqrt{a^2 + b^2}$

In the set of complex numbers $z_1 > z_2$ or $z_1 < z_2$ are meaningless but

$$|z_1| > |z_2| \text{ or } |z_1| < |z_2|$$

are meaningful because $|z_1|$ and $|z_2|$ are real numbers.

5.1.10 *Properties of modulus of a complex number*

1. $|z| = 0 \Leftrightarrow z = 0$ i.e., Re $(z) = 0$ and Im $(z) = 0$
2. $|z| = |\overline{z}| = |-z|$
3. $-|z| \le \text{Re}\,(z) \le |z|$ and $-|z| \le \text{Im}\,(z) \le |z|$
4. $z\,\overline{z} = |z|^2,\ |z^2| = |\overline{z}|^2$
5. $|z_1 z_2| = |z_1|.|z_2|,\ \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}\ (z_2 \ne 0)$
6. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}\,(z_1\,\overline{z}_2)$
7. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}\,(z_1\,\overline{z}_2)$
8. $|z_1 + z_2| \le |z_1| + |z_2|$
9. $|z_1 - z_2| \ge |z_1| - |z_2|$
10. $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)\,(|z_1|^2 + |z_2|^2)$

    In particular:

    $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2\,(|z_1|^2 + |z_2|^2)$
11. As stated earlier multiplicative inverse (reciprocal) of a complex number $z = a + ib\ (\ne 0)$ is

$$\frac{1}{z} = \frac{a - ib}{a^2 + b^2} = \frac{\overline{z}}{|z|^2}$$

5.2 Argand Plane

A complex number $z = a + ib$ can be represented by a unique point P (a, b) in the cartesian plane referred to a pair of rectangular axes. The complex number $0 + 0i$ represent the origin 0 (0, 0). A purely real number a, i.e., $(a + 0i)$ is represented by the point $(a, 0)$ on x - axis. Therefore, x-axis is called real axis. A purely imaginary number

ib, i.e., $(0 + ib)$ is represented by the point $(0, b)$ on y-axis. Therefore, y-axis is called imaginary axis.

Similarly, the representation of complex numbers as points in the plane is known as **Argand diagram**. The plane representing complex numbers as points is called complex plane or Argand plane or Gaussian plane.

If two complex numbers z_1 and z_2 be represented by the points P and Q in the complex plane, then

$$|z_1 - z_2| = PQ$$

5.2.1 *Polar form of a complex number*

Let P be a point representing a non-zero complex number $z = a + ib$ in the Argand plane. If OP makes an angle θ with the positive direction of x-axis, then $z = r(\cos\theta + i\sin\theta)$ is called the polar form of the complex number, where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = \dfrac{b}{a}$. Here θ is called argument or amplitude of z and we write it as $\arg(z) = \theta$.

The unique value of θ such that $-\pi \le \theta \le \pi$ is called the principal argument.

$$\arg(z_1 \,.\, z_2) = \arg(z_1) + \arg(z_2)$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

5.2.2 *Solution of a quadratic equation*

The equations $ax^2 + bx + c = 0$, where a, b and c are numbers (real or complex, $a \neq 0$) is called the general quadratic equation in variable x. The values of the variable satisfying the given equation are called roots of the equation.

The quadratic equation $ax^2 + bx + c = 0$ with real coefficients has two roots given by $\dfrac{-b+\sqrt{D}}{2a}$ and $\dfrac{-b-\sqrt{D}}{2a}$, where $D = b^2 - 4ac$, called the discriminant of the equation.

Notes

1. When $D = 0$, roots of the quadratic equation are real and equal. When $D > 0$, roots are real and unequal.
   Further, if $a, b, c \in \mathbf{Q}$ and D is a perfect square, then the roots of the equation are rational and unequal, and if $a, b, c \in \mathbf{Q}$ and D is not a perfect square, then the roots are irrational and occur in pair.

When D < 0, roots of the quadratic equation are non real (or complex).

2. Let α, β be the roots of the quadratic equation $ax^2 + bx + c = 0$, then sum of the roots

$(\alpha + \beta) = \dfrac{-b}{a}$ and the product of the roots $(\alpha \, . \, \beta) = \dfrac{c}{a}$.

3. Let S and P be the sum of roots and product of roots, respectively, of a quadratic equation. Then the quadratic equation is given by $x^2 - Sx + P = 0$.

5.2 Solved Exmaples

Short Answer Type

Example 1 Evaluate : $(1 + i)^6 + (1 - i)^3$

Solution $(1 + i)^6 = \{(1 + i)^2\}^3 = (1 + i^2 + 2i)^3 = (1 - 1 + 2i)^3 = 8\, i^3 = -8i$

and $(1 - i)^3 = 1 - i^3 - 3i + 3i^2 = 1 + i - 3i - 3 = -2 - 2i$

Therefore, $(1 + i)^6 + (1 - i)^3 = -8i - 2 - 2i = -2 - 10i$

Example 2 If $(x+iy)^{\frac{1}{3}} = a + ib$, where $x, y, a, b \in \mathrm{R}$, show that $\dfrac{x}{a} - \dfrac{y}{b} = -2\,(a^2 + b^2)$

Solution $(x+iy)^{\frac{1}{3}} = a + ib$

$\Rightarrow$ $x + iy = (a + ib)^3$

i.e., $x + iy = a^3 + i^3 b^3 + 3iab\,(a + ib)$

$= a^3 - ib^3 + i3a^2b - 3ab^2$

$= a^3 - 3ab^2 + i\,(3a^2b - b^3)$

$\Rightarrow$ $x = a^3 - 3ab^2$ and $y = 3a^2b - b^3$

Thus $\dfrac{x}{a} = a^2 - 3b^2$ and $\dfrac{y}{b} = 3a^2 - b^2$

So, $\dfrac{x}{a} - \dfrac{y}{b} = a^2 - 3b^2 - 3a^2 + b^2 = -2a^2 - 2b^2 = -2\,(a^2 + b^2)$.

Example 3 Solve the equation $z^2 = \bar{z}$, where $z = x + iy$

Solution $z^2 = \bar{z} \Rightarrow x^2 - y^2 + i2xy = x - iy$

Therefore, $x^2 - y^2 = x$ ... (1) and $2xy = -y$ ... (2)

From (2), we have $y = 0$ or $x = -\frac{1}{2}$

When $y = 0$, from (1), we get $x^2 - x = 0$, i.e., $x = 0$ or $x = 1$.

When $x = -\frac{1}{2}$, from (1), we get $y^2 = \frac{1}{4} + \frac{1}{2}$ or $y^2 = \frac{3}{4}$, i.e., $y = \pm\frac{\sqrt{3}}{2}$.

Hence, the solutions of the given equation are

$$0 + i0,\ 1 + i0,\ -\frac{1}{2} + i\,\frac{\sqrt{3}}{2},\ -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Example 4 If the imaginary part of $\frac{2z+1}{iz+1}$ is -2, then show that the locus of the point representing z in the argand plane is a straight line.

Solution Let $z = x + iy$. Then

$$\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1} = \frac{(2x+1)+i2y}{(1-y)+ix}$$

$$= \frac{\{(2x+1)+i2y\}}{\{(1-y)+ix\}} \times \frac{\{(1-y)-ix\}}{\{(1-y)-ix\}}$$

$$= \frac{(2x+1-y)+i\,(2y-2y^2-2x^2-x)}{1+y^2-2y+x^2}$$

Thus $$\operatorname{Im}\left(\frac{2z+1}{iz+1}\right) = \frac{2y-2y^2-2x^2-x}{1+y^2-2y+x^2}$$

But $$\operatorname{Im}\left(\frac{2z+1}{iz+1}\right) = -2 \qquad \text{(Given)}$$

So $$\frac{2y-2y^2-2x^2-x}{1+y^2-2y+x^2} = -2$$

$\Rightarrow$ $2y - 2y^2 - 2x^2 - x = -2 \quad -2y^2 + 4y - 2x^2$

i.e., $x + 2y - 2 = 0$, which is the equation of a line.

Example 5 If $\left|z^2-1\right| = \left|z\right|^2 + 1$, then show that z lies on imaginary axis.

Solution Let $z = x + iy$. Then $|z^2 - 1| = |z|^2 + 1$

$\Rightarrow \qquad \left|x^2 - y^2 - 1 + i\,2xy\right| = \left|x + iy\right|^2 + 1$

$\Rightarrow \qquad (x^2 - y^2 - 1)^2 + 4x^2y^2 = (x^2 + y^2 + 1)^2$

$\Rightarrow \qquad 4x^2 = 0 \qquad$ i.e., $\qquad x = 0$

Hence z lies on y-axis.

Example 6 Let z_1 and z_2 be two complex numbers such that $\bar{z}_1 + i\bar{z}_2 = 0$ and arg $(z_1\, z_2) = \pi$. Then find arg (z_1).

Solution Given that $\bar{z}_1 + i\bar{z}_2 = 0$

$\Rightarrow \qquad z_1 = i z_2$, i.e., $z_2 = -\,i z_1$

Thus $\qquad$ arg $(z_1\, z_2)$ = arg z_1 + arg $(-\,i z_1) = \pi$

$\Rightarrow \qquad \arg\,(-i z_1^2) = \pi$

$\Rightarrow \qquad \arg\,(-\,i) + \arg\,(z_1^2) = \pi$

$\Rightarrow \qquad \arg\,(-\,i) + 2\arg\,(z_1) = \pi$

$$\Rightarrow \qquad \frac{-\pi}{2} + 2\arg\,(z_1) = \pi$$

$$\Rightarrow \qquad \arg\,(z_1) = \frac{3\pi}{4}$$

Example 7 Let z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$.

Then show that arg (z_1) – arg (z_2) = 0.

Solution Let $z_1 = r_1\,(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2\,(\cos\theta_2 + i\sin\theta_2)$

where $\qquad r_1 = |z_1|$, arg $(z_1) = \theta_1$, $r_2 = |z_2|$, arg $(z_2) = \theta_2$.

We have, $\qquad |z_1 + z_2| = |z_1| + |z_2|$

$\qquad = \left|r_1(\cos\theta_1 + \cos\theta_2) + r_2(\cos\theta_2 + \sin\theta_2)\right| = r_1 + r_2$

$\qquad = r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2) = (r_1 + r_2)^2 \Rightarrow \cos\,(\theta_1 - \theta_2) = 1$

$\qquad \Rightarrow \theta_1 - \theta_2$ i.e. arg z_1 = arg z_2

Example 8 If z_1, z_2, z_3 are complex numbers such that

$|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$, then find the value of $|z_1 + z_2 + z_3|$.

Solution $|z_1| = |z_2| = |z_3| = 1$

$$\Rightarrow \qquad |z_1|^2 = |z_2|^2 = |z_3|^2 = 1$$

$$\Rightarrow \qquad z_1 \bar{z}_1 = z_2 \bar{z}_2 = z_3 \bar{z}_3 = 1$$

$$\Rightarrow \qquad \bar{z}_1 = \frac{1}{z_1},\ \bar{z}_2 = \frac{1}{z_2},\ \bar{z}_3 = \frac{1}{z_3}$$

Given that $\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$

$$\Rightarrow \qquad |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 1, \text{ i.e., } \left|\overline{z_1 + z_2 + z_3}\right| = 1$$

$$\Rightarrow \qquad |z_1 + z_2 + z_3| = 1$$

Example 9 If a complex number z lies in the interior or on the boundary of a circle of radius 3 units and centre (– 4, 0), find the greatest and least values of $|z+1|$.

Solution Distance of the point representing z from the centre of the circle is $|z-(-4+i0)| = |z+4|$.

According to given condition $|z+4| \le 3$.

Now $|z+1| = |z+4-3| \le |z+4| + |-3| \le 3+3 = 6$

Therefore, greatest value of $|z + 1|$ is 6.

Since least value of the modulus of a complex number is zero, the least value of $|z+1| = 0$.

Example 10 Locate the points for which $3 < |z| < 4$

Solution $|z| < 4 \Rightarrow x^2 + y^2 < 16$ which is the interior of circle with centre at origin and radius 4 units, and $|z| > 3 \Rightarrow x^2 + y^2 > 9$ which is exterior of circle with centre at origin and radius 3 units. Hence $3 < |z| < 4$ is the portion between two circles $x^2 + y^2 = 9$ and $x^2 + y^2 = 16$.

Example 11 Find the value of $2x^4 + 5x^3 + 7x^2 - x + 41$, when $x = -2 - \sqrt{3}\, i$

Solution $x + 2 = -\sqrt{3}\, i \Rightarrow x^2 + 4x + 7 = 0$

Therefore $\quad 2x^4 + 5x^3 + 7x^2 - x + 41 = (x^2 + 4x + 7)(2x^2 - 3x + 5) + 6$

$$= 0 \times (2x^2 - 3x + 5) + 6 = 6.$$

Example 12 Find the value of P such that the difference of the roots of the equation $x^2 - Px + 8 = 0$ is 2.

Solution Let α, β be the roots of the equation $x^2 - Px + 8 = 0$

Therefore $\quad \alpha + \beta = P$ and $\alpha . \beta = 8$.

Now $\quad \alpha - \beta = \pm\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

Therefore $\quad 2 = \pm\sqrt{P^2 - 32}$

$\Rightarrow \quad P^2 - 32 = 4$, i.e., $P = \pm 6$.

Example 13 Find the value of a such that the sum of the squares of the roots of the equation $x^2 - (a - 2)x - (a + 1) = 0$ is least.

Solution Let α, β be the roots of the equation

Therefore, $\quad \alpha + \beta = a - 2$ and $\alpha\beta = -(a + 1)$

Now

$$\begin{aligned}\alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= (a - 2)^2 + 2(a + 1) \\ &= (a - 1)^2 + 5\end{aligned}$$

Therefore, $\quad \alpha^2 + \beta^2$ will be minimum if $(a - 1)^2 = 0$, i.e., $a = 1$.

Long Answer Type

Example 14 Find the value of k if for the complex numbers z_1 and z_2,

$$|1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = k(1 - |z_1|^2)(1 - |z_2|^2)$$

Solution

$$\begin{aligned}\text{L.H.S.} &= |1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 \\ &= (1 - \bar{z}_1 z_2)\,\overline{(1 - \bar{z}_1 z_2)} - (z_1 - z_2)\,\overline{(z_1 - z_2)} \\ &= (1 - \bar{z}_1 z_2)\,(1 - z_1 \bar{z}_2) - (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\ &= 1 + z_1 \bar{z}_1\, z_2 \bar{z}_2 - z_1 \bar{z}_1 - z_2 \bar{z}_2 \\ &= 1 + |z_1|^2 \cdot |z_2|^2 - |z_1|^2 - |z_2|^2 \\ &= (1 - |z_1|^2)\,(1 - |z_2|^2)\end{aligned}$$

$$\text{R.H.S.} = k\,(1 - |z_1|^2)\,(1 - |z_2|^2)$$

$\Rightarrow \quad k = 1$

Hence, equating LHS and RHS, we get $k = 1$.

Example 15 If z_1 and z_2 both satisfy $z + \overline{z} = 2|z-1|$ arg $(z_1 - z_2) = \frac{\pi}{4}$, then find Im $(z_1 + z_2)$.

Solution Let $z = x + iy$, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Then $\quad z + \overline{z} = 2|z-1|$

$\Rightarrow \quad (x + iy) + (x - iy) = 2\,|x-1+iy|$

$\Rightarrow \quad 2x = 1 + y^2 \qquad \ldots (1)$

Since z_1 and z_2 both satisfy (1), we have

$$2x_1 = 1 + y_1^2 \ \ldots \text{ and } 2x_2 = 1 + y_2^2$$

$\Rightarrow \quad 2\,(x_1 - x_2) = (y_1 + y_2)\,(y_1 - y_2)$

$\Rightarrow \quad 2 = (y_1 + y_2)\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \qquad \ldots (2)$

Again $\quad z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$

Therefore, $\quad \tan\theta = \frac{y_1 - y_2}{x_1 - x_2}$, where $\theta = \arg(z_1 - z_2)$

$\Rightarrow \quad \tan\frac{\pi}{4} = \frac{y_1 - y_2}{x_1 - x_2} \qquad \left(\text{since } \theta = \frac{\pi}{4}\right)$

i.e., $\quad 1 = \frac{y_1 - y_2}{x_1 - x_2}$

From (2), we get $2 = y_1 + y_2$, i.e., Im $(z_1 + z_2) = 2$

Objective Type Questions

Example 16 Fill in the blanks:

(i) The real value of 'a' for which $3i^3 - 2ai^2 + (1 - a)i + 5$ is real is ________.

(ii) If $|z| = 2$ and arg $(z) = \frac{\pi}{4}$, then $z =$ ________.

(iii) The locus of z satisfying arg $(z) = \frac{\pi}{3}$ is _______.

(iv) The value of $(-\sqrt{-1})^{4n-3}$, where $n \in \mathbf{N}$, is ______.

(v) The conjugate of the complex number $\frac{1-i}{1+i}$ is _____.

(vi) If a complex number lies in the third quadrant, then its conjugate lies in the _______.

(vii) If $(2 + i)(2 + 2i)(2 + 3i) \ldots (2 + ni) = x + iy$, then $5.8.13 \ldots (4 + n^2) =$ _______.

Solution

(i) $3i^3 - 2ai^2 + (1 - a)i + 5 = -3i + 2a + 5 + (1 - a)i$

$= 2a + 5 + (- a - 2)\, i$, which is real if $- a - 2 = 0$ i.e. $a = - 2$.

(ii) $z = |z|\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(1+i)$

(iii) Let $z = x + iy$. Then its polar form is $z = r(\cos\theta + i\sin\theta)$, where $\tan\theta = \frac{y}{x}$ and θ is arg (z). Given that $\theta = \frac{\pi}{3}$. Thus.

$\tan\frac{\pi}{3} = \frac{y}{x} \Rightarrow y = \sqrt{3}x$, where $x > 0, y > 0$.

Hence, locus of z is the part of $y = \sqrt{3}x$ in the first quadrant except origin.

(iv) Here $(-\sqrt{-1})^{4n-3} = (-i)^{4n-3} = (-i)^{4n}\,(-i)^{-3} = \frac{1}{(-i)^3}$

$$= \frac{1}{-i^3} = \frac{1}{i} = \frac{i}{i^2} = -i$$

(v) $\frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{1+i^2-2i}{1-i^2} = \frac{1-1-2i}{1+1} = -i$

Hence, conjugate of $\frac{1-i}{1+i}$ is i.

(vi) Conjugate of a complex number is the image of the complex number about the x-axis. Therefore, if a number lies in the third quadrant, then its image lies in the second quadrant.

(vii) Given that $(2 + i)(2 + 2i)(2 + 3i) \ldots (2 + ni) = x + iy$ ... (1)

$\Rightarrow \quad \overline{(2+i)}\;\overline{(2+2i)}\;\overline{(2+3i)} \ldots \overline{(2+ni)} = \left(\overline{x+iy}\right) = (x-iy)$

i.e., $\quad (2 - i)(2 - 2i)(2 - 3i) \ldots (2 - ni) = x - iy$ ... (2)

Multiplying (1) and (2), we get 5.8.13 ... $(4 + n^2) = x^2 + y^2$.

Example 17 State true or false for the following:

(i) Multiplication of a non-zero complex number by i rotates it through a right angle in the anti- clockwise direction.

(ii) The complex number $\cos\theta + i\sin\theta$ can be zero for some θ.

(iii) If a complex number coincides with its conjugate, then the number must lie on imaginary axis.

(iv) The argument of the complex number $z = (1 + i\sqrt{3})(1 + i)(\cos\theta + i\sin\theta)$ is $\frac{7\pi}{12} + \theta$

(v) The points representing the complex number z for which $|z+1| < |z-1|$ lies in the interior of a circle.

(vi) If three complex numbers z_1, z_2 and z_3 are in A.P., then they lie on a circle in the complex plane.

(vii) If n is a positive integer, then the value of $i^n + (i)^{n+1} + (i)^{n+2} + (i)^{n+3}$ is 0.

Solution

(i) True. Let $z = 2 + 3i$ be complex number represented by OP. Then $iz = -3 + 2i$, represented by OQ, where if OP is rotated in the anticlockwise direction through a right angle, it coincides with OQ.

(ii) False. Because $\cos\theta + i\sin\theta = 0 \Rightarrow \cos\theta = 0$ and $\sin\theta = 0$. But there is no value of θ for which $\cos\theta$ and $\sin\theta$ both are zero.

(iii) False, because $x + iy = x - iy \Rightarrow y = 0 \Rightarrow$ number lies on x-axis.

(iv) True, arg (z) = arg $(1 + i\sqrt{3})$ + arg $(1 + i)$ + arg $(\cos\theta + i\sin\theta)$

$$\frac{\pi}{3} + \frac{\pi}{4} + \theta = \frac{7\pi}{12} + \theta$$

(v) False, because $|x + iy + 1| < |x + iy - 1|$

$\Rightarrow$ $(x + 1)^2 + y^2 < (x - 1)^2 + y^2$ which gives $4x < 0$.

(vi) False, because if z_1, z_2 and z_3 are in A.P., then $z_2 = \frac{z_1 + z_3}{2} \Rightarrow z_2$ is the midpoint of z_1 and z_3, which implies that the points z_1, z_2, z_3 are collinear.

(vii) True, because $i^n + (i)^{n+1} + (i)^{n+2} + (i)^{n+3}$

$$= i^n(1 + i + i^2 + i^3) = i^n(1 + i - 1 - i)$$

$$= i^n(0) = 0$$

Example 18 Match the statements of column A and B.

Column A	Column B
(a) The value of $1+i^2+i^4+i^6+\ldots i^{20}$ is	(i) purely imaginary complex number
(b) The value of i^{-1097} is	(ii) purely real complex number
(c) Conjugate of $1+i$ lies in	(iii) second quadrant
(d) $\frac{1+2i}{1-i}$ lies in	(iv) Fourth quadrant
(e) If $a, b, c \in$ R and $b^2 - 4ac < 0$, then the roots of the equation $ax^2 + bx + c = 0$ are non real (complex) and	(v) may not occur in conjugate pairs
(f) If $a, b, c \in$ R and $b^2 - 4ac > 0$, and $b^2 - 4ac$ is a perfect square, then the roots of the equation $ax^2 + bx + c = 0$	(vi) may occur in conjugate pairs

Solution

(a) $\Leftrightarrow$ (ii), because $1 + i^2 + i^4 + i^6 + \ldots + i^{20}$
$= 1 - 1 + 1 - 1 + \ldots + 1 = 1$ (which is purely a real complex number)

(b) $\Leftrightarrow$ (i), because $i^{-1097} = \frac{1}{(i)^{1097}} = \frac{1}{i^{4\times 274+1}} = \frac{1}{\{(i)^4\}^{274}(i)} = \frac{1}{i} = \frac{i}{i^2} = -i$

which is purely imaginary complex number.

(c) $\Leftrightarrow$ (iv), conjugate of $1 + i$ is $1 - i$, which is represented by the point $(1, -1)$ in the fourth quadrant.

(d) $\Leftrightarrow$ (iii), because $\frac{1+2i}{1-i} = \frac{1+2i}{1-i} \times \frac{1+i}{1+i} = \frac{-1+3i}{2} = -\frac{1}{2} + \frac{3}{2}i$, which is represented by the point $\left(-\frac{1}{2}, \frac{3}{2}\right)$ in the second quadrant.

(e) $\Leftrightarrow$ (vi), If $b^2 - 4ac < 0 = \text{D} < 0$, i.e., square root of D is a imaginary number, therefore, roots are $x = \frac{-b \pm \text{Imaginary Number}}{2a}$, i.e., roots are in conjugate pairs.

(f) $\Leftrightarrow$ (v), Consider the equation $x^2 - (5 + \sqrt{2})\,x + 5\sqrt{2} = 0$, where $a = 1$, $b = -(5 + \sqrt{2})$, $c = 5\sqrt{2}$, clearly $a, b, c \in$ R.

Now $D = b^2 - 4ac = \{-(5 + \sqrt{2})\}^2 - 4.1.5\sqrt{2} = (5 - \sqrt{2})^2$.

Therefore $x = \dfrac{5+\sqrt{2} \pm 5-\sqrt{2}}{2} = 5, \sqrt{2}$ which do not form a conjugate pair.

Example 19 What is the value of $\dfrac{i^{4n+1} - i^{4n-1}}{2}$?

Solution i, because $\dfrac{i^{4n+1} - i^{4n-1}}{2} = \dfrac{i^{4n}i - i^{4n}i^{-i}}{2}$

$$= \frac{i - \frac{1}{i}}{2} = \frac{i^2 - 1}{2i} = \frac{-2}{2i} = i$$

Example 20 What is the smallest positive integer n, for which $(1 + i)^{2n} = (1 - i)^{2n}$?

Solution $n = 2$, because $(1 + i)^{2n} = (1 - i)^{2n} = \left(\dfrac{1+i}{1-i}\right)^{2n} = 1$

$\Rightarrow \qquad (i)^{2n} = 1$ which is possible if $n = 2 \qquad (\because i^4 = 1)$

Example 21 What is the reciprocal of $3 + \sqrt{7}\,i$

Solution Reciprocal of $z = \dfrac{\bar{z}}{|z|^2}$

Therefore, reciprocal of $3 + \sqrt{7}\,i = \dfrac{3 - \sqrt{7}\,i}{16} = \dfrac{3}{16} - \dfrac{\sqrt{7}\,i}{16}$

Example 22 If $z_1 = \sqrt{3} + i\sqrt{3}$ and $z_2 = \sqrt{3} + i$, then find the quadrant in which $\left(\dfrac{z_1}{z_2}\right)$ lies.

Solution $\dfrac{z_1}{z_2} = \dfrac{\sqrt{3} + i\sqrt{3}}{\sqrt{3} + i} = \left(\dfrac{3+\sqrt{3}}{4}\right) + \left(\dfrac{3-\sqrt{3}}{4}\right)i$

which is represented by a point in first quadrant.

Example 23 What is the conjugate of $\dfrac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$?

Solution Let

$$z = \frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}} \times \frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}+\sqrt{5-12i}}$$

$$= \frac{5+12i+5-12i+2\sqrt{25+144}}{5+12i-5+12i}$$

$$= \frac{3}{2i} = \frac{3i}{-2} = 0-\frac{3}{2}i$$

Therefore, the conjugate of $z = 0 + \dfrac{3}{2}i$

Example 24 What is the principal value of amplitude of $1 - i$?

Solution Let θ be the principle value of amplitude of $1 - i$. Since

$$\tan\theta = -1 \Rightarrow \tan\theta = \tan\left(-\frac{\pi}{4}\right) \Rightarrow \theta = -\frac{\pi}{4}$$

Example 25 What is the polar form of the complex number $(i^{25})^3$?

Solution $z = (i^{25})^3 = (i)^{75} = i^{4\times18+3} = (i^4)^{18}\ (i)^3$

$= i^3 = -i = 0 - i$

Polar form of $z = r(\cos\theta + i\sin\theta)$

$$= 1\left\{\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right\}$$

$$= \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$$

Example 26 What is the locus of z, if amplitude of $z - 2 - 3i$ is $\dfrac{\pi}{4}$?

Solution Let $z = x + iy$. Then $z - 2 - 3i = (x - 2) + i(y - 3)$

Let θ be the amplitude of $z - 2 - 3i$. Then $\tan\theta = \dfrac{y-3}{x-2}$

$$\Rightarrow \qquad \tan\frac{\pi}{4} = \frac{y-3}{x-2} \left(\text{since } \theta = \frac{\pi}{4}\right)$$

$$\Rightarrow \qquad 1 = \frac{y-3}{x-2} \text{ i.e. } x - y + 1 = 0$$

Hence, the locus of z is a straight line.

Example 27 If $1 - i$, is a root of the equation $x^2 + ax + b = 0$, where $a, b \in \mathbf{R}$, then find the values of a and b.

Solution Sum of roots $\frac{-a}{1} = (1 - i) + (1 + i) \Rightarrow a = -2$.

(since non real complex roots occur in conjugate pairs)

Product of roots, $\frac{b}{1} = (1-i)(1+i) \Rightarrow b = 2$

Choose the correct options out of given four options in each of the Examples from 28 to 33 (M.C.Q.).

Example 28 $1 + i^2 + i^4 + i^6 + ... + i^{2n}$ is

(A) positive (B) negative

(C) 0 (D) can not be evaluated

Solution (D), $1 + i^2 + i^4 + i^6 + ... + i^{2n} = 1 - 1 + 1 - 1 + ... (-1)^n$

which can not be evaluated unless n is known.

Example 29 If the complex number $z = x + iy$ satisfies the condition $|z+1| = 1$, then z lies on

(A) x-axis

(B) circle with centre (1, 0) and radius 1

(C) circle with centre (–1, 0) and radius 1

(D) y-axis

Solution (C), $|z+1| = 1 \Rightarrow |(x+1)+iy| = 1$

$$\Rightarrow \qquad (x+1)^2 + y^2 = 1$$

which is a circle with centre (–1, 0) and radius 1.

Example 30 The area of the triangle on the complex plane formed by the complex numbers z, $-iz$ and $z + iz$ is:

(A) $|z|^2$ (B) $|\bar{z}|^2$

(C) $\frac{|z|^2}{2}$ (D) none of these

Solution (C), Let $z = x + iy$. Then $-iz = y - ix$. Therefore,

$$z + iz = (x - y) + i\,(x + y)$$

Required area of the triangle $= \dfrac{1}{2}(x^2 + y^2) = \dfrac{|z|^2}{2}$

Example 31 The equation $|z+1-i| = |z-1+i|$ represents a

(A) straight line (B) circle

(C) parabola (D) hyperbola

Solution (A), $|z+1-i| = |z-1+i|$

$\Rightarrow$ $|z-(-1+i)| = |z-(1-i)|$

$\Rightarrow$ PA = PB, where A denotes the point (–1, 1), B denotes the point (1, –1) and P denotes the point (x, y)

$\Rightarrow$ z lies on the perpendicular bisector of the line joining A and B and perpendicular bisector is a straight line.

Example 32 Number of solutions of the equation $z^2 + |z|^2 = 0$ is

(A) 1 (B) 2

(C) 3 (D) infinitely many

Solution (D), $z^2 + |z|^2 = 0,\ z \neq 0$

$\Rightarrow$ $x^2 - y^2 + i2xy + x^2 + y^2 = 0$

$\Rightarrow$ $2x^2 + i2xy = 0 \Rightarrow 2x\,(x + iy) = 0$

$\Rightarrow$ $x = 0$ or $x + iy = 0$ (not possible)

Therefore, $x = 0$ and $z \neq 0$

So y can have any real value. Hence infinitely many solutions.

Example 33 The amplitude of $\sin\dfrac{\pi}{5} + i\left(1 - \cos\dfrac{\pi}{5}\right)$ is

(A) $\dfrac{2\pi}{5}$ (B) $\dfrac{\pi}{5}$ (C) $\dfrac{\pi}{15}$ (D) $\dfrac{\pi}{10}$

Solution (D), Here $r\cos\theta = \sin\left(\dfrac{\pi}{5}\right)$ and $r\sin\theta = 1 - \cos\dfrac{\pi}{5}$

Therefore, $$\tan\theta = \frac{1-\cos\frac{\pi}{5}}{\sin\frac{\pi}{5}} = \frac{2\sin^2\left(\frac{\pi}{10}\right)}{2\sin\left(\frac{\pi}{10}\right).\cos\left(\frac{\pi}{10}\right)}$$

$\Rightarrow$ $$\tan\theta = \tan\left(\frac{\pi}{10}\right) \text{ i.e., } \theta = \frac{\pi}{10}$$

5.3 EXERCISE

Short Answer Type

1. For a positive integer n, find the value of $(1-i)^n\left(1-\frac{1}{i}\right)^n$

2. Evaluate $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $n \in \mathbf{N}$.

3. If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$, then find (x, y).

4. If $\frac{(1+i)^2}{2-i} = x + iy$, then find the value of $x + y$.

5. If $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$, then find (a, b).

6. If $a = \cos\theta + i\sin\theta$, find the value of $\frac{1+a}{1-a}$.

7. If $(1 + i)\, z = (1 - i)\, \bar{z}$, then show that $z = -i\bar{z}$.

8. If $z = x + iy$, then show that $z\bar{z} + 2(z + \bar{z}) + b = 0$, where $b \in \mathbf{R}$, represents a circle.

9. If the real part of $\frac{\bar{z}+2}{\bar{z}-1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle.

10. Show that the complex number z, satisfying the condition $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ lies on a circle.

11. Solve the equation $|z| = z + 1 + 2i$.

Long Answer Type

12. If $|z+1| = z + 2(1+i)$, then find z.

13. If arg $(z - 1)$ = arg $(z + 3i)$, then find $x - 1 : y$. where $z = x + iy$

14. Show that $\left|\dfrac{z-2}{z-3}\right| = 2$ represents a circle. Find its centre and radius.

15. If $\dfrac{z-1}{z+1}$ is a purely imaginary number $(z \neq -1)$, then find the value of $|z|$.

16. z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and arg (z_1) + arg (z_2) = π, then show that $z_1 = -\bar{z}_2$.

17. If $|z_1| = 1$ $(z_1 \neq -1)$ and $z_2 = \dfrac{z_1 - 1}{z_1 + 1}$, then show that the real part of z_2 is zero.

18. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then find

$$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right).$$

19. If $|z_1| = |z_2| = \ldots = |z_n| = 1$, then

show that $|z_1 + z_2 + z_3 + \ldots + z_n| = \left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3} + \ldots + \dfrac{1}{z_n}\right|$.

20. If for complex numbers z_1 and z_2, arg (z_1) – arg (z_2) = 0, then show that $|z_1 - z_2| = |z_1| - |z_2|$

21. Solve the system of equations Re $(z^2) = 0$, $|z| = 2$.

22. Find the complex number satisfying the equation $z + \sqrt{2}\,|(z+1)| + i = 0$.

23. Write the complex number $z = \dfrac{1-i}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$ in polar form.

24. If z and w are two complex numbers such that $|zw| = 1$ and arg (z) – arg (w) = $\dfrac{\pi}{2}$, then show that $\bar{z}w = -i$.

Objective Type Questions

25. Fill in the blanks of the following

(i) For any two complex numbers z_1, z_2 and any real numbers a, b, $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = ...$

(ii) The value of $\sqrt{-25} \times \sqrt{-9}$ is ....

(iii) The number $\dfrac{(1-i)^3}{1-i^3}$ is equal to ....

(iv) The sum of the series $i + i^2 + i^3 + ...$ upto 1000 terms is ....

(v) Multiplicative inverse of $1 + i$ is ....

(vi) If z_1 and z_2 are complex numbers such that $z_1 + z_2$ is a real number, then $z_2 = ...$

(vii) arg (z) + arg $\overline{z}$ $(\overline{z} \neq 0)$ is ....

(viii) If $|z+4| \leq 3$, then the greatest and least values of $|z+1|$ are ..... and .....

(ix) If $\left|\dfrac{z-2}{z+2}\right| = \dfrac{\pi}{6}$, then the locus of z is ....

(x) If $|z| = 4$ and arg $(z) = \dfrac{5\pi}{6}$, then z = ...

26. State True or False for the following :

(i) The order relation is defined on the set of complex numbers.

(ii) Multiplication of a non zero complex number by $-i$ rotates the point about origin through a right angle in the anti-clockwise direction.

(iii) For any complex number z the minimum value of $|z| + |z-1|$ is 1.

(iv) The locus represented by $|z-1| = |z-i|$ is a line perpendicular to the join of (1, 0) and (0, 1).

(v) If z is a complex number such that $z \neq 0$ and Re $(z) = 0$, then Im $(z^2) = 0$.

(vi) The inequality $|z-4| < |z-2|$ represents the region given by $x > 3$.

(vii) Let z_1 and z_2 be two complex numbers such that $|z_1+z_2|=|z_1|+|z_2|$, then arg $(z_1 - z_2) = 0$.

(viii) 2 is not a complex number.

27. Match the statements of Column A and Column B.

Column A	Column B
(a) The polar form of $i+\sqrt{3}$ is	(i) Perpendicular bisector of segment joining (– 2, 0) and (2, 0)
(b) The amplitude of $-1+\sqrt{-3}$ is	(ii) On or outside the circle having centre at (0, – 4) and radius 3.
(c) If $\lvert z+2\rvert=\lvert z-2\rvert$, then locus of z is	(iii) $\dfrac{2\pi}{3}$
(d) If $\lvert z+2i\rvert=\lvert z-2i\rvert$, then locus of z is	(iv) Perpendicular bisector of segment joining (0, – 2) and (0, 2).
(e) Region represented by $\lvert z+4i\rvert\geq 3$ is	(v) $2\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)$
(f) Region represented by $\lvert z+4\rvert\leq 3$ is	(vi) On or inside the circle having centre (– 4, 0) and radius 3 units.
(g) Conjugate of $\dfrac{1+2i}{1-i}$ lies in	(vii) First quadrant
(h) Reciprocal of $1-i$ lies in	(viii) Third quadrant

28. What is the conjugate of $\dfrac{2-i}{(1-2i)^2}$?

29. If $|z_1|=|z_2|$, is it necessary that $z_1 = z_2$?

30. If $\dfrac{(a^2+1)^2}{2a-i} = x+iy$, what is the value of x^2+y^2?

31. Find z if $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$.

32. Find $\left|(1+i)\frac{(2+i)}{(3+i)}\right|$

33. Find principal argument of $(1 + i\sqrt{3})^2$.

34. Where does z lie, if $\left|\frac{z-5i}{z+5i}\right| = 1$.

Choose the correct answer from the given four options indicated against each of the Exercises from 35 to 50 (M.C.Q)

35. $\sin x + i \cos 2x$ and $\cos x - i \sin 2x$ are conjugate to each other for:

(A) $x = n\pi$ (B) $x = \left(n+\frac{1}{2}\right)\frac{\pi}{2}$

(C) $x = 0$ (D) No value of x

36. The real value of α for which the expression $\frac{1-i\sin\alpha}{1+2i\sin\alpha}$ is purely real is :

(A) $(n+1)\frac{\pi}{2}$ (B) $(2n + 1)\frac{\pi}{2}$

(C) $n\pi$ (D) None of these, where $n \in \mathbf{N}$

37. If $z = x + iy$ lies in the third quadrant, then $\frac{\bar{z}}{z}$ also lies in the third quadrant if

(A) $x > y > 0$ (B) $x < y < 0$

(C) $y < x < 0$ (D) $y > x > 0$

38. The value of $(z + 3)(\bar{z} + 3)$ is equivalent to

(A) $|z+3|^2$ (B) $|z-3|$

(C) $z^2 + 3$ (D) None of these

39. If $\left(\frac{1+i}{1-i}\right)^x = 1$, then

(A) $x = 2n+1$ (B) $x = 4n$

(C) $x = 2n$ (D) $x = 4n + 1$, where $n \in \mathrm{N}$

40. A real value of x satisfies the equation $\left(\frac{3-4ix}{3+4ix}\right) = \alpha - i\beta\ (\alpha, \beta \in \mathbf{R})$ if $\alpha^2 + \beta^2 =$

(A) 1 (B) -1 (C) 2 (D) -2

41. Which of the following is correct for any two complex numbers z_1 and z_2?

(A) $|z_1 z_2| = |z_1||z_2|$ (B) $\arg(z_1 z_2) = \arg(z_1).\arg(z_2)$

(C) $|z_1 + z_2| = |z_1| + |z_2|$ (D) $|z_1 + z_2| \geq |z_1| - |z_2|$

42. The point represented by the complex number $2 - i$ is rotated about origin through an angle $\frac{\pi}{2}$ in the clockwise direction, the new position of point is:

(A) $1 + 2i$ (B) $-1 - 2i$ (C) $2 + i$ (D) $-1 + 2i$

43. Let $x, y \in \mathbf{R}$, then $x + iy$ is a non real complex number if:

(A) $x = 0$ (B) $y = 0$ (C) $x \neq 0$ (D) $y \neq 0$

44. If $a + ib = c + id$, then

(A) $a^2 + c^2 = 0$ (B) $b^2 + c^2 = 0$

(C) $b^2 + d^2 = 0$ (D) $a^2 + b^2 = c^2 + d^2$

45. The complex number z which satisfies the condition $\left|\frac{i+z}{i-z}\right| = 1$ lies on

(A) circle $x^2 + y^2 = 1$ (B) the x-axis

(C) the y-axis (D) the line $x + y = 1$.

46. If z is a complex number, then

(A) $|z^2| > |z|^2$ (B) $|z^2| = |z|^2$

(C) $|z^2| < |z|^2$ (D) $|z^2| \geq |z|^2$

47. $|z_1 + z_2| = |z_1| + |z_2|$ is possible if

(A) $z_2 = \overline{z_1}$ (B) $z_2 = \frac{1}{z_1}$

(C) $\arg(z_1) = \arg(z_2)$ (D) $|z_1| = |z_2|$

48. The real value of θ for which the expression $\dfrac{1+i\cos\theta}{1-2i\cos\theta}$ is a real number is:

(A) $n\pi+\dfrac{\pi}{4}$ (B) $n\pi+(-1)^n\dfrac{\pi}{4}$

(C) $2n\pi\pm\dfrac{\pi}{2}$ (D) none of these.

49. The value of arg (x) when $x < 0$ is:

(A) 0 (B) $\dfrac{\pi}{2}$

(C) π (D) none of these

50. If $f(z) = \dfrac{7-z}{1-z^2}$, where $z = 1 + 2i$, then $|f(z)|$ is

(A) $\dfrac{|z|}{2}$ (B) $|z|$

(C) $2|z|$ (D) none of these.