Chapter 5. Complex Numbers and Quadratic Equations

Question-1

If $z_1, z_2 \in C$, show that $(z_1 + z_2)^2 = z_1^2 + 2z_1z_2 + z_2^2$

Solution:

Let $z_1 = x_1 + iy_1$

$z^2 = x_2 + iy_2$

$(z_1 + z_2)^2 = [(x_1 + iy_1) + (x_2 + iy_2)]^2$

$= [(x_1 + x_2) + i(y_1 + y_2)]^2$

$= (x_1 + x_2)^2 + 2i(x_1 + x_2)(y_1 + y_2) - (y_1 + y_2)^2$

$= x_1^2 + 2x_1x_2 + x_2^2 + 2ix_1y_1 + 2ix_1y_2 + 2ix_2y_1 + 2ix_2y_2 - y_1^2 - 2y_1y_2 - y_2^2$

$= x_1^2 + 2ix_1y_1 - y_1^2 + x_2^2 + 2ix_2y_2 - y_2^2 + 2(x_1 + iy_1)(x_2 + iy_2)$

$= x_1^2 + 2ix_1y_1 + (iy_1)^2 + x_2^2 + 2ix_2y_2 + (iy_2)^2 + 2(x_1 + iy_1)(x_2 + iy_2)$

$= (x_1 + iy_1)^2 + (x_2 + iy_2)^2 + 2z_1z_2$

$= z_1^2 + 2z_1z_2 + z_2^2$

Question-2

Write the following as complex numbers

i. $\sqrt{-16}$

ii. $1 + \sqrt{-1}$

iii. $-1 - \sqrt{-5}$

iv $\frac{\sqrt{3}}{2} - \frac{\sqrt{-2}}{\sqrt{7}}$

v. $\sqrt{x}$, $(x>0)$

vi. $-b + \sqrt{-4ac}$, $(a, c > 0)$

Solution:

i. $\sqrt{-16} = \sqrt{-1 \times 16} = \sqrt{-1}\sqrt{16} = i\sqrt{16}$

ii. $1 + \sqrt{-1} = 1 + i$

iii. $-1 - \sqrt{-5} = -1 - \sqrt{-1}\sqrt{5} = -1 - i\sqrt{5}$

iv. $\frac{\sqrt{3}}{2} - \frac{\sqrt{-2}}{\sqrt{7}} = \frac{\sqrt{3}}{2} - \frac{\sqrt{-1}\sqrt{2}}{\sqrt{7}} = \frac{\sqrt{3}}{2} - \frac{i\sqrt{2}}{\sqrt{7}}$

v. $\sqrt{x} = \sqrt{x} + i0$

vi. $-b + \sqrt{-4ac} = -b + \sqrt{-1}\sqrt{4ac} = -b + 2i\sqrt{ac}$

Question-3

Obtain a quadratic equation whose root are 2 and 3.

Solution:

Let α, β be the roots of the equation.

Sum of the roots $\alpha + \beta = 2+3 = 5$

Product of the roots $\alpha x \beta = 2x3 = 6$

∴ The equation is given by

x^2 - (sum of roots)x + product of roots = 0

∴ Equation is $x^2 - 5x + 6 = 0$.

Question-4

Solve the following equation: $25x^2 - 30x + 9 = 0$.

Solution:

$25x^2 - 30x + 9 = 0$

$D = b^2 - 4ac = 900 - 4 \times 25 \times 9 = 900 - 900 = 0$

Hence the two real equal roots of the equation are : $\frac{30}{50}, \frac{30}{50}$

i.e $\frac{3}{5}, \frac{3}{5}$

Question-5

Write the real and imaginary parts of the following complex numbers below:

- **i.** $\frac{\sqrt{17}}{2} + \frac{i2}{\sqrt{70}}$
- **ii.** $-\frac{1}{5} + \frac{i}{5}$
- **iii.** $\sqrt{37} + \sqrt{-19}$
- **iv.** $\sqrt{3} + i\frac{\sqrt{2}}{76}$
- **v.** 7
- **vi.** 3i

Solution:

i. Let $z = \frac{\sqrt{17}}{2} + \frac{i2}{\sqrt{70}}$

Re z = $\frac{\sqrt{17}}{2}$, Im z = $\frac{2}{\sqrt{70}}$

ii. Let $z = -\frac{1}{5} + \frac{i}{5}$

Re z = $-\frac{1}{5}$, Im z = $\frac{1}{5}$

iii. Let $z = \sqrt{37} + \sqrt{-19} = \sqrt{37} + i\sqrt{19}$

Re z = $\sqrt{37}$, Im z = $\sqrt{19}$

iv. $\sqrt{3} + i\frac{\sqrt{2}}{76}$

Re z = $\sqrt{3}$, Im z = $\frac{\sqrt{2}}{76}$

v. 7

Re z = 7, Im z = 0

vi. 3i

Re z = 0, Im z = 3

Question-6

Without computing the roots of $3x^2 + 2x+6 = 0$, find (i) $\frac{1}{\alpha} + \frac{1}{\beta}$ (ii) α2 +β2 (iii) $\alpha^3+\beta^3$

Solution:

If α and β an the root of the equation $3x^2 +2x+6 = 0$

Sum of the roots $\alpha +\beta = \frac{-b}{a} = \frac{-2}{3}$

Product of roots $\alpha\beta = \frac{c}{a} = \frac{6}{3} = 2$

(i) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{-2}{3} \times \frac{1}{2} = \frac{-1}{3}$

(ii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \left(\frac{-2}{3}\right)^2 - 2\times2 = \frac{4}{9} - 4 = \frac{4-36}{9} = -\frac{32}{9}$

(iii $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = \left(\frac{-2}{3}\right)^3 - 3(2)\left(\frac{-2}{3}\right) = \frac{-8}{27} + 4 = 3\frac{19}{27}$.

Question-7

Show that $(1-i)^2 = -2i$.

Solution:

$(1-i)^2 = 1^2 - 2(i)(1) + (i)^2 = 1-2(-i) + (i)^2 = 1-2i-1 = -2i$

Question-8

Solve the following equation: $2x^2-2\sqrt{3}x + 1 = 0$.

Solution:

$2x^2-2\sqrt{3}x + 1 = 0$

$D = b^2-4ac = 12 - 4\times 2\times 1 = 12 - 8 = 4 > 0$

$\sqrt{D} = 2$

Hence the two real and unequal roots are : $\frac{2\sqrt{3}+2}{4}, \frac{2\sqrt{3}-2}{4}$

i.e $\frac{\sqrt{3}+1}{2}, \frac{\sqrt{3}-1}{2}$

Question-9

Solve the equation $\sqrt{x}$ = (x - 2) in C.

Solution:

Squaring both sides

$(\sqrt{x})^2=(x-2)^2$

$x = x^2 - 2(x)(2)+4$

$0 = x^2 - 4x+ 4 - x$

$= x^2 - 5x + 4$

$x^2 - 5x + 4 = 0$

$x^2 - 4x - x + 4 = 0$

$x(x - 4) - (x - 4) = 0$

$x = 4$ or $x = 1$

x=1 doesn't satisfy the equation

$\therefore x = 4$.

Question-10

Find the conjugate of the following complex numbers

i. 3 + i
ii. 3 – i
iii. $-\sqrt{5} - i\sqrt{7}$
iv. $-i\sqrt{5}$

v. 4/5

vi. 49 – i/7

Solution:

i. Conjugate of 3 + i is 3 – i.
ii. Conjugate of 3 – i is 3 + i.
iii. Conjugate of $-\sqrt{5} - i\sqrt{7}$ is $-\sqrt{5} + i\sqrt{7}$
iv. Conjugate of $-i\sqrt{5}$ is $i\sqrt{5}$.

v. Conjugate of 4/5 is 4/5.

vi. Conjugate of 49 – i/7 is 49 + i/7.

Question-11

Solve the following equation: $\sqrt{3x+1}-\sqrt{x-1}=2$.

Solution:

$\sqrt{3x+1}-\sqrt{x-1}=2$

Squaring,

$(3x+1)+(x-1) - 2\sqrt{3x+1}\times\sqrt{x-1} = 4$

$4x - 2\sqrt{3x+1}\times\sqrt{x-1} = 4$

$\sqrt{3x+1}\times\sqrt{x-1} = 2x-2$

Squaring,

$3x^2 - 2x - 1 = (2x-2)^2$

$3x^2-2x-1 = 4x^2 - 8x+4$

$x^2-6x+5=0$

$D = b^2-4ac = 36 - 4\times 1\times 5 = 16 > 0$

$\sqrt{D} = 4$

Hence the two real and unequal roots are : $\frac{6+4}{2}, \frac{6-4}{2}$

i.e 5,1

Question-12

Find the conjugate of $\frac{1-i}{1+i}$.

Solution:

$\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{(1-i)^2}{1^2-(i)^2} = \frac{1-2i+i^2}{1+1} = \frac{1-2i-1}{2} = \frac{-2i}{2} = -i$.

$\therefore$Conjugate of $\frac{1-i}{1+i} = i$

Question-13

Show that if a,b,c,d,∈ R, $\overline{(a+ib)(c+id)} = (a-ib)(c-id)$ **= (a-ib) (c-id).**

Solution:

$(a+ib)(c+id) = ac + iad + ibc + i^2bd = ac + i(ad+bc) - bd = (ac-bd) + i(ad+bc)$

$\therefore \overline{ac - bd + i(ad + bc)}$

$= (ac-bd) - i(ad+bc)$ ---- (1)

$(a-ib)(c-id) = (ac-bd) - i(bc+ad)$ ------(2)

From (1) and (2) $\overline{(a+ib)(c+id)} = (a-ib)(c-id)$

Question-14

Find the value of x and y, if 4x + i(3x - y) = 3 – i6.

Solution:

$4x + i(3x - y) = 3 - i6$

Equating the real and imaginary, we have

$4x = 3$

$x = 3/4$

$3x - y = -6$

$3(3/4) - y = -6$

$9/4 - y = -6$

$-y = -6 - 9/4$

$y = 33/4$

Question-15

Solve the following equation: $2x^2+1 = 0$.

Solution:

$2x^2+1 = 0 \Rightarrow x^2 = -1/2$

Hence the complex roots of the equation are $+i\frac{\sqrt{2}}{2}, -i\frac{\sqrt{2}}{2}$.

Question-16

Does the equation $2x^2 - 4x + 3 = 0$ have equal roots? Find the roots.

Solution:

The given equation is $2x^2 - 4x + 3 = 0$.

Comparing with $ax^2 + bx + c = 0$

$a = 2, b = -4, c = 3$

$b^2 - 4ac = (-4)^2 - 4\times 2\times 3 = 16 - 24 = -8 < 0$

The roots are not equal.

Hence the roots of the given equation is $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{-8}}{2\times 2} = \frac{4 \pm 2\sqrt{2}i}{4} = 1 \pm \frac{1}{\sqrt{2}}i$

Question-17

Find the value of x and y, if (3y - 2) + i(7 − 2x) = 0.

Solution:

$(3y - 2) + i(7 - 2x) = 0$

Equating the real and imaginary, we have

$3y - 2 = 0$

$y = 2/3$

$7 - 2x = 0$

$2x = 7$

$x = 7/2$

The value of $x = 7/2$ and $y = 2/3$.

Question-18

If two complex numbers $z_1 z_2$ are such that $|z_1| = |z_2|$, is it then necessary that $z_1 = z_2$?

Solution:

$|z_1| = |x_1 + iy| = \sqrt{x_1^2 + y_1^2}$

$|z_2| = |x_2 + iy_2| = \sqrt{x_2^2 + y_2^2}$

$\therefore \quad |z_1| = |z_2|$

$\sqrt{x_1^2 + y_1^2} = \sqrt{x_2^2 + y_2^2}$

$x_1^2 + y_1^2 = x2^2 + y_2^2$

$x_1^2 = x_2^2$ and $y_1^2 = y_2^2$

$x_1 = \pm x_2 \; y_1 = \pm y$

$\therefore z_1$ need not be z_2

Question-19

Solve the following equation: $x^2 - 4x + 7 = 0$.

Solution:

$x^2 - 4x + 7 = 0$

$D = b^2 - 4ac = 16 - 4\times 1\times 7 = 16 - 28 = -12 < 0$

$\sqrt{D} = 2\sqrt{3}\,i$

Hence the two complex roots are : $\frac{4+2\sqrt{3}i}{2}, \frac{4-2\sqrt{3}i}{2}$

i.e $2+\sqrt{3}\,i$, $2+\sqrt{3}\,i$

Question-20

For what values of a is one of the roots of the equation $x^2 + (2a+1)x + a^2 + 2 = 0$ twice the value of the other.

Solution:

Let the roots be α, 2α.

$\alpha + 2\alpha = \frac{-(2a+1)}{1}$

$\Rightarrow 3\alpha = -2\alpha - 1$

$\Rightarrow \alpha = \frac{-2a-1}{3}$

$\alpha . 2\alpha = \frac{a^2+2}{1}$

$2\alpha^2 = \frac{a^2+2}{1}$

$2\left(\frac{-(2\alpha+1)}{3}\right)^2 = \alpha^2 + 2$

$2(4\alpha^2 + 4\alpha + 1) = 9(\alpha^2 + 2)$

$8\alpha^2 + 8\alpha + 2 = 9\alpha^2 + 18$

$-\alpha^2 + 8\alpha - 16 = 0$

$\alpha^2 - 8\alpha + 16 = 0$

$\alpha^2 - 4\alpha - 4\alpha + 16 = 0$

$\alpha(\alpha - 4) - 4(\alpha - 4) = 0$

$\therefore \alpha = 4$ or $\alpha = 4$

Question-21

If the difference of the root of $x^2 - bx + c = 0$ is the same as that of the roots of $x^2 - cx + b = 0$ then b+c+4 = 0 unless b - c = 0.

Solution:

Let α,β be the root of the equation x^2-bx+c = 0; γ, δand be the roots of the equation $x^2 - cx + b = 0$.

Then a + b = b, αβ=c , γ+ δ=c and γ δ = b

Given that a - b = g - δ

$(\alpha - \beta)^2 = (\gamma - \delta)^2$

$(\alpha + \beta)^2 - 4\alpha\beta = (\gamma + \delta)^2 - 4\gamma\delta$

$b^2 - 4c = c^2 - 4b$

$b^2 - c^2 + 4b - 4c = 0$

(b-c)(b+c)+4(b-c) = 0

(b-c) (b+c+4) = 0

Hence b-c = 0 or b+c+4 = 0

(ie) b+c+4 = 0 or b = c

Question-22

Find the value of x and y, if $\left(\frac{3}{\sqrt{5}} x - 5\right) + i2\sqrt{5}\, y = \sqrt{2}$.

Solution:

$\left(\frac{3}{\sqrt{5}} x - 5\right) + i2\sqrt{5}\, y = \sqrt{2}$

Equating the real and imaginary, we have

$\left(\frac{3}{\sqrt{5}} x - 5\right) = \sqrt{2}$

$\frac{3}{\sqrt{5}} x = \sqrt{2} + 5$

$x = \sqrt{5}\,(\sqrt{2} + 5)/3$

$2\sqrt{5}\, y = 0$

y = 0

The value of $x = \sqrt{5}\,(\sqrt{2} + 5)/3$ and y = 0.

Question-23

If z_1, z_2, z_3 are 3 complex numbers such that there exists a z with $|z_1 - z| = |z_2 - z| = |z_3 - z|$ show that z_1 , z_2 , z_3 lie on a circle in the plane diagram.

Solution:

Let z_1, z_2, z_3 be $x_1 + iy_1$, $x_2 + iy_2$ and $x_3 + iy_3$ respectively.

Representing points P, Q, R

Let the z be point O given by x + iy.

$|z_1 - z| = \sqrt{(x_1 - x)^2 + (y_1 - y)^2}$ = OP

Similarly $|z_2 - z|$ = OQ

and $|z_3 - z|$ = OR

$|z_1 - z| = |z_2 - z| = |z_3 - z|$

OP = OQ = OR = r

This means P, Q, R are points on a circle with centre O and radius r.

Or z_1, z_2, z_3 lie on a circle.

Question-24

Solve the following equation: $x^2+x+1=0$.

Solution:

$x^2+x+1=0$

$D = b^2-4ac = 1 - 4 = -3 < 0$

$\sqrt{D} = \sqrt{3}\,i$

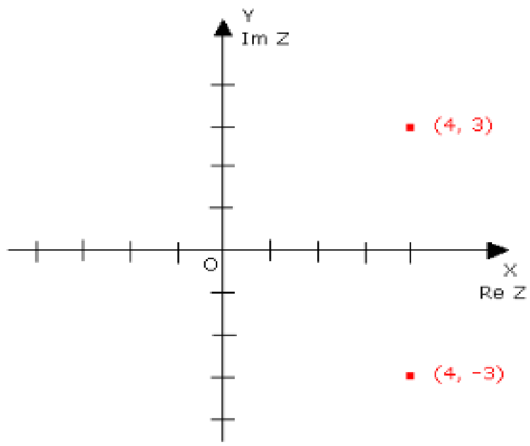
Hence the two complex roots are : $\frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}$

Question-25

Plot the following number and their complex conjugates on a complex number plane and find their absolute values: 4 – i3.

Solution:

Conjugate of 4 – 3i is 4 + 3i.



The absolute value of 4 – 3i

$= \sqrt{4^2 + 3^2}$

$= \sqrt{16 + 9}$

$= \sqrt{25}$

= 5

Question-26

A group of students decided to buy a tape-records from 170 to 195 rupee. But at the last moment two student backed out of the decision so that the remaining student had to pay 1rupee more than they had planned. What was the price of the tape recorder if the student paid equal shares ?

Solution:

Let the price of the tape recorder be Rs. x

Let no. of student be n.

At the last moment

No. of students = (n-2)

Increased contribution = $\frac{x}{n-2}$

Original contribution = $\frac{x}{n}$

According to the question

$\frac{x}{n-2} = \frac{x}{n} + 1$

$\frac{x}{n-2} = \frac{x+n}{n}$

$nx = (n-2)(x+n) = nx + n^2 - 2x - 2n$

$n^2 - 2n = 2x$

$x = \frac{n^2-2n}{2}$, Also $170 < x < 195$

$170 < \frac{n^2-2n}{2} < 195$

$\Rightarrow 340 \leq n^2 - 2n \leq 390$

Either $340 \leq n^2 - 2n$

$n^2 - 2n - 340 \geq 0$

Roots are given by

$n = \frac{2 \pm \sqrt{4+1360}}{2} = \frac{1 \pm \sqrt{341}}{2}$

$n \geq 1+\sqrt{341}$ or $n \leq 1 - \sqrt{341}$

n = 20 or $n^2 - 2n - 390 < 0$ ----- (1)

$n = \frac{2 \pm \sqrt{1564}}{2}$ $n = \frac{1 \pm \sqrt{391}}{2}$

$\therefore 1 - \sqrt{391} \leq n < 1 + \sqrt{391}$

Since n is a natural no. n = 1, 2, 3 ....20 ---- (2)

From (1) and (2),

n = 20

Cost of tape - recorder $x = \frac{n^2-2n}{2} = \frac{20^2-2(20)}{2}$ = = Rs. 180.

Question-27

Solve the following equation: $x^2 + 2x + 2 = 0$.

Solution:

$x^2 + 2x + 2 = 0$

$D = b^2 - 4ac = 4 - 4 \times 2 = -4 < 0$

$\sqrt{D} = 2i$

Hence the two complex roots are : $\frac{-2+2i}{2}, \frac{-2-2i}{2}$

i.e -1-i , -1+i

Question-28

Plot the following number and their complex conjugates on a complex number plane and find their absolute values: −3 + i5

Solution:

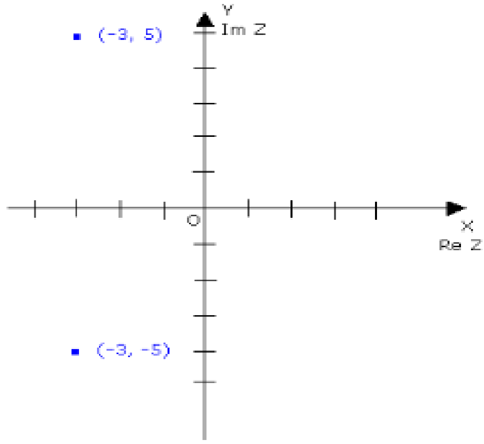
Conjugate of −3 + i5 is −3 - i5.

The absolute value of −3 + i5

$= \sqrt{(-3)^2 + 5^2}$

$= \sqrt{9 + 25}$

$= \sqrt{34}$



Question-29

Solve $(x^2-5x+7)^2$ - (x -2)(x-3) =1.

Solution:

$(x^2 - 5x + 7)^2 - (x - 2)(x - 3) = 1$

$(x^2 - 5x + 7)^2 - [x^2 - (2 + 3)x + 2 \times 3] = 1$

$(x^2-5x+7)^2 - [x^2 -5x+6] -1 = 0$

Let $x^2 - 5x = y$ -------------(1)

$(y+7)^2 - (y+6) -1 = 0$

$y^2 + 14y + 49 - y - 6 - 1 = 0$

$y^2 + 13y + 42 = 0$

$y = \frac{-13 \pm \sqrt{13^2 - 4(1)(42)}}{2 \times 1}$

$= \frac{-13 \pm \sqrt{169-168}}{2}$

$= \frac{-13 \pm \sqrt{1}}{2}$

$= \frac{-13+1}{2}$ or $\frac{-13-1}{2}$

$= \frac{-12}{2}$ or $\frac{-14}{2}$

= -6 or -7

∴ y = -6 or -7

Substituting y = -6 in (1)

$x^2 - 5x = -6$

$x^2 - 5x + 6=0$

x = 3 or x = 2

Substituting y = -7 in (1)

$x^2 - 5x = -7$

$x^2 - 5x + 7=0$

$x = \frac{5 \pm \sqrt{3}i}{2}$.

Question-30

Solve the following equation: $25x^2 - 30x + 11 = 0$.

Solution:

$25x^2 - 30x + 11 = 0$

$D = b^2 - 4ac = 900 - 4 \times 25 \times 11 = -200 < 0$

$\sqrt{D} = 10\sqrt{2}\, i$

Hence the two complex roots are : $\frac{30+10\sqrt{2}i}{50}, \frac{30-10\sqrt{2}i}{50}$

i.e $\frac{3+\sqrt{2}i}{5}, \frac{3-\sqrt{2}i}{5}$

Question-31

Prove that $x^4+4 = (x+1+i)(x+1-i)(x-1+i)(x-1-i)$.

Solution:

$(x+1+i)\,(x+1-i)\,(x-1+i)\,(x-1-i) = [(x+1)^2-i^2]\,[(x-1)^2 - i^2]$

$= (x^2+2x+1+1)\,(x^2-2x+1+1)$

$= [(x^2+2)+2x]\,[(x+2)-2x]$

$= (x^2+2)^2 - 4x^2$

$= x^4+4x^2+4-4x^2$

$= x^4+4$

Question-32

Solve the following equation: $5x^2 - 6x + 2 = 0$.

Solution:

$5x^2 - 6x + 2 = 0$

$D = b^2 - 4ac = 36 - 4\times 5\times 2 = -4 < 0$

$\sqrt{D} = 2i$

Hence the two complex roots are : $\frac{6+2i}{10}, \frac{6-2i}{10}$

i.e $\frac{3+i}{5}, \frac{3-i}{5}$

Question-33

Plot the following number and their complex conjugates on a complex number plane and find their absolute values: 5

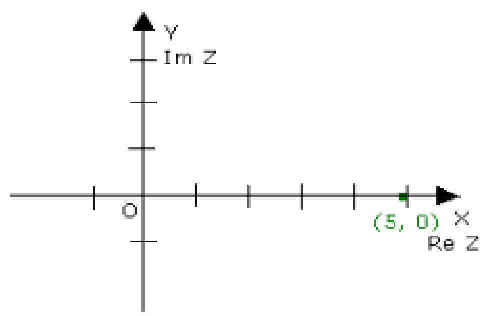
Solution:

Conjugate of 5 is 5.

The absolute value of 5 = $\sqrt{5^2+0^2}$

= $\sqrt{25}$

= 5



Question-34

If $(1+x)^n = p_0+p_1x+p_2x^2+ \ldots\ldots\ldots\ldots p_nx^n$, prove that :$p_0+p_3+p_6+\ldots\ldots\ldots\ldots = \frac{1}{3}(2^n + 2\cos\frac{n\pi}{3})$.

Solution:

$(1+x)^n = p_o+p_1 x + p_2x^2+ \ldots\ldots\ldots\ldots p_nx^n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$

Put x = 1,w,w^2 in (1) and add

[$1+w = -w^2$ and $1+w^2 = -w$]

$3(p_o+p_3+p_6 \ldots\ldots\ldots\ldots) = 2^n+(-w^2)^n + (-w)^n \ldots\ldots\ldots\ldots\ldots$ (2)

Now $w = \frac{-1+i\sqrt{3}}{2}$

$\therefore -w = \frac{1}{2} - i\frac{\sqrt{3}}{2} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$, ($\because r = 1, \theta = \frac{\pi}{3}$)

$-w^2 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$\therefore (-w)^n + (-w^2)^n = 2\cos\frac{n\pi}{3}$ (Demoivre' s Theorem)

Substituting in (2),$3(p_o+p_3+p_6\ldots\ldots\ldots\ldots) = 2^n + 2\cos\frac{n\pi}{3}$

or $p_o+p_3+p_6\ldots\ldots\ldots\ldots = \frac{1}{3}(2^n + 2\cos\frac{n\pi}{3})$.

Question-35

From an equation whose roots are the squares of the sum and difference of the roots of
$2x^2 + 2(m + n)x + m^2 + n^2 = 0$.

Solution:

Let α , β be the roots of the equation $2x^2 + 2(m + n)x + m^2 + n^2 = 0$.

Then $\alpha + \beta = -2(m + n)/2 = -(m + n)$

$\alpha\beta = (m^2 + n^2)/2$

The roots of the required equation are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$

Sum of the roots $= (\alpha + \beta)^2 + (\alpha - \beta)^2 = (\alpha + \beta)^2 + [(\alpha + \beta)^2 - 4\alpha\beta]$

$= (m + n)^2 + [(m + n)^2 - \frac{4(m^2 + n^2)}{2}]$

$= 4mn$

Product of the roots $= (\alpha + \beta)^2 (\alpha - \beta)^2 = (\alpha + \beta)^2 [(\alpha + \beta)^2 - 4\alpha\beta]$

$= (m + n)^2 [(m + n)^2 - \frac{4(m^2 + n^2)}{2}]$

$= (m + n)^2 [2mn - m^2 - n^2]$

The required equation is $x^2 - 4mnx + (m + n)^2[2mn - m^2 - n^2] = 0$

or $x^2 - 4mnx + (m + n)^2[-(m - n)^2] = 0$

or $x^2 - 4mnx - (m^2 - n^2)^2 = 0$

Question-36

Find the values of the root $\sqrt{1 - i}$.

Solution:

$1 - i = \sqrt{2}(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})$

$= \sqrt{2}\left[\cos(2n\pi + \frac{\pi}{4})\right] - i\sin(2n\pi + \frac{\pi}{4})$

$= \sqrt{2}(\cos(8n + 1)\frac{\pi}{4} - \sin(8n + 1)\frac{\pi}{4})$

$\sqrt{1 - i} = 2^{1/4}[\cos(8n+1)\frac{\pi}{8} - i\sin(8n + 1)\frac{\pi}{8}]$

$= 2^{1/4}(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8})$ for n = 0

$= 2^{1/4}(\cos(\pi + \frac{\pi}{8}) - i\sin(\pi + \frac{\pi}{8})$ for n = 1

$= -2^{1/4}(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8})$ where

$\cos\frac{\pi}{8} = \sqrt{\left(\frac{\sqrt{2} + 1}{2\sqrt{2}}\right)}$, $\sin\frac{\pi}{8} = \sqrt{\left(\frac{\sqrt{2} - 1}{2\sqrt{2}}\right)}$

Question-37

Solve the following equation: $3x^2 - 7x + 5 = 0$.

Solution:

$3x^2 - 7x + 5 = 0$

$D = b^2 - 4ac = 49 - 4\times 3\times 5 = -11 < 0$

$\sqrt{D} = \sqrt{11}\,i$

Hence the two complex roots are : $\frac{7+\sqrt{11}i}{6}, \frac{7-\sqrt{11}i}{6}$

Question-38

Solve the equation $25x^2 - 30x + 9 = 0$.

Solution:

$$X = \frac{+30 \pm \sqrt{30^2 - 4(25)(9)}}{2\times 25} = \frac{30 \pm \sqrt{900-900}}{50} = \frac{30}{50}$$

$$X = \frac{3}{5}, \frac{3}{5}$$

Question-39

Plot the following number and their complex conjugates on a complex number plane and find their absolute values: 2i

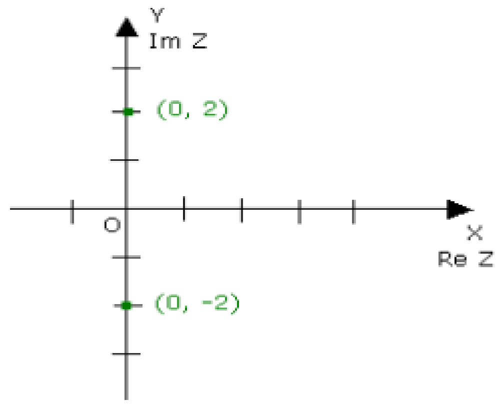
Solution:

Conjugate of 2i is -2i.

The absolute value of 2i = $\sqrt{0^2 + 2^2}$

= $\sqrt{4}$

= 2



Question-40

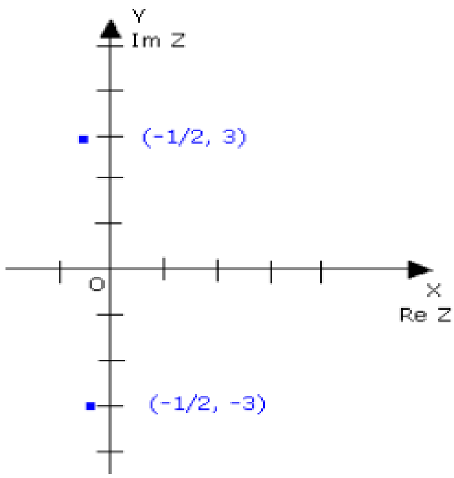
Plot the following number and their complex conjugates on a complex number plane and find their absolute values: -1/2 - 3i

Solution:

Conjugate of $-\frac{1}{2} - i3$ is $-\frac{1}{2} + i3$

The absolute value of $-\frac{1}{2} + i3$

$$= \sqrt{\left(-\frac{1}{2}\right)^2 + 3^2}$$
$$= \sqrt{\frac{1}{4} + 9}$$
$$= \sqrt{\frac{37}{4}}$$



Question-41

If the roots of $x^2 - lx + m = 0$ differ by 1, then prove that $l^2 = 4m + 1$.

Solution:

Let a , b be the roots of the equation $x^2 - lx + m = 0$.

$\alpha + \beta = l$

$\alpha\beta = m$

$\alpha - \beta = 1$

$(\alpha + \beta)^2 = (\alpha - \beta)^2 + 4\alpha\beta$

$l^2 = 1 + 4m$

Question-42

Solve the following equation: $13x^2 - 7x + 1 = 0$.

Solution:

$13x^2 - 7x + 1 = 0$

$D = b^2 - 4ac = 49 - 4\times 13\times 1 = -3 < 0$

$\sqrt{D} = \sqrt{3}\,i$

Hence the two complex roots are : $\frac{7+\sqrt{3}i}{26}, \frac{7-\sqrt{3}i}{26}$

Question-43

If z = x+iy and $z^{1/3}$= a-ib then show that $\frac{x}{a}-\frac{y}{b}=4(a^2-b^2)$.

Solution:

z = x+iy and $2^{1/3}$ = a-ib

$(x+iy)^{1/3}$ = a-ib

Cubing both sides,

$x+iy = (a-ib)^3$

$= a^3+b^3i-3abi(a-ib)$

$= a^3+b^3i-3a^2bi-3ab^2$

Equating the real and imaginary,

$x = a^3-3ab^2$

$y = b^3-3a^2b$

$\frac{x}{y}-\frac{y}{b}=a^2-3b^2-b^2+3a^2$

$= 4(a^2-b^2)$

Question-44

$(1-w+w^2)(1-w^2+w^4)(1-w^4+w^8)$.......... to 2n factors = 2^{2n}.

Solution:

$(1-w+w^2)(1-w^2+w^4)(1-w^4+w^8)$ .... 2n factors.

$= (1-w+w^2)(1-w^2+w)(1-w+w^2)$ .... 2n factors.

(since $w^4=w$, $w^8 = w^2$ ....)

$= (-2w)(-2w^2)(-2w)(-2w^2)$ .... 2n factors.

$= (2^2w^3)(2^2w^3)$ .... n factors.

$= (2^2)^n = 2^{2n}$

Question-45

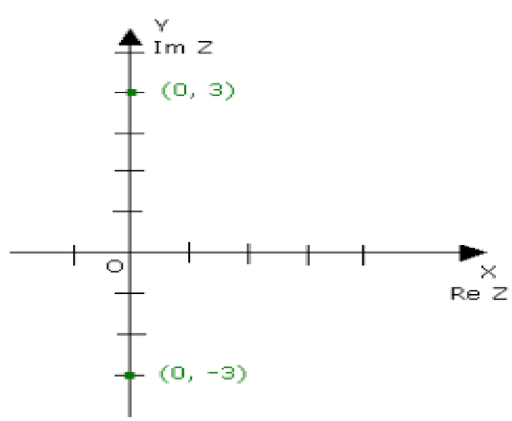
Plot the following number and their complex conjugates on a complex number plane and find their absolute values: √(-3)

Solution:

$\sqrt{-3}$ = 3i

Conjugate of 3i is −3i.

The absolute value of 3i = $\sqrt{3^2}$ = $\sqrt{9}$ = 3



Question-46

Solve the following equation: $9x^2+10x+3=0$.

Solution:

$9x^2+10x+3=0$

$D = b^2-4ac = 100-4\times 9\times 3 = -8 < 0$

$\sqrt{D} = 2\sqrt{2}i$

Hence the two complex roots are : $\frac{-10+2\sqrt{2}i}{18}, \frac{-10-2\sqrt{2}i}{18}$

i.e $\frac{-5+\sqrt{2}i}{9}, \frac{-5-\sqrt{2}i}{9}$

Complex Numbers & Quadratic Equations

1. For a positive integer n, find the value of $(1-i)^n \left(1-\frac{1}{i}\right)^n$

2. Evaluate $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $n \in \mathbf{N}$.

3. If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$, then find (x, y).

4. If $\frac{(1+i)^2}{2-i} = x + iy$, then find the value of $x + y$.

5. If $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$, then find (a, b).

6. If $a = \cos\theta + i\sin\theta$, find the value of $\frac{1+a}{1-a}$.

7. If $(1 + i)\, z = (1 - i)\, \bar{z}$, then show that $z = -i\bar{z}$.

8. If $z = x + iy$, then show that $z\bar{z} + 2(z + \bar{z}) + b = 0$, where $b \in \mathbf{R}$, represents a circle.

9. If the real part of $\frac{\bar{z}+2}{\bar{z}-1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle.

10. Show that the complex number z, satisfying the condition $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ lies on a circle.

11. Solve the equation $|z| = z + 1 + 2i$.

12. If $|z+1| = z + 2(1 + i)$, then find z.

13. If $\arg(z - 1) = \arg(z + 3i)$, then find $x - 1 : y$. where $z = x + iy$

14. Show that $\left|\frac{z-2}{z-3}\right| = 2$ represents a circle. Find its centre and radius.

15. If $\frac{z-1}{z+1}$ is a purely imaginary number $(z \neq -1)$, then find the value of $|z|$.

16. z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$, then show that $z_1 = -\bar{z}_2$.

17. If $|z_1| = 1\ (z_1 \neq -1)$ and $z_2 = \frac{z_1 - 1}{z_1 + 1}$, then show that the real part of z_2 is zero.

18. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then find $\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right)$.

19. If $|z_1| = |z_2| = \ldots = |z_n| = 1$, then

show that $|z_1 + z_2 + z_3 + \ldots + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \ldots + \frac{1}{z_n}\right|$.

20. If for complex numbers z_1 and z_2, $\arg(z_1) - \arg(z_2) = 0$, then show that $|z_1 - z_2| = |z_1| - |z_2|$

21. Solve the system of equations $\text{Re}(z^2) = 0$, $|z| = 2$.

22. Find the complex number satisfying the equation $z + \sqrt{2}\ |(z + 1)| + i = 0$.

23. Write the complex number $z = \frac{1-i}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$ in polar form.

24. If z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then show that $\bar{z}w = -i$.

25. What is the conjugate of $\frac{2-i}{(1-2i)^2}$?

26. If $|z_1|=|z_2|$, is it necessary that $z_1 = z_2$?

27. If $\frac{(a^2+1)^2}{2a-i} = x + iy$, what is the value of $x^2 + y^2$?

28. Find z if $|z|=4$ and $\arg(z) = \frac{5\pi}{6}$.

29. Find $\left|(1+i)\frac{(2+i)}{(3+i)}\right|$

30. Find principal argument of $(1 + i\sqrt{3})^2$.

31. Where does z lie, if $\left|\frac{z-5i}{z+5i}\right| = 1$.

32. $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other for:

(A) $x = n\pi$

(B) $x = \left(n+\frac{1}{2}\right)\frac{\pi}{2}$

(C) $x = 0$

(D) No value of x

33. The real value of α for which the expression $\frac{1-i\sin\alpha}{1+2i\sin\alpha}$ is purely real is :

(A) $(n+1)\frac{\pi}{2}$

(B) $(2n+1)\frac{\pi}{2}$

(C) $n\pi$

(D) None of these, where $n \in \mathbf{N}$

34. If $z = x + iy$ lies in the third quadrant, then $\frac{\bar{z}}{z}$ also lies in the third quadrant if

(A) $x > y > 0$

(B) $x < y < 0$

(C) $y < x < 0$

(D) $y > x > 0$

35. The value of $(z+3)(\bar{z}+3)$ is equivalent to

(A) $|z+3|^2$

(B) $|z-3|$

(C) z^2+3

(D) None of these

36. If $\left(\frac{1+i}{1-i}\right)^x = 1$, then

(A) $x = 2n+1$

(B) $x = 4n$

(C) $x = 2n$

(D) $x = 4n+1$, where $n \in \mathbf{N}$

37. A real value of x satisfies the equation $\left(\frac{3-4ix}{3+4ix}\right) = \alpha - i\beta$ $(\alpha, \beta \in \mathbf{R})$ if $\alpha^2 + \beta^2 =$

(A) 1

(B) -1

(C) 2

(D) -2

38. Which of the following is correct for any two complex numbers z_1 and z_2?

(A) $|z_1 z_2| = |z_1||z_2|$

(B) $\arg(z_1 z_2) = \arg(z_1).\arg(z_2)$

(C) $|z_1 + z_2| = |z_1| + |z_2|$

(D) $|z_1 + z_2| \geq |z_1| - |z_2|$

39. The point represented by the complex number $2 - i$ is rotated about origin through

an angle $\frac{\pi}{2}$ in the clockwise direction, the new position of point is:

(A) $1 + 2i$ (B) $-1 - 2i$ (C) $2 + i$ (D) $-1 + 2i$

40. Let $x, y \in \mathbf{R}$, then $x + iy$ is a non real complex number if:

(A) $x = 0$ (B) $y = 0$ (C) $x \neq 0$ (D) $y \neq 0$

41. If $a + ib = c + id$, then

(A) $a^2 + c^2 = 0$ (B) $b^2 + c^2 = 0$

(C) $b^2 + d^2 = 0$ (D) $a^2 + b^2 = c^2 + d^2$

42. The complex number z which satisfies the condition $\left|\frac{i+z}{i-z}\right| = 1$ lies on

(A) circle $x^2 + y^2 = 1$ (B) the x-axis

(C) the y-axis (D) the line $x + y = 1$.

43. If z is a complex number, then

(A) $|z^2| > |z|^2$ (B) $|z^2| = |z|^2$

(C) $|z^2| < |z|^2$ (D) $|z^2| \geq |z|^2$

44. $|z_1 + z_2| = |z_1| + |z_2|$ is possible if

(A) $z_2 = \overline{z_1}$ (B) $z_2 = \frac{1}{z_1}$

(C) $\arg(z_1) = \arg(z_2)$ (D) $|z_1| = |z_2|$

45. The real value of θ for which the expression $\frac{1 + i\cos\theta}{1 - 2i\cos\theta}$ is a real number is:

(A) $n\pi + \frac{\pi}{4}$ (B) $n\pi + (-1)^n \frac{\pi}{4}$

(C) $2n\pi \pm \frac{\pi}{2}$ (D) none of these.

46. The value of arg (x) when $x < 0$ is:

(A) 0 (B) $\frac{\pi}{2}$

(C) π

(D) none of these

47. If $f(z) = \frac{7-z}{1-z^2}$, where $z = 1 + 2i$, then $|f(z)|$ is

(A) $\frac{|z|}{2}$

(B) $|z|$

(C) $2|z|$

(D) none of these.

CBSE Class 11 Mathematics

Important Questions

Chapter 5

Complex Numbers and Quadratic Equations

1 Marks Questions

1. Evaluate i^{-39}

Ans. $i^{-39}=\frac{1}{i^{39}}=\frac{1}{(i^4)^9 . i^3}$

$=\frac{1}{1\times(-i)}$ $\quad\left[\because i^4=1,\ i^3=-i\right.$

$=\frac{1}{-i}\times\frac{i}{i}$

$=\frac{i}{-i^2}=\frac{i}{-(-1)}=i$ $\quad\left[\because i^2=-1\right.$

2. Solved the quadratic equation $x^2+x+\frac{1}{\sqrt{2}}=0$

Ans. $\frac{x^2}{1}+\frac{x}{1}+\frac{1}{\sqrt{2}}=0$

$\frac{\sqrt{2}x^2+\sqrt{2}x+1}{\sqrt{2}}=\frac{0}{1}$

$\sqrt{2}x^2+\sqrt{2}x+1=0$

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

$$= \frac{-\sqrt{2} \pm \sqrt{2 - 4\sqrt{2}}}{2 \times \sqrt{2}}$$

$$= \frac{-\sqrt{2} \pm \sqrt{2}\sqrt{1 - 2\sqrt{2}}}{2\sqrt{2}}$$

$$= \frac{-1 \pm \sqrt{2\sqrt{2} - 1}\ i}{2}$$

3. If $\left(\frac{1+i}{1-i}\right)^m$ = 1, then find the least positive integral value of m.

Ans. $\left(\frac{1+i}{1-i}\right)^m = 1$

$$\left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^m = 1$$

$$\left(\frac{1+i^2+2i}{1-i^2}\right)^m = 1$$

$$\left(\frac{\not{1} - \not{1} + 2i}{2}\right)^m = 1 \qquad \left[\ \because i^2 = -1\right.$$

$$i^m = 1$$

m=4

4. Evaluate (1+ i)^4

Ans. $(1+i)^4 = \left[(1+i)^2\right]^2$

$= \left(1+i^2+2i\right)^2$

$= (1-1+2i)^2$

$= (2i)^2 = 4i^2$

$= 4(-1) = -4$

5. Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$

Ans. Let z = $\frac{1+i}{1-i} - \frac{1-i}{1+i}$

$= \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)}$

$= \frac{4i}{2}$

$= 2i$

$z = 0 + 2i$

$|z| = \sqrt{(0)^2 + (2)^2}$

$= 2$

6. Express in the form of a + ib. $(1+3i)^{-1}$

Ans. $(1+3i)^{-1} = \frac{1}{1+3i} \times \frac{1-3i}{1-3i}$

$$= \frac{1-3i}{(1)^2-(3i)^2}$$

$$= \frac{1-3i}{1-9i^2}$$

$$= \frac{1-3i}{1+9} \qquad [i^2 = -1$$

$$= \frac{1-3i}{10}$$

$$= \frac{1}{10} - \frac{3i}{10}$$

7. Explain the fallacy in -1 = i. i. = $\sqrt{-1}.\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$

Ans. $1 = \sqrt{1} = \sqrt{(-1)(-1)}$ is okay but

$\sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1}$ is wrong.

8. Find the conjugate of $\frac{1}{2-3i}$

Ans. Let z = $\frac{1}{2-3i}$

$$z = \frac{1}{2-3i} \times \frac{2+3i}{2+3i} i$$

$$= \frac{2+3i}{(2)^2-(3i)^2}$$

$$= \frac{2+3i}{4+9}$$

$$= \frac{2+3i}{13}$$

$$z = \frac{2}{13} + \frac{3}{13} i$$

$$\bar{z} = \frac{2}{13} - \frac{3}{13}$$

9. Find the conjugate of – 3i – 5.

Ans. Let z = 3i – 5

$$\bar{z} = 3i - 5$$

10. Let z_1 = 2 – i, z_2 = -2+i Find Re $\left(\frac{z_1 z_2}{\overline{z_1}} \right)$

Ans. $z_1 z_2$ = (2 – i)(-2 + i)

$$= -4 + 2i + 2i - i^2$$

$$= -4 + 4i + 1$$

$$= 4i - 3$$

$$\overline{z_1} = 2 + i$$

$$\frac{z_1 z_2}{\overline{z}_1} = \frac{4i - 3}{2 + i} \times \frac{2 - i}{2 - i}$$

$$= \frac{8i - 6 - 4i^2 + 3i}{4 - i^2}$$

$$= \frac{11i - 2}{5}$$

$$\frac{z_1 z_2}{z_1} = \frac{11}{5} i - \frac{2}{5}$$

$$\text{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = -\frac{2}{5}$$

11. Express in the form of a + ib (3i-7) + (7-4i) – (6+3i) + i^{23}

Ans. Let

Z = $\cancel{3i} - \cancel{7} + \cancel{7} - 4i - 6 - \cancel{3i} + \left(i^4\right)^5 .i^3$

$= -4i - 6 - i \qquad \left[\begin{array}{l} \because\ i^4 = 1 \\ i^3 = -i \end{array}\right.$

$= -5i - 6$

$= -6 + \left(-5i\right)$

12. Find the conjugate of $\sqrt{-3} + 4i^2$

Ans. $\text{Let } z = \sqrt{-3} + 4i^2$

$= \sqrt{3}\ i\ - 4$

$\bar{z} = -\sqrt{3}\ i\ - 4$

13. Solve for x and y, 3x + (2x-y) i= 6 – 3i

Ans. 3x = 6

x = 2

2x – y = - 3

2 × 2 – y = - 3

- y = - 3 – 4

y = 7

14. Find the value of 1+i^2 + i^4 + i^6 + i^8 + ---- + i^{20}

Ans. $1+i^2+(i^2)^2+(i^2)^3+(i^2)^4+-----+(i^2)^{10}=1$ $\left[\because i^2 = -1\right.$

15. Multiply 3-2i by its conjugate.

Ans.Let z = 3 – 2i

$$\bar{z}=3+2i$$
$$z\ \bar{z}=(3-2i)(3+2i)$$
$$= 9 + \not{6i} - \not{6i} - 4i^2$$
$$= 9 - 4(-1)$$
$$= 13$$

16. Find the multiplicative inverse 4 – 3i.

Ans. Let z = 4 – 3i

$$\bar{z}=4+3i$$
$$|z|=\sqrt{16+9}=5i$$
$$z^{-1}=\frac{\bar{z}}{|z|^2}$$
$$=\frac{4+3i}{25}$$
$$=\frac{4}{25}+\frac{3}{25}$$

17. Express in term of a + ib

$$\frac{\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)}{\left(\sqrt{3}+\sqrt{2}i\right)-\left(\sqrt{3}-i\sqrt{2}\right)}$$

Ans. $= \frac{(3)^2-\left(i\sqrt{5}\right)^2}{\sqrt{3}+\sqrt{2}i-\sqrt{3}+i\sqrt{2}}$

$$=\frac{9+5}{2\sqrt{2}i}=\frac{14}{2\sqrt{2}i}$$

$$=\frac{7}{\sqrt{2}i}\times\frac{\sqrt{2}i}{\sqrt{2}i}=\frac{7\sqrt{2}i}{-2}$$

18. Evaluate $i^n+i^{n+1}+i^{n+2}+i^{n+3}$

Ans. $=i^n+i^n.i^1+i^n.i^2+i^n.i^3$

$$=i^n+i^n.i-i^n+i^n.(-i) \quad \left[\begin{array}{l} i^3=-i \\ i^2=-1 \end{array}\right.$$

$$=0$$

19. If 1, w, w^2 are three cube root of unity, show that (1 – w + w^2) (1 + w – w^2) = 4

Ans.(1 – w + w^2) (1 + w – w^2)

(1 + w^2 - w) (1 + w – w^2)

$$(-w-w)\left(-w^2-w^2\right) \quad \left[\begin{array}{l} \because 1+w=-w^2 \\ 1+w^2=-w \end{array}\right.$$

$$(-2w)\left(-2w^2\right)$$

$$4w^3 \quad \left[w^3=1\right.$$

4×1

$=4$

20. Find that sum product of the complex number $-\sqrt{3}+\sqrt{-2}$ and $2\sqrt{3}-i$

Ans. $z_1+z_2=-\sqrt{3}+\sqrt{2}i+2\sqrt{3}-i$

$=\sqrt{3}+\left(\sqrt{2}-1\right)i$

$z_1z_2=\left(-\sqrt{3}+\sqrt{2}i\right)\left(2\sqrt{3}-i\right)$

$=-6+\sqrt{3}i+2\sqrt{6}i-\sqrt{2}\ i^2$

$=-6+\sqrt{3}\ i+2\sqrt{6}\ i+\sqrt{2}$

$=\left(-6+\sqrt{2}\right)+\left(\sqrt{3}+2\sqrt{6}\right)i$

21. Write the real and imaginary part 1 – 2i^2

Ans. Let z = 1 – 2i^2

=1 – 2 (-1)

= 1 + 2

= 3

= 3 + 0.i

Re (z) = 3, Im (z) = 0

22. If two complex number z_1, z_2 are such that $|z_1| = |z_2|$, is it then necessary that z_1 = z_2

Ans. Let z_1 = a + ib

$|z_1| = \sqrt{a^2 + b^2}$

$z_2 = b + ia$

$|z_2| = \sqrt{b^2 + a^2}$

Hence $|z_1| = |z_2|$ but $z_1 \neq z_2$

23. Find the conjugate and modulus of $\overline{9-i} + \overline{6+i^3} - \overline{9+i^2}$

Ans. Let $z = \overline{9-i} + \overline{6-i} \;- \overline{9-1}$

$= 9 + i + 6 + i - 0$

$= 5 + 2i$

$\bar{z} = 5 - 2i$

$|z| = \sqrt{(5)^2 + (-2)^2}$

$= \sqrt{25 + 4}$

$= \sqrt{29}$

24. Find the number of non zero integral solution of the equation $|1\text{-}i|^x = 2^x$

Ans. $|1-i|^x = 2^x$

$\left(\sqrt{(1)^2 + (-1)^2}\right)^x = 2^x$

$\left(\sqrt{2}\right)^x = 2^x$

$(2)^{\frac{1}{2}x} = 2^x$

$$\frac{1}{2}x = x$$

$$\frac{1}{2} = 1$$

$$1 = 2$$

Which is false no value of x satisfies.

25. If (a + ib) (c + id) (e + if) (g + ih) = A + iB then show that

$$\left(a^2+b^2\right)\left(c^2+d^2\right)\left(e^2+f^2\right)\left(g^2+h^2\right) = A^2+B^2$$

Ans. $\left(a+ib\right)\left(c+id\right)\left(e+if\right)\left(g+ih\right) = A+iB$

$$\Rightarrow \left|\left(a+ib\right)\left(c+id\right)\left(e+if\right)\left(g+ih\right)\right| = \left|A+iB\right|$$

$$\left|a+ib\right|\left|c+id\right|\left|e+if\right|\left|g+ih\right| = \left|A+iB\right|$$

$$\left(\sqrt{a^2+b^2}\right)\left(\sqrt{c^2+d^2}\right)\left(\sqrt{e^2+f^2}\right)\left(\sqrt{g^2+h^2}\right) = \sqrt{A^2+B^2}$$

sq. both side

$$\left(a^2+b^2\right)\left(c^2+d^2\right)\left(e^2+f^2\right)\left(g^2+h^2\right) = A^2+B^2$$

CBSE Class 12 Mathematics

Important Questions

Chapter 5

Complex Numbers and Quadratic Equations

4 Marks Questions

1.If x + ι y = $\frac{a+ib}{a-ib}$ **Prove that** $x^2 + y^2 = 1$

Ans. $x+iy=\frac{a+ib}{a-ib}$ (i) (Given)

taking conjugate both side

$x-iy=\frac{a-ib}{a+ib}$ (ii)

(i) × (ii)

$(x+iy)(x-iy)=\left(\frac{a+ib}{a-ib}\right)\times\left(\frac{a-ib}{a+ib}\right)$

$(x)^2-(iy)^2=1$

$x^2 + y^2 = 1$

$[i^2 = -1$

2.Find real θ such that $\frac{3+2i\ \mathrm{Sin}\,\theta}{1-2i\ \mathrm{Sin}\,\theta}$ **is purely real.**

Ans. $\frac{3+2i\ Sin\theta}{1-2i\ Sin\theta}=\frac{3+2i\ Sin\theta}{1-2i\ Sin\theta}\times\frac{1+2i\ Sin\theta}{1+2i\ Sin\theta}$

$=\frac{3+6i\ Sin\theta+2i\ Sin\theta-4Sin^2\theta}{1+4Sin^2\theta}$

$$= \frac{3-4\ Sin^2\theta}{1+4\ Sin^2\theta} + \frac{8i\ Sin\theta}{1+4\ Sin^2\theta}$$

For purely real

Im (z) = 0

$$\frac{8Sin\theta}{1+4Sin^2\theta} = 0$$

$$Sin\theta = 0$$

$$\theta = n\pi$$

3.Find the modulus of $\frac{(1+i)(2+i)}{3+i}$

Ans. $\left|\frac{(1+i)(2+i)}{3+i}\right| = \frac{|(1+i)||2+i|}{|3+i|}$

$$= \frac{\left(\sqrt{1^2+1^2}\right)\left(\sqrt{4+1}\right)}{\sqrt{(3)^2+(1)^2}}$$

$$= \frac{\left(\sqrt{2}\right)\left(\sqrt{5}\right)}{\sqrt{10}}$$

$$= \frac{\sqrt{2}\times\sqrt{5}}{\sqrt{2}\times\sqrt{5}}$$

$$= 1$$

4.If $|a+ib| = 1$, **then Show that** $\frac{1+b+ai}{1+b-ai} = b+ai$

Ans. $|a+ib| = 1$

$$\sqrt{a^2+b^2}=1$$

$$a^2+b^2=1$$

$$\frac{1+b+ai}{1+b-ai}=\frac{(1+b)+ai}{(1+b)-ai}\times\frac{(1+b)+ai}{(1+b)+ai}$$

$$=\frac{(1+b)^2+(ai)^2+2(1+b)(ai)}{(1+b)^2-(ai)^2}$$

$$=\frac{1+b^2+2b-a^2+2ai+2abc}{1+b^2+2a-a^2}$$

$$=\frac{(a^2+b^2)+b^2+2b-a^2+2ai+2abi}{(a^2+b^2)+b^2+2b-a^2}$$

$$=\frac{2b^2+2b+2ai+2abi}{2b^2+2b}$$

$$=\frac{b^2+b+ai+abi}{b^2+b}$$

$$=\frac{b(b+1)+ai(b+1)}{b(b+1)}$$

$$=b+ai$$

5.If x – iy = $\sqrt{\frac{a-ib}{c-id}}$ **Prove that** $(x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}$

Ans. $x-iy=\sqrt{\frac{a-ib}{c-id}}$ (1) (Given)

Taking conjugate both side

$$x+iy=\sqrt{\frac{a+ib}{c+id}} \quad \text{(ii)}$$

(i) $\times$ (ii)

$$(x-iy)\times(x+iy)=\sqrt{\frac{a-ib}{c-id}}\times\sqrt{\frac{a+ib}{c+id}}$$

$$(x)^2-(iy)^2=\sqrt{\frac{(a)^2-(ib)^2}{(c)^2-(id)^2}}$$

$$x^2+y^2=\sqrt{\frac{a^2+b^2}{c^2+d^2}}$$

squaring both side

$$\left(x^2+y^2\right)^2=\frac{a^2+b^2}{c^2+d^2}$$

6.If $a+ib=\frac{c+i}{c-i}$, where a, b, c are real prove that $a^2+b^2=1$ and $\frac{b}{a}=\frac{2c}{c^2-1}$

Ans. $a+ib=\frac{c+i}{c-i}$ (Given) (i)

$$a+ib=\frac{c+i}{c-i}\times\frac{c+i}{c+i}$$

$$a+ib=\frac{c^2+2ci+i^2}{c^2-i^2}$$

$$a+ib=\frac{c^2-1}{c^2+1}+\frac{2c}{c^2+1}i$$

$$a=\frac{c^2-1}{c^2+1},\ b=\frac{2c}{c^2+1}$$

$$a^2+b^2=\left(\frac{c^2-1}{c^2+1}\right)^2+\frac{4c^2}{\left(c^2+1\right)^2}$$

$$=\frac{\left(c^2+1\right)^2}{\left(c^2+1\right)^2}$$

$a^2 + b^2 = 1$

$$\frac{b}{a}=\frac{2c}{c^2-1}$$

$$\frac{b}{a}=\frac{\frac{2c}{c^2+1}}{\frac{c^2-1}{c^2+1}}$$

7.If z_1 = 2-i and Z_2 = 1+i Find $\left|\frac{z_1+z_2+1}{z_1-z_2+i}\right|$

Ans.$z_1 + z_2 + 1 = 2 - i + 1 + i + 1 = 4$

$$z_1-z_2+i=2-i-1-i+i=1-i$$

$$\left|\frac{z_1+z_2+1}{z_1-z_2+i}\right|=\left|\frac{4}{1-i}\right|$$

$$=\frac{|4|}{|1-i|}$$

$$=\frac{4}{\sqrt{1^2+(-1)^2}}$$

$$=\frac{4}{\sqrt{2}}$$

$$=\frac{4}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$

$$=\frac{4\sqrt{2}}{2}$$

$$=2\sqrt{2}$$

8.If (p + iq)^2 = x + iy Prove that (p^2 + q^2)^2 = x^2 + y^2

Ans.(p + iq)^2 = x + iy (i)

Taking conjugate both side

(p – iq)^2 = x –iy (ii)

(i) × (ii)

$$(p+iq)^2(p-iq)^2=(x+iy)(x-iy)$$

$$[(p+iq)(p-iq)]^2=(x)^2-(iy)^2$$

$$[(p)^2-(iq)^2]^2=x^2-i^2y^2$$

$$\Rightarrow(p^2+q^2)^2=x^2+y^2$$

9.If $a+ib=\frac{(x+i)^2}{2x^2+1}$ Prove that $a^2+b^2=\frac{(x^2+1)^2}{(2x^2+1)^2}$

Ans. $a+ib=\frac{(x+i)^2}{2x^2+1}$ (i) (Given)

Taking conjugate both side

$$a-ib=\frac{(x-i)^2}{2x^2+1} \quad \text{(ii)}$$

(i) × (ii)

$$(a+ib)(a-ib)=\left(\frac{(x+i)^2}{2x^2+1}\right)\times\left(\frac{(x-i)^2}{2x^2+1}\right)$$

$$(a)^2-(ib)^2=\frac{(x^2-i^2)^2}{(2x^2+1)^2}$$

$$a^2+b^2=\frac{(x^2+1)^2}{(2x^2+1)^2} \quad \text{proved.}$$

10.If $(x+iy)^3=u+iv$ then show that $\frac{u}{x}+\frac{v}{y}=4(x^2-y^2)$

Ans. $(x+iy)^3=4+iv$

$$x^3+(iy)^3+3x^2(iy)+3.x(iy)^2=u+iv$$

$$x^3-iy^3+3x^2yi-3xy^2=u+iv$$

$$x^3-3xy^2+(3x^2y-y^3)i=u+iv$$

$$x(x^2-3y^2)+y(3x^2-y^2)i=u+iv$$

$$x(x^2-3y^2)=u,\ y(3x^2-y^2)=v$$

$$x^2-3y^2=\frac{u}{x} \quad \text{(i)} \quad \Big| \quad 3x^2-y^2=\frac{v}{y} \quad \text{(ii)}$$

(i) + (ii)

$4x^2 - 4y^2 = \frac{u}{x} + \frac{v}{y}$

$4\left(x^2 - y^2\right) = \frac{u}{x} + \frac{v}{y}$

11.Solve $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

Ans. $\sqrt{3}\ x^2 - \sqrt{2}\ x + 3\ \sqrt{3} = 0$

$a = \sqrt{3}, b = -\sqrt{2}, c = 3\sqrt{3}$

$D = b^2 - 4ac$

$= \left(-\sqrt{2}\right)^2 - 4 \times \sqrt{3}\left(3\sqrt{3}\right)$

$= 2 - 36$

$= -34$

$x = \frac{-b \pm \sqrt{D}}{2a}$

$= \frac{-\left(-\sqrt{2}\right) \pm \sqrt{-34}}{2 \times \sqrt{3}}$

$= \frac{\sqrt{2} \pm \sqrt{34}\ i}{2\sqrt{3}}$

12.Find the modulus $i^{25} + \left(1+3i\right)^3$

Ans.$i^{25} + (1+3i)^3$

$= \left(i^4\right)^6 .i + 1 + 27i^3 + 3(1)(3i)(1+3i)$

$= i + \left(1 - 27i + 9i + 27i^2\right)$

$= i + 1 - 18i - 27$

$= -26 - 17i$

$\left|i^{25} + (1+3i)^3\right| = |-26 - 17i|$

$= \sqrt{(-26)^2 + (-17)^2}$

$= \sqrt{676 + 289}$

$= \sqrt{965}$

13.If $a + ib = \dfrac{(x+i)^2}{2x-i}$ prove that $a^2 + b^2 = \dfrac{(x^2+1)^2}{4x^2+1}$

Ans. $a + ib = \dfrac{(x+i)^2}{2x-i}$ (i) (Given)

$a - ib = \dfrac{(x-i)^2}{2x+i}$ (ii) [taking conjugate both side

(i) × (ii)

$(a+ib)(a-ib) = \dfrac{(x+i)^2}{(2x-i)} \times \dfrac{(x-i)^2}{(2x+i)}$

$a^2 + b^2 = \dfrac{(x^2+1)^2}{4x^2+1}$ proved.

14.Evaluate $\left[i^{18} + \left(\dfrac{1}{i}\right)^{25}\right]^3$

Ans. $\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3$

$\left[\left(i^4\right)^4.i^2+\frac{1}{i^{25}}\right]^3$

$\left[i^2+\frac{1}{\left(i^4\right)^6.i}\right]^3$

$\left[-1+\frac{1}{i}\right]^3$

$\left[-1+\frac{i^3}{i^4}\right]^3$

$[-1-i]^3=-(1+i)^3$

$=-\left[1^3+i^3+3.1.i(1+i)\right]$

$=-\left[1-i+3i+3i^2\right]$

$=-[1-i+3i-3]$

$=-[-2+2i]=2-2i$

15.Find that modulus and argument $\frac{1+i}{1-i}$

Ans. $\frac{1+i}{1-i}=\frac{1+i}{1-i}\times\frac{1+i}{1+i}$

$=\frac{(1+i)^2}{1^2-i^2}$

$$= \frac{1+i^2+2i}{1+1}$$

$$= \frac{2i}{2}$$

$= i$

$z = 0 + i$

$r = |z| = \sqrt{(0)^2 + (1)^2} = 1$

Let α be the acute $\angle$s

$\tan\ \alpha = \left|\frac{1}{0}\right|$

$\alpha = \pi/2$

$\arg\ (z) = \pi/2$

$r = 1$

16.For what real value of x and y are numbers equal (1+i) y^2 + (6+i) and (2+i) x

Ans.(1+i) y^2 + (6 + i) = (2 + i) x

$y^2 + iy^2 + 6 + i = 2x + xi$

$(y^2 + 6) + (y^2 + 1)\ i = 2x + xi$

$y^2 + 6 = 2x$

$y^2 + 1 = x$

$y^2 = x - 1$

$x - 1 + 6 = 2x$

$5 = x$

$y = \pm 2$

17.If x + iy = $\sqrt{\frac{1+i}{1-i}}$, prove that $x^2 + y^2 = 1$

Ans. $x + iy = \sqrt{\frac{1+i}{1-i}}$ (i) (Given)

taking conjugate both side

$x - iy = \sqrt{\frac{1-i}{1+i}}$ (ii)

(i) × (ii)

$(x+iy)(x-iy) = \sqrt{\frac{1+i}{1-i}} \times \sqrt{\frac{1-i}{1+i}}$

$(x)^2 - (iy)^2 = 1$

$x^2 + y^2 = 1$

Proved.

18.Convert in the polar form $\frac{1+7i}{(2-i)^2}$

Ans. $\frac{1+7i}{(2-i)^2} = \frac{1+7i}{4+i^2-4i} = \frac{1+7i}{3-4i}$

$= \frac{1+7i}{3-4i} \times \frac{3+4i}{3+4i}$

$= \frac{3+4i+21i+28i^2}{9+16}$

$$= \frac{25i - 25}{25} = i - 1$$

$$= -1 + i$$

$$r = |z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

Let α be the acute $\angle$s

$$\text{ten } \alpha = \left|\frac{1}{-1}\right|$$

$$\alpha = \pi/4$$

since Re (z) < 0, Im (z) > 0

$$\theta = \pi - \alpha$$

$$= \pi - \frac{\pi}{4} = 3\pi/4$$

$$z = r(\text{Cos}\theta + \text{i Sin}\theta)$$

$$= \sqrt{2}\left(\text{Cos } \frac{3\pi}{4} + i Sin \frac{3\pi}{4}\right)$$

19.Find the real values of x and y if (x - iy) (3 + 5i) is the conjugate of – 6 – 24i

Ans.

(x – iy) (3 + 5i) = - 6 + 24i

3x + 5xi – 3yi – 5yi^2 = - 6 + 24i

$$(3x + 5y) + (5x - 3y)i = -6 + 24i$$

$$3x + 5y = -6$$

$$5x - 3y = 24$$

$x = 3$

$y = -3$

20.If $|z_1| = |z_2| = 1$, prove that $\left|\frac{1}{z_1} + \frac{1}{z_2}\right| = |z_1 + z_2|$

Ans. If $|z_1| = |z_2| = 1$ (Given)

$\Rightarrow |z_1|^2 = |z_2|^2 = 1$

$\Rightarrow z_1 \ \overline{z_1} = 1$

$\overline{z_1} = \frac{1}{z_1}$ (1)

$z_2 \ \overline{z_2} = 1$

$\overline{z_2} = \frac{1}{z_2}$ (2)

$\left[\ \because z \ \overline{z} = |z|^2 \right.$

$\left|\frac{1}{z_1} + \frac{1}{z_2}\right| = |\overline{z_1} + \overline{z_2}|$

$= |\overline{z_1 + z_2}|$

$= |z_1 + z_2|$

$\left[\because |\overline{z}| = |z| \right.$ proved.

CBSE Class 12 Mathematics

Important Questions

Chapter 5

Complex Numbers and Quadratic Equations

6 Marks Questions

1.If z = x + i y and w = $\frac{1-i^2}{z-i}$ **Show that |w| = 1** $\Rightarrow$ z is purely real.

Ans. w = $\frac{1-iz}{z-i}$

$$=\frac{1-i(x+iy)}{x+iy-i}$$

$$=\frac{1-ix-i^2y}{x+i(y-1)}$$

$$=\frac{(1+y)-ix}{x+i(y-1)}$$

$$\therefore \quad |w|=1$$

$$\Rightarrow\left|\frac{(1+y)-ix}{x+i(y-1)}\right|=1$$

$$\frac{|(1+y)-ix|}{|x+i(y-1)|}=1$$

$$\frac{\sqrt{(1+y)^2+(-x)^2}}{\sqrt{x^2+(y-1)^2}}=1$$

$$1+y^2+2y+x^2=x^2+y^2+1-2y$$

$$4y=0$$

$y = 0$

$\therefore\ z = x + \ i$

is purely real

2.Convert into polar form $\dfrac{-16}{1+i\sqrt{3}}$

Ans. $\dfrac{-16}{1+i\sqrt{3}} = \dfrac{-16}{1+i\sqrt{3}} \times \dfrac{1-i\sqrt{3}}{1-i\sqrt{3}}$

$$= \frac{-16\left(1-i\sqrt{3}\right)}{(1)^2 - \left(i\sqrt{3}\right)^2}$$

$$= \frac{-16\left(1-i\sqrt{3}\right)}{1+3}$$

$$= -4\left(1-i\sqrt{3}\right)$$

$$z = -4 + i4\sqrt{3}$$

$$r = \ |z| = \sqrt{(-4)^2 + \left(4\sqrt{3}\right)^2}$$

$$= \sqrt{16+48}$$

$$= \sqrt{64}$$

$$= 8$$

Let α be the acute $\angle S$

$$\tan\ \alpha = \left|\frac{\cancel{4}\sqrt{3}}{-\cancel{4}}\right|$$

$$\tan \alpha = \tan\frac{\pi}{3}$$

$\alpha = \pi/3$

Since Re (z) < o, and Im (z) > o

$\theta = \pi - \alpha$

$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$z = 8\left(Cos\ \frac{2\pi}{3} + i\ \text{Sin}\ \frac{2\pi}{3} \right)$

3.Find two numbers such that their sum is 6 and the product is 14.

Ans. Let x and y be the no.

x + y = 6

xy = 14

$x^2 - 6x + 14 = 0$

$D = -20$

$x = \frac{-(-6) \pm \sqrt{-20}}{2 \times 1}$

$= \frac{6 \pm 2\sqrt{5}\ \text{i}}{2}$

$= 3 \pm \sqrt{5}\ \text{i}$

$\text{x} = 3 + \sqrt{5}\ \text{i}$

$y = 6 - \left(3 + \sqrt{5}\ \text{i}\right)$

$= 3 - \sqrt{5}\ \text{i}$

when $x = 3-\sqrt{5}\,i$

$y = 6-\left(3-\sqrt{5}\,i\right)$

$= 3+\sqrt{5}\,i$

4.Convert into polar form $z = \dfrac{i-1}{\cos\dfrac{\pi}{3}+i\ \text{Sin}\dfrac{\pi}{3}}$

Ans. $z = \dfrac{i-1}{\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i}$

$= \dfrac{2(i-1)}{1+\sqrt{3}i}\times\dfrac{1-\sqrt{3}i}{1-\sqrt{3}i}$

$z = \dfrac{\sqrt{3}-1}{2}+\dfrac{\sqrt{3}+1}{2}i$

$r = |z| = \left(\dfrac{\sqrt{3}-1}{2}\right)^2+\left(\dfrac{\sqrt{3}+1}{2}\right)^2$

$r = 2$

Let α be the acule $\angle$s

$\tan\ \alpha = \left|\dfrac{\dfrac{\sqrt{3}+1}{2}}{\dfrac{\sqrt{3}-1}{2}}\right|$

$= \left|\dfrac{\sqrt{3}\left(1+\dfrac{1}{\sqrt{3}}\right)}{\sqrt{3}\left(1-\dfrac{1}{\sqrt{3}}\right)}\right|$

$$= \left| \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{6}}{1 - \tan\frac{\pi}{4}\tan\frac{\pi}{6}} \right|$$

$$\tan\alpha = \left| \tan\left(\frac{\pi}{4} + \frac{\pi}{6}\right) \right|$$

$$\alpha = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$$

$$z = 2\left(Cos\frac{5\pi}{12} + i\ \text{Sin}\ \frac{5\pi}{12} \right)$$

5.If α and β are different complex number with |β| = 1 Then find $\left| \frac{\beta - \alpha}{1 - \overline{\alpha}\beta} \right|$

Ans. $\left| \frac{\beta - \alpha}{1 - \overline{\alpha}\beta} \right|^2 = \left(\frac{\beta - \alpha}{1 - \overline{\alpha}\beta} \right)\left(\overline{\frac{\beta - \alpha}{1 - \overline{\alpha}\beta}} \right) \quad \left[\because |z|^2 = z\overline{z} \right.$

$$= \left(\frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right) \left(\frac{\bar{\beta} - \bar{\alpha}}{1 - \alpha\bar{\beta}} \right)$$

$$= \left(\frac{\beta\bar{\beta} - \beta\bar{\alpha} - \alpha\bar{\beta} + \alpha\bar{\alpha}}{1 - \alpha\bar{\beta} - \bar{\alpha}\beta + \alpha\bar{\alpha}\beta\bar{\beta}} \right)$$

$$= \left(\frac{|\beta|^2 - \beta\bar{\alpha} - \alpha\bar{\beta} + |\alpha|^2}{1 - \alpha\bar{\beta} - \bar{\alpha}\beta + |\alpha|^2 |\beta|^2} \right)$$

$$= \left(\frac{1 - \beta\bar{\alpha} - \alpha\bar{\beta} + |\alpha|^2}{1 - \alpha\bar{\beta} - \bar{\alpha}\beta + |\alpha|^2} \right) \qquad [\because |\beta| = 1$$

$$= 1$$

$$\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| = \sqrt{1}$$

$$\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| = 1$$