Chapter 13

LIMITS AND DERIVATIVES

13.1 Overview

13.1.1 *Limits of a function*

Let f be a function defined in a domain which we take to be an interval, say, I. We shall study the concept of limit of f at a point 'a' in I.

We say $\lim\limits_{x \to a^-} f(x)$ is the expected value of f at $x = a$ given the values of f near to the left of a. This value is called the *left hand limit* of f at a.

We say $\lim\limits_{x \to a^+} f(x)$ is the expected value of f at $x = a$ given the values of f near to the right of a. This value is called the *right hand limit* of f at a.

If the right and left hand limits coincide, we call the common value as the limit of f at $x = a$ and denote it by $\lim\limits_{x \to a} f(x)$.

Some properties of limits

Let f and g be two functions such that both $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist. Then

(i) $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

(ii) $\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

(iii) For every real number α

$$\lim_{x \to a} (\alpha f)(x) = \alpha \lim_{x \to a} f(x)$$

(iv) $\lim\limits_{x \to a} [f(x)\, g(x)] = [\lim\limits_{x \to a} f(x) \lim\limits_{x \to a} g(x)]$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}, \text{ provided } g(x) \neq 0$$

Limits of polynomials and rational functions

If f is a polynomial function, then $\lim_{x \to a} f(x)$ exists and is given by

$$\lim_{x \to a} f(x) = f(a)$$

An Important limit

An important limit which is very useful and used in the sequel is given below:

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

Remark The above expression remains valid for any rational number provided 'a' is positive.

Limits of trigonometric functions

To evaluate the limits of trigonometric functions, we shall make use of the following limits which are given below:

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (ii) $\lim_{x \to 0} \cos x = 1$ (iii) $\lim_{x \to 0} \sin x = 0$

13.1.2 *Derivatives* Suppose f is a real valued function, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{... (1)}$$

is called the **derivative** of f at x, provided the limit on the R.H.S. of (1) exists.

Algebra of derivative of functions Since the very definition of derivatives involve limits in a rather direct fashion, we expect the rules of derivatives to follow closely that of limits as given below:

Let f and g be two functions such that their derivatives are defined in a common domain. Then :

(i) Derivative of the sum of two function is the sum of the derivatives of the functions.

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

(ii) Derivative of the difference of two functions is the difference of the derivatives of the functions.

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

(iii) Derivative of the product of two functions is given by the following *product rule*.

$$\frac{d}{dx}[f(x)\cdot g(x)] = \left(\frac{d}{dx}f(x)\right)\cdot g(x) + f(x)\cdot\left(\frac{d}{dx}g(x)\right)$$

This is referred to as Leibnitz Rule for the product of two functions.

(iv) Derivative of quotient of two functions is given by the following *quotient rule* (wherever the denominator is non-zero).

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left(\frac{d}{dx}f(x)\right)\cdot g(x) - f(x)\cdot\left(\frac{d}{dx}g(x)\right)}{(g(x))^2}$$

13.2 Solved Examples

Short Answer Type

Example 1 Evaluate $\lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x}\right]$

Solution We have

$$\lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x}\right] = \lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{x(x-1)-2(2x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{x^2-5x+6}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{(x-2)(x-3)}{x(x-1)(x-2)}\right] \quad [x-2 \neq 0]$$

$$= \lim_{x\to 2}\left[\frac{x-3}{x(x-1)}\right] = \frac{-1}{2}$$

Example 2 Evaluate $\lim_{x \to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$

Solution Put $y = 2 + x$ so that when $x \to 0$, $y \to 2$. Then

$$\lim_{x \to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{y \to 2} \frac{y^{\frac{1}{2}} - 2^{\frac{1}{2}}}{y-2}$$

$$= \frac{1}{2}(2)^{\frac{1}{2}-1} = \frac{1}{2} \cdot 2^{-\frac{1}{2}} = \frac{1}{2\sqrt{2}}$$

Example 3 Find the positive integer n so that $\lim_{x \to 3} \frac{x^n - 3^n}{x-3} = 108$.

Solution We have

$$\lim_{x \to 3} \frac{x^n - 3^n}{x-3} = n(3)^{n-1}$$

Therefore, $n(3)^{n-1} = 108 = 4\,(27) = 4(3)^{4-1}$

Comparing, we get $n = 4$

Example 4 Evaluate $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

Solution Put $y = \frac{\pi}{2} - x$. Then $y \to 0$ as $x \to \frac{\pi}{2}$. Therefore

$$\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) = \lim_{y \to 0} [\sec(\frac{\pi}{2} - y) - \tan(\frac{\pi}{2} - y)]$$

$$= \lim_{y \to 0} (\operatorname{cosec} y - \cot y)$$

$$= \lim_{y \to 0} \left(\frac{1}{\sin y} - \frac{\cos y}{\sin y} \right)$$

$$= \lim_{y \to 0} \left(\frac{1 - \cos y}{\sin y} \right)$$

$$= \lim_{y \to 0} \frac{2\sin^2 \frac{y}{2}}{2\sin\frac{y}{2}\cos\frac{y}{2}} \qquad \left(\begin{array}{l} \text{since}, \sin^2 \frac{y}{2} = \frac{1-\cos y}{2} \\ \sin y = 2\sin\frac{y}{2}\cos\frac{y}{2} \end{array} \right)$$

$$= \lim_{\frac{y}{2} \to 0} \tan \frac{y}{2} = 0$$

Example 5 Evaluate $\lim_{x \to 0} \frac{\sin(2+x) - \sin(2-x)}{x}$

Solution (i) We have

$$\lim_{x \to 0} \frac{\sin(2+x) - \sin(2-x)}{x} = \lim_{x \to 0} \frac{2\cos\frac{(2+x+2-x)}{2}\sin\frac{(2+x-2+x)}{2}}{x}$$

$$= \lim_{x \to 0} \frac{2\cos 2 \sin x}{x}$$

$$= 2\cos 2 \lim_{x \to 0} \frac{\sin x}{x} = 2\cos 2 \quad \left(\text{as} \lim_{x \to 0} \frac{\sin x}{x} = 1 \right)$$

Example 6 Find the derivative of $f(x) = ax + b$, where a and b are non-zero constants, by first principle.

Solution By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a(x+h) + b - (ax+b)}{h} = \lim_{h \to 0} \frac{bh}{h} = b$$

Example 7 Find the derivative of $f(x) = ax^2 + bx + c$, where a, b and c are none-zero constant, by first principle.

Solution By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a(x+h)^2 + b(x+h) + c - ax^2 - bx - c}{h}$$

$$= \lim_{h \to 0} \frac{bh + ah^2 + 2axh}{h} = \lim_{h \to 0} ah + 2ax + b = b + 2ax$$

Example 8 Find the derivative of $f(x) = x^3$, by first principle.

Solution By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + h^3 + 3xh(x+h) - x^3}{h}$$

$$= \lim_{h \to 0} (h^2 + 3x(x+h)) = 3x^2$$

Example 9 Find the derivative of $f(x) = \frac{1}{x}$ by first principle.

Solution By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right)$$

$$= \lim_{h \to 0} \frac{-h}{h(x+h)x} = \frac{-1}{x^2}.$$

Example 10 Find the derivative of $f(x) = \sin x$, by first principle.

Solution By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}}{2 \cdot \frac{h}{2}}$$

$$= \lim_{h \to 0} \cos\frac{(2x+h)}{2} \cdot \lim_{h \to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= \cos x.1 = \cos x$$

Example 11 Find the derivative of $f(x) = x^n$, where n is positive integer, by first principle.

Solution By definition,

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{(x+h)^n - x^n}{h}$$

Using Binomial theorem, we have $(x + h)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} h + ... + {}^nC_n\, h^n$

Thus, $$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{h\,(nx^{n-1} + ... + h^{n-1}]}{h} = nx^{n-1}.$$

Example 12 Find the derivative of $2x^4 + x$.

Solution Let $y = 2x^4 + x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2x^4) + \frac{d}{dx}(x)$$

$$= 2 \times 4x^{4-1} + 1x^0$$

$$= 8x^3 + 1$$

Therefore, $$\frac{d}{dx}(2x^4 + x) = 8x^3 + 1.$$

Example 13 Find the derivative of $x^2 \cos x$.

Solution Let $y = x^2 \cos x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \cos x)$$

$$= x^2 \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^2)$$

$$= x^2 (-\sin x) + \cos x\,(2x)$$

$$= 2x \cos x - x^2 \sin x$$

Long Answer Type

Example 14 Evaluate $\lim\limits_{x \to \frac{\pi}{6}} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1}$

Solution Note that

$$2\sin^2 x + \sin x - 1 = (2\sin x - 1)(\sin x + 1)$$

$$2\sin^2 x - 3\sin x + 1 = (2\sin x - 1)(\sin x - 1)$$

Therefore, $$\lim_{x \to \frac{\pi}{6}} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1} = \lim_{x \to \frac{\pi}{6}} \frac{(2\sin x - 1)(\sin x + 1)}{(2\sin x - 1)(\sin x - 1)}$$

$$= \lim_{x \to \frac{\pi}{6}} \frac{\sin x + 1}{\sin x - 1} \qquad (\text{as } 2\sin x - 1 \neq 0)$$

$$= \frac{1 + \sin\frac{\pi}{6}}{\sin\frac{\pi}{6} - 1} = -3$$

Example 15 Evaluate $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{\sin^3 x}$

Solution We have

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}=\lim_{x\to 0}\frac{\sin x\left(\dfrac{1}{\cos x}-1\right)}{\sin^3 x}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{\cos x\sin^2 x}$$

$$=\lim_{x\to 0}\frac{2\sin^2\dfrac{x}{2}}{\cos x\left(4\sin^2\dfrac{x}{2}\cdot\cos^2\dfrac{x}{2}\right)}=\frac{1}{2}.$$

Example 16 Evaluate $\lim\limits_{x\to a}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

Solution We have $\lim\limits_{x\to a}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

$$=\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\times\frac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}}$$

$$=\lim_{x\to a}\frac{a+2x-3x}{\left(\sqrt{3a+x}-2\sqrt{x}\right)\left(\sqrt{a+2x}+\sqrt{3x}\right)}$$

$$=\lim_{x\to a}\frac{(a-x)\left(\sqrt{3a+x}+2\sqrt{x}\right)}{\left(\sqrt{a+2x}+\sqrt{3x}\right)\left(\sqrt{3a+x}-2\sqrt{x}\right)\left(\sqrt{3a+x}+2\sqrt{x}\right)}$$

$$=\lim_{x\to a}\frac{(a-x)\left[\sqrt{3a+x}+2\sqrt{x}\right]}{\left(\sqrt{a+2x}+\sqrt{3x}\right)(3a+x-4x)}$$

$$= \frac{4\sqrt{a}}{3 \times 2\sqrt{3a}} = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}.$$

Example 17 Evaluate $\lim\limits_{x \to 0} \dfrac{\cos ax - \cos bx}{\cos cx - 1}$

Solution We have $\lim\limits_{x \to 0} \dfrac{2\sin\left(\dfrac{(a+b)}{2}x\right) \sin \dfrac{(a-b)x}{2}}{2\sin^2 \dfrac{cx}{2}}$

$$= \lim_{x \to 0} \frac{2\sin \dfrac{(a+b)x}{2} \cdot \sin \dfrac{(a-b)x}{2}}{x^2} \cdot \frac{x^2}{\sin^2 \dfrac{cx}{2}}$$

$$= \lim_{x \to 0} \frac{\sin \dfrac{(a+b)x}{2}}{\dfrac{(a+b)x}{2} \cdot \left(\dfrac{2}{a+b}\right)} \cdot \frac{\sin \dfrac{(a-b)x}{2}}{\dfrac{(a-b)x}{2} \cdot \dfrac{2}{a-b}} \cdot \frac{\left(\dfrac{cx}{2}\right)^2 \times \dfrac{4}{c^2}}{\sin^2 \dfrac{cx}{2}}$$

$$= \left(\frac{a+b}{2} \times \frac{a-b}{2} \times \frac{4}{c^2}\right) = \frac{a^2 - b^2}{c^2}$$

Example 18 Evaluate $\lim\limits_{h \to 0} \dfrac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$

Solution We have $\lim\limits_{h \to 0} \dfrac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$

$$= \lim_{h \to 0} \frac{(a^2 + h^2 + 2ah)[\sin a \cos h + \cos a \sin h] - a^2 \sin a}{h}$$

$$= \lim_{h \to 0} \left[\frac{a^2 \sin a (\cos h - 1)}{h} + \frac{a^2 \cos a \sin h}{h} + (h + 2a)(\sin a \cos h + \cos a \sin h)\right]$$

$$= \lim_{h \to 0} \left[\frac{a^2 \sin a \left(-2\sin^2 \frac{h}{2}\right)}{\frac{h^2}{2}} \cdot \frac{h}{2} \right] + \lim_{h \to 0} \frac{a^2 \cos a \sin h}{h} + \lim_{h \to 0} (h + 2a) \sin (a + h)$$

$$= a^2 \sin a \times 0 + a^2 \cos a \ (1) + 2a \sin a$$

$$= a^2 \cos a + 2a \sin a.$$

Example 19 Find the derivative of $f(x) = \tan (ax + b)$, by first principle.

Solution We have $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\tan \big(a(x+h) + b\big) - \tan (ax + b)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\sin (ax + ah + b)}{\cos (ax + ah + b)} - \frac{\sin (ax + b)}{\cos (ax + b)}}{h}$$

$$= \lim_{h \to 0} \frac{\sin (ax + ah + b) \cos (ax + b) - \sin (ax + b) \cos (ax + ah + b)}{h \cos (ax + b) \cos (ax + ah + b)}$$

$$= \lim_{h \to 0} \frac{a \sin (ah)}{a \cdot h \cos (ax + b) \cos (ax + ah + b)}$$

$$= \lim_{h \to 0} \frac{a}{\cos (ax + b) \cos (ax + ah + b)} \lim_{ah \to 0} \frac{\sin ah}{ah} \quad [\text{as } h \to 0 \ \ ah \to 0]$$

$$= \frac{a}{\cos^2 (ax + b)} = a \sec^2 (ax + b).$$

Example 20 Find the derivative of $f(x) = \sqrt{\sin x}$, by first principle.

Solution By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{\sin(x+h)} - \sqrt{\sin x}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{\sin(x+h)} - \sqrt{\sin x}\right)\left(\sqrt{\sin(x+h)} + \sqrt{\sin x}\right)}{h\left(\sqrt{\sin(x+h)} + \sqrt{\sin x}\right)}$$

$$= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h\left(\sqrt{\sin(x+h)} + \sqrt{\sin x}\right)}$$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}}{2\cdot\frac{h}{2}\left(\sqrt{\sin(x+h)} + \sqrt{\sin x}\right)}$$

$$= \frac{\cos x}{2\sqrt{\sin x}} = \frac{1}{2}\cot x\sqrt{\sin x}$$

Example 21 Find the derivative of $\frac{\cos x}{1+\sin x}$.

Solution Let $y = \frac{\cos x}{1+\sin x}$

Differentiating both sides with respects to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\cos x}{1+\sin x}\right)$$

$$= \frac{(1+\sin x)\frac{d}{dx}(\cos x) - \cos x\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-(1+\sin x)}{(1+\sin x)^2} = \frac{-1}{1+\sin x}$$

Objective Type Questions

Choose the correct answer out of the four options given against each Example 22 to 28 (M.C.Q.).

Example22 $\lim\limits_{x\to 0} \frac{\sin x}{x(1+\cos x)}$ is equal to

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) –1

Solution (B) is the correct answer, we have

$$\lim_{x\to 0} \frac{\sin x}{x(1+\cos x)} = \lim_{x\to 0} \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{x\left(2\cos^2\frac{x}{2}\right)}$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{\tan\frac{x}{2}}{\frac{x}{2}} = \frac{1}{2}$$

Example23 $\lim\limits_{x\to \frac{\pi}{2}} \frac{1-\sin x}{\cos x}$ is equal to

(A) 0 (B) –1 (C) 1 (D) does not exit

Solution (A) is the correct answer, since

$$\lim_{x\to \frac{\pi}{2}} \frac{1-\sin x}{\cos x} = \lim_{y\to 0}\left[\frac{1-\sin\left(\frac{\pi}{2}-y\right)}{\cos\left(\frac{\pi}{2}-y\right)}\right] \left(\text{taking } \frac{\pi}{2}-x=y\right)$$

$$= \lim_{y\to 0}\frac{1-\cos y}{\sin y} = \lim_{y\to 0}\frac{2\sin^2\frac{y}{2}}{2\sin\frac{y}{2}\cos\frac{y}{2}}$$

$$= \lim_{y\to 0}\tan\frac{y}{2} = 0$$

Example 24 $\lim_{x\to 0}\frac{|x|}{x}$ is equal to

(A) 1 (B) –1 (C) 0 (D) does not exists

Solution (D) is the correct answer, since

$$\text{R.H.S} = \lim_{x\to 0^+}\frac{|x|}{x} = \frac{x}{x} = 1$$

and
$$\text{L.H.S} = \lim_{x\to 0^-}\frac{|x|}{x} = \frac{-x}{x} = -1$$

Example 25 $\lim_{x\to 1}[x-1]$, where [.] is greatest integer function, is equal to

(A) 1 (B) 2 (C) 0 (D) does not exists

Solution (D) is the correct answer, since

$$\text{R.H.S} = \lim_{x\to 1^+}[x-1] = 0$$

and
$$\text{L.H.S} = \lim_{x\to 1^-}[x-1] = -1$$

Example 26 $\lim_{x\to 0} x\sin\frac{1}{x}$ is equals to

(A) 0 (B) 1 (C) $\frac{1}{2}$ (D) does not exist

Solution (A) is the correct answer, since

$\lim_{x\to 0} x = 0$ and $-1 \le \sin\frac{1}{x} \le 1$, by Sandwitch Theorem, we have

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0$$

Example 27 $\lim_{n\to\infty}\frac{1+2+3+...+n}{n^2}$, $n \in \mathbf{N}$, is equal to

(A) 0 (B) 1 (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

Solution (C) is the correct answer. As $\lim_{x\to\infty}\frac{1+2+3+...+n}{n^2}$

$$= \lim_{n\to\infty}\frac{n(n+1)}{2n^2} = \lim_{x\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2}$$

Example 28 If $f(x) = x \sin x$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

(A) 0 (B) 1 (C) –1 (D) $\frac{1}{2}$

Solution (B) is the correct answer. As $f'(x) = x\cos x + \sin x$

So, $$f'\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 1$$

13.3 EXERCISE

Short Answer Type

Evaluate :

1. $\lim_{x\to 3}\frac{x^2-9}{x-3}$
2. $\lim_{x\to\frac{1}{2}}\frac{4x^2-1}{2x-1}$
3. $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$
4. $\lim_{x\to 0}\frac{(x+2)^{\frac{1}{3}}-2^{\frac{1}{3}}}{x}$
5. $\lim_{x\to 1}\frac{(1+x)^6-1}{(1+x)^2-1}$
6. $\lim_{x\to a}\frac{(2+x)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{x-a}$

7. $\lim_{x\to 1} \frac{x^4 - \sqrt{x}}{\sqrt{x} - 1}$

8. $\lim_{x\to 2} \frac{x^2 - 4}{\sqrt{3x-2} - \sqrt{x+2}}$

9. $\lim_{x\to \sqrt{2}} \frac{x^4 - 4}{x^2 + 3\sqrt{2}x - 8}$

10. $\lim_{x\to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$

11. $\lim_{x\to 0} \frac{\sqrt{1+x^3} - \sqrt{1-x^3}}{x^2}$

12. $\lim_{x\to -3} \frac{x^3 + 27}{x^5 + 243}$

13. $\lim_{x\to \frac{1}{2}} \left(\frac{8x-3}{2x-1} - \frac{4x^2+1}{4x^2-1} \right)$

14. Find 'n', if $\lim_{x\to 2} \frac{x^n - 2^n}{x-2} = 80$, $n \in \mathbf{N}$

15. $\lim_{x\to a} \frac{\sin 3x}{\sin 7x}$

16. $\lim_{x\to 0} \frac{\sin^2 2x}{\sin^2 4x}$

17. $\lim_{x\to 0} \frac{1 - \cos 2x}{x^2}$

18. $\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x^3}$

19. $\lim_{x\to 0} \frac{1 - \cos mx}{1 - \cos nx}$

20. $\lim_{x\to \frac{\pi}{3}} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2}\left(\frac{\pi}{3} - x\right)}$

21. $\lim_{x\to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$

22. $\lim_{x\to \frac{\pi}{6}} \frac{\sqrt{3}\sin x - \cos x}{x - \frac{\pi}{6}}$

23. $\lim_{x\to 0} \frac{\sin 2x + 3x}{2x + \tan 3x}$

24. $\lim_{x\to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$

25. $\lim_{x\to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$

26. $\lim_{x\to 0} \frac{\sqrt{2} - \sqrt{1+\cos x}}{\sin^2 x}$

27. $\lim_{x\to 0} \frac{\sin x - 2\sin 3x + \sin 5x}{x}$

28. If $\lim_{x\to 1} \frac{x^4 - 1}{x - 1} = \lim_{x\to k} \frac{x^3 - k^3}{x^2 - k^2}$, then find the value of k.

Differentiate each of the functions w. r. to x in Exercises 29 to 42.

29. $\frac{x^4 + x^3 + x^2 + 1}{x}$

30. $\left(x + \frac{1}{x}\right)^3$

31. $(3x + 5)(1 + \tan x)$

32. $(\sec x - 1)(\sec x + 1)$ **33.** $\dfrac{3x+4}{5x^2-7x+9}$ **34.** $\dfrac{x^5 - \cos x}{\sin x}$

35. $\dfrac{x^2 \cos\frac{\pi}{4}}{\sin x}$ **36.** $(ax^2 + \cot x)(p + q\cos x)$

37. $\dfrac{a + b\sin x}{c + d\cos x}$ **38.** $(\sin x + \cos x)^2$ **39.** $(2x - 7)^2 (3x + 5)^3$

40. $x^2 \sin x + \cos 2x$ **41.** $\sin^3 x \cos^3 x$ **42.** $\dfrac{1}{ax^2 + bx + c}$

Long Answer Type

Differentiate each of the functions with respect to 'x' in Exercises 43 to 46 using first principle.

43. $\cos(x^2 + 1)$ **44.** $\dfrac{ax+b}{cx+d}$ **45.** $x^{\frac{2}{3}}$

46. $x \cos x$

Evaluate each of the following limits in Exercises 47 to 53.

47. $\lim\limits_{y \to 0} \dfrac{(x+y)\sec(x+y) - x\sec x}{y}$

48. $\lim\limits_{x \to 0} \dfrac{(\sin(\alpha+\beta)x + \sin(\alpha-\beta)x + \sin 2\alpha\, x)}{\cos 2\beta x - \cos 2\alpha x} \cdot x$

49. $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$ **50.** $\lim\limits_{x \to \pi} \dfrac{1 - \sin\frac{x}{2}}{\cos\frac{x}{2}\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)}$

51. Show that $\lim\limits_{x \to 4} \dfrac{|x-4|}{x-4}$ does not exists

52. Let $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x} & \text{when } x \neq \dfrac{\pi}{2} \\ 3 & x = \dfrac{\pi}{2} \end{cases}$ and if $\lim\limits_{x \to \frac{\pi}{2}} f(x) = f\left(\dfrac{\pi}{2}\right)$,

find the value of k.

53. Let $f(x) = \begin{cases} x+2 & x \leq -1 \\ cx^2 & x > -1 \end{cases}$, find '$c$' if $\lim\limits_{x \to -1} f(x)$ exists.

Objective Type Questions

Choose the correct answer out of 4 options given against each Exercise 54 to 76 (M.C.Q).

54. $\lim\limits_{x \to \pi} \dfrac{\sin x}{x - \pi}$ is

(A) 1 (B) 2 (C) –1 (D) –2

55. $\lim\limits_{x \to 0} \dfrac{x^2 \cos x}{1 - \cos x}$ is

(A) 2 (B) $\dfrac{3}{2}$ (C) $\dfrac{-3}{2}$ (D) 1

56. $\lim\limits_{x \to 0} \dfrac{(1+x)^n - 1}{x}$ is

(A) n (B) 1 (C) $-n$ (D) 0

57. $\lim\limits_{x \to 1} \dfrac{x^m - 1}{x^n - 1}$ is

(A) 1 (B) $\dfrac{m}{n}$ (C) $-\dfrac{m}{n}$ (D) $\dfrac{m^2}{n^2}$

58. $\lim\limits_{x \to 0} \dfrac{1 - \cos 4\theta}{1 - \cos 6\theta}$ is

(A) $\frac{4}{9}$ (B) $\frac{1}{2}$ (C) $\frac{-1}{2}$ (D) –1

59. $\lim\limits_{x\to 0} \frac{\operatorname{cosec} x - \cot x}{x}$ is

(A) $\frac{-1}{2}$ (B) 1 (C) $\frac{1}{2}$ (D) 1

60. $\lim\limits_{x\to 0} \frac{\sin x}{\sqrt{x+1} - \sqrt{1-x}}$ is

(A) 2 (B) 0 (C) 1 (D) –1

61. $\lim\limits_{x\to \frac{\pi}{4}} \frac{\sec^2 x - 2}{\tan x - 1}$ is

(A) 3 (B) 1 (C) 0 (D) $\sqrt{2}$

62. $\lim\limits_{x\to 1} \frac{\left(\sqrt{x} - 1\right)\left(2x - 3\right)}{2x^2 + x - 3}$ is

(A) $\frac{1}{10}$ (B) $\frac{-1}{10}$ (C) 1 (D) None of these

63. If $f(x) = \begin{cases} \frac{\sin[x]}{[x]}, & [x] \neq 0 \\ 0 & ,[x] = 0 \end{cases}$, where [.] denotes the greatest integer function ,

then $\lim\limits_{x\to 0} f(x)$ is equal to

(A) 1 (B) 0 (C) –1 (D) None of these

64. $\lim\limits_{x\to 0} \frac{|\sin x|}{x}$ is

(A) 1 (B) –1 (C) does not exist (D) None of these

65. Let $f(x) = \begin{cases} x^2 - 1, & 0 < x < 2 \\ 2x + 3, & 2 \le x < 3 \end{cases}$, the quadratic equation whose roots are $\lim\limits_{x\to 2^-} f(x)$ and

$\lim\limits_{x\to 2^+} f(x)$ is

(A) $x^2 - 6x + 9 = 0$ (B) $x^2 - 7x + 8 = 0$

(C) $x^2 - 14x + 49 = 0$ (D) $x^2 - 10x + 21 = 0$

66. $\lim_{x \to 0} \frac{\tan 2x - x}{3x - \sin x}$ is

(A) 2 (B) $\frac{1}{2}$ (C) $\frac{-1}{2}$ (D) $\frac{1}{4}$

67. Let $f(x) = x - [x]$; $\in$ **R**, then $f'\left(\frac{1}{2}\right)$ is

(A) $\frac{3}{2}$ (B) 1 (C) 0 (D) –1

68. If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, then $\frac{dy}{dx}$ at $x = 1$ is

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) 0

69. If $f(x) = \frac{x-4}{2\sqrt{x}}$, then $f'(1)$ is

(A) $\frac{5}{4}$ (B) $\frac{4}{5}$ (C) 1 (D) 0

70. If $y = \frac{1 + \frac{1}{x^2}}{1 - \frac{1}{x^2}}$, then $\frac{dy}{dx}$ is

(A) $\frac{-4x}{(x^2-1)^2}$ (B) $\frac{-4x}{x^2-1}$ (C) $\frac{1-x^2}{4x}$ (D) $\frac{4x}{x^2-1}$

71. If $y = \frac{\sin x + \cos x}{\sin x - \cos x}$, then $\frac{dy}{dx}$ at $x = 0$ is

(A) –2 (B) 0 (C) $\frac{1}{2}$ (D) does not exist

72. If $y = \dfrac{\sin(x+9)}{\cos x}$ then $\dfrac{dy}{dx}$ at $x = 0$ is

(A) cos 9 (B) sin 9 (C) 0 (D) 1

73. If $f(x) = 1 + x + \dfrac{x^2}{2} + \ldots + \dfrac{x^{100}}{100}$, then $f'(1)$ is equal to

(A) $\dfrac{1}{100}$ (B) 100 (C) does not exist (D) 0

74. If $f(x) = \dfrac{x^n - a^n}{x - a}$ for some constant 'a', then $f'(a)$ is

(A) 1 (B) 0 (C) does not exist (D) $\dfrac{1}{2}$

75. If $f(x) = x^{100} + x^{99} + \ldots + x + 1$, then $f'(1)$ is equal to

(A) 5050 (B) 5049 (C) 5051 (D) 50051

76. If $f(x) = 1 - x + x^2 - x^3 \ldots - x^{99} + x^{100}$, then $f'(1)$ is euqal to

(A) 150 (B) –50 (C) –150 (D) 50

Fill in the blanks in Exercises 77 to 80.

77. If $f(x) = \dfrac{\tan x}{x - \pi}$, then $\lim\limits_{x \to \pi} f(x)$ = ______________

78. $\lim\limits_{x \to 0}\left(\sin mx \cot \dfrac{x}{\sqrt{3}}\right) = 2$, then m = ______________

79. if $y = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$, then $\dfrac{dy}{dx}$ = ______________

80. $\lim\limits_{x \to 3^+} \dfrac{x}{[x]}$ = ______________