DCAM classes
Dynamic Classes for Academic Mastery

Chapter 12. Limits and Derivatives

Question-1

Find the indicated limit: $\lim_{x \to 1} \frac{x^2 + 2x + 5}{x^2 + 1}$

Solution:

$\lim_{x \to 1} \frac{x^2 + 2x + 5}{x^2 + 1} = \frac{1 + 2 + 5}{1 + 1}$ by direct substitution

$= \frac{8}{2} =$

Question-2

Find the indicated limit: $\lim_{x \to 2} \frac{x - 2}{\sqrt{2 - x}}$

Solution:

$\lim_{x \to 2} \frac{x - 2}{\sqrt{2 - x}} = \lim_{x \to 2} \frac{-(2 - x)}{\sqrt{2 - x}} = \lim_{x \to 2} -(\sqrt{2 - x}) = 0$

Question-3

Find the indicated limit: $\lim_{h \to 0} \frac{(x + h)^2 - x^2}{h}$

Solution:

$\lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{h(2x + h)}{h} = \lim_{h \to 0} (2x + h) = 2x$

Question-4

Find the indicated limit: $\lim_{x \to 1} \frac{x^m - 1}{x - 1}$

Solution:

$$\lim_{x \to 1} \frac{x^m - 1}{x - 1} = \lim_{x \to 1} \frac{x^m - 1^m}{x - 1} = m(1)^{m-1} = m$$

Question-5

Find the indicated limit: $\lim_{x \to 4} \frac{\sqrt{2x+1} - 3}{\sqrt{x-1} - \sqrt{2}}$

Solution:

$$\lim_{x \to 4} \frac{\sqrt{2x+1} - 3}{\sqrt{x-1} - \sqrt{2}} = \lim_{x \to 4} \frac{\sqrt{2x+1} - 3}{\sqrt{x-1} - \sqrt{2}} \times \frac{\sqrt{x-2} + \sqrt{2}}{\sqrt{x-2} + \sqrt{2}} \times \frac{\sqrt{2x+1} + 3}{\sqrt{2x+1} + 3}$$

$$= \lim_{x \to 4} \frac{(2x + 1 - 9)}{(x - 2 - 2)} \frac{\sqrt{x-2} + \sqrt{2}}{\sqrt{2x+1} + 3}$$

$$= \lim_{x \to 4} \frac{2x - 8}{x - 4} \frac{\sqrt{x-2} + \sqrt{2}}{\sqrt{2x+1} + 3}$$

$$= \lim_{x \to 4} 2\left\{\frac{\sqrt{x-2} + \sqrt{2}}{\sqrt{2x+1} + 3}\right\}$$

$$= 2\,\frac{2\sqrt{2}}{6}$$

$$= \frac{2\sqrt{2}}{3}$$

Question-6

Find the indicated limit: $\lim\limits_{x \to 0} \dfrac{\sqrt{x^2 + p^2} - p}{\sqrt{x^2 + q^2} - q}$

Solution:

$$\lim_{x \to 0} \frac{\sqrt{x^2 + p^2} - p}{\sqrt{x^2 + q^2} - q}$$

$$= \lim_{x \to 0} \frac{\sqrt{x^2 + p^2} - p}{\sqrt{x^2 + q^2} - q} \times \frac{\sqrt{x^2 + q^2} + q}{\sqrt{x^2 + q^2} + q} \times \frac{\sqrt{x^2 + p^2} + p}{\sqrt{x^2 + p^2} + p}$$

$$= \lim_{x \to 0} \frac{x^2 + p^2 - p^2}{x^2 + q^2 - q^2} \times \frac{\sqrt{x^2 + p^2} + q}{\sqrt{x^2 + q^2} + p}$$

$$= \lim_{x \to 0} \frac{2q}{2p} = \frac{q}{p}$$

Question-7

Find the indicated limit: $\lim\limits_{x \to a} \dfrac{\sqrt[m]{x} - \sqrt[m]{a}}{x - a}$

Solution:

$$\lim_{x \to a} \frac{\sqrt[m]{x} - \sqrt[m]{a}}{x - a} = \lim_{x \to a} \frac{x^{1/m} - a^{1/m}}{x - a} = \frac{1}{m} a^{1/m - 1}$$ (na^{n-1} formula)

Question-8

Find the indicated limit: $\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$

Solution:

$$\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1} = \lim_{x \to 1} \frac{x^{1/3} - 1}{x^{1/2} - 1}$$

$$= \lim_{x \to 1} \frac{x^{1/3} - 1}{x - 1} \times \frac{x - 1}{x^{1/2} - 1} = \lim_{x \to 1} \frac{x^{1/3} - x^{1/3}}{x - 1} \times \frac{x - 1}{x^{1/2} - 1^{1/2}}$$

$$= \frac{1}{3}(1)^{1/3 - 1} \times \frac{1}{\frac{1}{2}(1)^{1/2 - 1}} = \frac{1/3}{1/2} = \frac{2}{3}$$

Question-9

Find the indicated limit: $\lim_{x \to 0} \frac{\sqrt{1 + x + x^2} - 1}{x}$

Solution:

$$\lim_{x \to 0} \frac{\sqrt{1 + x + x^2} - 1}{x} = \lim_{x \to 0} \frac{\sqrt{1 + x + x^2} - 1}{x} \times \frac{\sqrt{1 + x + x^2} + 1}{\sqrt{1 + x + x^2} + 1} = \lim_{x \to 0} \frac{1 + x + x^2 - 1}{x\left[\sqrt{1 + x + x^2} + 1\right]}$$

$$= \lim_{x \to 0} \frac{x(x + 1)}{x(\sqrt{1 + x + x^2} + 1)}$$

$$= \frac{1}{\sqrt{1} + 1} = \frac{1}{2}$$

Question-10

Find the indicated limit: $\lim_{x \to 0} \frac{\sin^2\left(\frac{x}{3}\right)}{x^2}$

Solution:

$$\lim_{x \to 0} \frac{\sin^2\left(\frac{x}{3}\right)}{x^2} = \lim_{x \to 0} \left(\frac{\sin x/3}{x/3}\right)^2 \frac{(x/3)^2}{x^2} = 1 \times \frac{1}{9} = \frac{1}{9}$$

Question-11

Find the indicated limit: $\lim_{x \to 0} \frac{\sin(a+x)-\sin(a-x)}{x}$

Solution:

$$\lim_{x \to 0} \frac{\sin(a+x)-\sin(a-x)}{x} = \lim_{x \to 0} \frac{2\cos a \sin x}{x} = \lim_{x \to 0} 2\cos a \left(\frac{\sin x}{x}\right) = 2\cos$$

Question-12

Find the indicated limit: $\lim_{x \to 0} \frac{\log(1+\alpha x)}{x}$

Solution:

$$\lim_{x \to 0} \frac{\log(1+\alpha x)}{x} = \lim_{x \to 0} \frac{\alpha x - \frac{(\alpha x)^2}{2} + \frac{(\alpha x)^3}{3}}{x} \ldots\ldots = \lim_{x \to 0} \alpha - \frac{\alpha^2}{2}x + \frac{\alpha^3}{3}x^2 \ldots\ldots$$

Question-13

Find the indicated limit: $\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^{n+5}$

Solution:

$$\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^{n+5} = \lim_{n \to \infty} \left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)^{5} = e.\,(1)^5 = e$$

Question-14

Evaluate the left and right limits of f(x) = $\frac{x^3-27}{x-3}$ at x = 3. Does the limit of f(x) = x → 3 exist? Justify your answer.

Solution:

$\lim_{x \to 3}\left(\frac{x^3-27}{x-3}\right)$

Let x = 3 + h

Then $\lim_{h \to 0}\left(\frac{(3+h)^3-27}{(3+h)-3}\right) = \left(\frac{9h^2+27h+h^3}{h}\right) = \lim_{h \to 0} 27+9h+h^2 = 27$

$\lim_{h \to 3}\frac{x^3-27}{x-3}$

Let x = 3 – h

Then $\lim_{h \to 0}\frac{(3-h)^3-27}{(3-h-3)} = \lim_{h \to 0}\frac{-27h^2+9h^2-h^3}{-h} = 27$

Also, $\lim_{x \to 3}\frac{x^3-27}{x-3} = \lim_{x \to 3}\frac{(x-3)(x^2+3x+9)}{x-3} = 9+9+9 = 27$

Question-15

Find the positive integer n such that $\lim_{x \to 3}\frac{x^n-3^n}{x-3} = 108$.

Solution:

$\lim_{x \to 3}\frac{x^n-3^n}{x-3} = 108$

$n\,3^{n-1} = 108$

Put n = 4, then $4.3^3 = 4 \times 27 = 108$

$\therefore n = 4$

Question-16

Evaluate $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$ [Hint: Take e^x or $e^{\sin x}$ as common factor in numerator]

Solution:

$$\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \lim_{x \to 0} \frac{e^x\left(1 - \frac{e^{\sin x}}{e^x}\right)}{(x - \sin x)}$$

$$= \frac{e^x(1 - e^{\sin x - x})}{(x - \sin x)} = \frac{e^x(1 - e^{-(x - \sin x)})}{(x - \sin x)}$$

$$= e^x \left[\frac{1 - (1 - (x - \sin x) + x - \sin x)^2 + \ldots\ldots}{(x - \sin x)}\right]$$

$$= e^x \left[\frac{(x - \sin x) - (x - \sin x)^2 + \ldots\ldots}{(x - \sin x)}\right]$$

$$= \lim_{x \to 0} e^x [1 - (x - \sin x) \ldots\ldots] = e^0 = 1$$

Question-17

If $f(x) = \frac{ax^2 + b}{x^2 - 1}$, $\lim_{x \to 0} f(x) = 1$ and $\lim_{x \to \infty} f(x) = 1$

Solution:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{ax^2 + b}{x^2 - 1} = b/-1 = -b = 1$$

$\therefore b = -1$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{a + \frac{b}{x^2}}{1 - \frac{1}{x^2}} = a = 1$$

$\therefore a = 1$

$$\therefore f(x) = \frac{x^2 - 1}{x^2 - 1} = 1$$

$f(2) = 1$ and $f(-2) = 1$

Question-18

Evaluate $\lim_{x \to 0} \frac{|x|}{x}$ and $\lim_{x \to 0} \frac{|x|}{x}$. What can you say about $\lim_{x \to 0} \frac{|x|}{x}$?

Solution:

Let $f(x) = \frac{x}{|x|}$ where $|x| = x \quad x \geq 0$

$= -x \quad x < 0$

Then, $\underset{x \to 0}{\text{Lt}} f(x) = \underset{x \to 0}{\text{Lt}} \frac{x}{|x|} = \underset{x \to 0}{\text{Lt}} \frac{x}{x} = 1$

$\underset{x \to 0}{\text{Lt}} f(x) = \underset{x \to 0}{\text{Lt}} \frac{x}{|x|} = \underset{x \to 0}{\text{Lt}} \frac{x}{-x} = -1$

$\therefore \underset{x \to 0}{\text{Lt}} f(x) \neq \underset{x \to 0}{\text{Lt}} f(x)$

$\therefore \underset{x \to 0}{\text{Lt}} \frac{x}{|x|}$ does not exist.

Question-19

Compute $\lim_{x \to 0} \frac{a^x - b^x}{x}, a, b > 0$. Hence evaluate $\lim_{x \to 0} \frac{5^x - 6^x}{x}$.

Solution:

$\lim_{x \to 0} \frac{a^x - b^x}{x} = \lim_{x \to 0} \frac{(a^x - 1) - (b^x - 1)}{x} = \lim_{x \to 0} \left(\frac{a^x - 1}{x}\right) - \lim_{x \to 0} \left(\frac{b^x - 1}{x}\right) = \log a - \log b = \log$
$\left(\frac{a}{b}\right)$

$\therefore \lim_{x \to 0} \frac{5^x - 6^x}{x} = \log\left(\frac{5}{6}\right)$

Question-20

Without using the series expansion of log(1 + x), prove that $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

Solution:

$\lim_{x \to 0} \frac{\log(1+x)}{x}$

Let $y = \log(1 + x)$ Then as $x \to 0$, $y \to 0$

$\lim_{y \to 0} \frac{y}{e^y - 1} = \lim_{y \to 0} \frac{y}{1 + y + \frac{y^2}{2} + \ldots\ldots - 1} = \lim_{y \to 0} \frac{y}{y\left(1 + \frac{y}{2} + \frac{y^2}{3} + \ldots\infty\right)} = \frac{1}{1} = 1$

Question-21

Differentiate the following with respect to x:

(i) $x^{7} + e^{x}$

(ii) $\log_7 x + 200$

(iii) $3 \sin x + 4 \cos x - e^{x}$

(iv) $e^{x} + 3 \tan x + \log x^{6}$

(v) $\sin 5 + \log_{10} x + 2 \sec x$

(vi) $x^{-3/2} + 8e + 7\tan x$

(vii) $\left(x + \frac{1}{x}\right)^3$

(viii) $\frac{(x-3)(2x^2-4)}{x}$

Solution:

(i) $y = x^7 + e^x$

$\frac{dy}{dx} = 7x^6 + e^x$

(ii) $y = \log_7 x + 200$

$= \log_e x \,.\, \log_{10} e + 200$

$\frac{dy}{dx} = \log_{10} e\left(\frac{1}{x}\right)$

(iii) $y = 3 \sin x + 4 \cos x - e^x$

$\frac{dy}{dx} = 3 \cos x - 4 \sin x - e^x$

(iv) $y = e^x + 3 \tan x + 6 \log x$

$\frac{dy}{dx} = e^x + 3 \sec^2 x + \frac{6}{x}$

(v) $y = \sin 5 + \log_{10} x + 2 \sec x$

$= \sin 5 + \log_e x \log_{10} e + 2 \sec x$

$\frac{dy}{dx} = 0 + \frac{\log_{10} e}{x} + 2 \sec x \tan x$

(vi) $y = x^{-3/2} + 8e + 7\tan x$

$\frac{dy}{dx} = -\frac{3}{2} x^{-5/2} + 7 \sec^2 x$

(vii) $y = \left(x + \frac{1}{x}\right)^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} = x^3 + 3x + 3x^{-1} + x^{-3}$

$\frac{dy}{dx} = 3x^2 + 3 - 3x^{-2} - 3x^{-4}$

(viii) $\frac{(x-3)(2x^2-4)}{x} = \frac{2x^3 - 6x^2 - 4x + 12}{x}$

$y = 2x^2 - 6x - 4 + \frac{12}{x}$

$\frac{dy}{dx} = 4x - 6 - \frac{12}{x^2}$

Question-22

Differentiate the following function using quotient rule.

$$\frac{1}{ax^2 + bx + c}$$

Solution:

Let $y = \frac{1}{ax^2 + bx + c}$

$$\frac{dy}{dx} = \frac{(ax^2 + bx + c).0 - 1(2ax + b)}{(ax^2 + bx + c)^2} = -\frac{(2ax + b)}{(ax^2 + bx + c)^2}$$

Question-23

Differentiate the following function using quotient rule.

$\frac{\tan x + 1}{\tan x - 1}$

Solution:

Let y = $\frac{\tan x + 1}{\tan x - 1}$

$$\frac{dy}{dx} = \frac{(\tan x - 1)(\sec^2 x) - (\tan x + 1)\sec^2 x}{(\tan x - 1)^2} = \frac{-2\sec^2 x}{(\tan x - 1)^2}$$

Question-24

Differentiate the following function using quotient rule.

$\frac{\sin x + x\cos x}{x\sin x - \cos x}$

Solution:

Let y = $\frac{\sin x + x\cos x}{x\sin x - \cos x}$

$$\frac{dy}{dx} = \frac{(x\sin x - \cos x)(\cos x - x\sin x + \cos x) - (\sin x + x\cos x)(x\cos x + \sin x + \sin x)}{(x\sin x - \cos x)^2}$$

$$= \frac{x\sin x\cos x - \cos^2 x - x^2\sin^2 x + x\sin x\cos x + x\sin\cos x - \cos^2 x - x\sin x\cos x - x^2\cos^2 x - \sin^2 x + x\sin x\cos x - \sin^2 x - x\sin x\cos x)}{(x\sin x - \cos x)^2}$$

$$= \frac{-2\cos^2 x - 2\sin^2 x - x^2\sin^2 x + x^2\cos^2 x}{(x\sin x - \cos x)^2}$$

$$= \frac{-(2 + x^2)}{(x\sin x - \cos x)^2}$$

Question-25

Differentiate the following function using quotient rule.

Solution:

Let y = $\frac{\log x^2}{e^x} = \frac{2\log x}{e^x}$

$\frac{dy}{dx} = \frac{e^x \frac{2}{x} - 2\log x . e^x}{(e^x)^2} = \frac{2e^x\left(\frac{1}{x} - \log x\right)}{(e^x)^2} = \frac{2\left(\frac{1}{x} - \log x\right)}{e^x} = e^{-x}\left(\frac{2}{x} - 2\log x\right)$

Question-26

Differentiate the following function with respect to x.
log(sinx)

Solution:

y = log(sinx)

Let u = sinx

$\frac{du}{dx}$ = cosx

y = log u

$\frac{dy}{du} = \frac{1}{u} = \frac{1}{\sin x}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u}.\cos x = \frac{1}{\sin x}\cos x = \cot x$

Question-27

Differentiate the following function with respect to x.
$e^{\sin x}$

Solution:

$y = e^{\sin x}$

Put u = sinx

$\frac{du}{dx} = \cos x$

$y = e^u$

$\frac{dy}{du} = e^u = e^{\sin x}$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^{\sin x} . \cos x$

Question-28

Differentiate the following function with respect to x.
$\sqrt{1 + \cot x}$

Solution:

$y = \sqrt{1 + \cot x}$

Put u = 1 + cotx

$\frac{du}{dx} = -\operatorname{cosec}^2 x$

$y = u^{1/2}$

$\frac{dy}{du} = \frac{1}{2} u^{-1/2} = \frac{1}{2}(1 + \cot x)^{-1/2}$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2}(1 + \cot x)^{1/2}(-\operatorname{cosec}^2 x)$

Question-29

Differentiate the following function with respect to x.
tan(logx)

Solution:

y = tan(logx)

Put u = logx

$$\frac{du}{dx} = \frac{1}{x}$$

y = tan u

$$\frac{dy}{du} = \sec^2 u$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \sec^2 u \cdot \frac{1}{x} = \frac{\sec^2(\log x)}{x}$$

Question-30

Differentiate the following function with respect to x.

$$\frac{e^{bx}}{\cos(a+b)}$$

Solution:

$$y = \frac{e^{bx}}{\cos(a+b)}$$

$$\frac{dy}{dx} = \frac{\cos(ax+b) \cdot \frac{d}{dx}(e^{bx}) - e^{bx}\frac{d}{dx}(\cos(ax+b))}{\cos^2(ax+b)}$$

$$= \frac{\cos(ax+b) \cdot e^{bx} \cdot b + e^{bx} \cdot \sin(ax+b)a}{\cos^2(ax+b)}$$

$$= \frac{e^{bx}(b\cos(ax+b) + a\sin(ax+b))}{\cos^2(ax+b)}$$

Question-31

Differentiate the following function with respect to x.

log sec $\left(\frac{\pi}{4}+\frac{x}{2}\right)$

Solution:

$y = \log \sec\left(\frac{\pi}{4}+\frac{x}{2}\right)$

Put $u = \frac{\pi}{4}+\frac{x}{2}$

$\frac{du}{dx} = \frac{1}{2}$

$y = \log \sec u$

$y = \log v$

Put $v = \sec u$

$\frac{dv}{du} = \sec u \tan u$

$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$

$$= \frac{1}{v} \cdot \sec\left(\frac{\pi}{4}+\frac{x}{2}\right) \tan\left(\frac{\pi}{4}+\frac{x}{2}\right) \cdot \frac{1}{2}$$

$$= \frac{1}{\sec\left(\frac{\pi}{4}+\frac{x}{2}\right)} \sec\left(\frac{\pi}{4}+\frac{x}{2}\right) \cdot \tan\left(\frac{\pi}{4}+\frac{x}{2}\right) \cdot \frac{1}{2}$$

$$= \frac{1}{2}\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

Question-32

Differentiate the following function with respect to x.
log sin(e^x + 4x + 5)

Solution:

y = log sin(e^x + 4x + 5)

$$\frac{dy}{dx} = \frac{1}{\sin(e^x + 4x + 5)} \cos(e^x + 4x + 5)(e^x + 4)$$

$$= (e^x + 4)\frac{\cos(e^x + 4x + 5)}{\sin(e^x + 4x + 5)}$$

$$= (e^x + 4)\cot(e^x + 4x + 5)$$

Question-33

Differentiate the following function with respect to x.
sin($x^{3/2}$)

Solution:

y = sin($x^{3/2}$)

Put u = $x^{3/2}$

y = sin u

$$\frac{du}{dx} = \frac{3}{2}x^{1/2}$$

$$\frac{dy}{du} = \cos u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u. \frac{3}{2} x^{1/2} = \cos x^{3/2}\left(\frac{3}{2}x^{1/2}\right)$$

Question-34

Differentiate the following function with respect to x.
cos($\sqrt{x}$)

Solution:

y = cos u

Put u = $\sqrt{x}$

$\frac{dy}{du}$ = -sin u

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin u \cdot \frac{1}{2\sqrt{x}} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$

Question-35

Differentiate the following function with respect to x.
$e^{\sin(\log x)}$

Solution:

$y = e^{\sin(\log x)}$

Put u = logx

$\frac{du}{dx} = \frac{1}{x}$

$y = e^{\sin u}$

Put v = sin u

$\frac{dv}{du} = \cos u = \cos(\log x)$

Put $y = e^v$

$\frac{dy}{dv} = e^v$

Hence $\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = e^v \cdot \cos(\log x) \cdot \frac{1}{x} = e^{\sin(\log x)} \cos^{(\log s)} \cdot \frac{1}{x}$

Question-36

Find the indicated limit: $\lim_{x \to 1} \frac{x^2 + 2x + 5}{x^2 + 1}$

Solution:

$\lim_{x \to 1} \frac{x^2 + 2x + 5}{x^2 + 1} = \frac{1 + 2 + 5}{1 + 1}$ by direct substitution

$= \frac{8}{2} = 4$

Question-37

Find the indicated limit: $\lim_{x \to 2} \frac{x-2}{\sqrt{2-x}}$

Solution:

$$\lim_{x \to 2} \frac{x-2}{\sqrt{2-x}} = \lim_{x \to 2} \frac{-(2-x)}{\sqrt{2-x}} = \lim_{x \to 2} -(\sqrt{2-x}) = 0$$

Question-38

Find the indicated limit: $\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$

Solution:

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0} (2x+h) = 2x$$

Question-39

Find the indicated limit: $\lim_{x \to 1} \frac{x^m - 1}{x-1}$

Solution:

$$\lim_{x \to 1} \frac{x^m - 1}{x-1} = \lim_{x \to 1} \frac{x^m - 1^m}{x-1} = m(1)^{m-1} = m$$

Question-40

Find the indicated limit: $\lim_{x \to 4} \frac{\sqrt{2x+1} - 3}{\sqrt{x-1} - \sqrt{2}}$

Solution:

$$\lim_{x \to 4} \frac{\sqrt{2x+1} - 3}{\sqrt{x-1} - \sqrt{2}} = \lim_{x \to 4} \frac{\sqrt{2x+1} - 3}{\sqrt{x-1} - \sqrt{2}} \times \frac{\sqrt{x-2} + \sqrt{2}}{\sqrt{x-2} + \sqrt{2}} \times \frac{\sqrt{2x+1} + 3}{\sqrt{2x+1} + 3} = \lim_{x \to 4} \frac{(2x+1-9)}{(x-2-2)} \frac{\sqrt{x-2} + \sqrt{2}}{\sqrt{2x+1} + 3}$$

$$= \lim_{x \to 4} \frac{2x-8}{x-4} \frac{\sqrt{x-2} + \sqrt{2}}{\sqrt{2x+1} + 3}$$

$$= \lim_{x \to 4} 2\left\{\frac{\sqrt{x-2} + \sqrt{2}}{\sqrt{2x+1} + 3}\right\}$$

$$= 2\,\frac{2\sqrt{2}}{6}$$

$$= \frac{2\sqrt{2}}{3}$$

Question-41

Find the indicated limit: $\lim_{x \to 0} \frac{\sqrt{x^2+p^2}-p}{\sqrt{x^2+q^2}-q}$

Solution:

$$\lim_{x \to 0} \frac{\sqrt{x^2+p^2}-p}{\sqrt{x^2+q^2}-q}$$

$$= \lim_{x \to 0} \frac{\sqrt{x^2+p^2}-p}{\sqrt{x^2+q^2}-q} \times \frac{\sqrt{x^2+q^2}+q}{\sqrt{x^2+q^2}+q} \times \frac{\sqrt{x^2+p^2}+p}{\sqrt{x^2+p^2}+p}$$

$$= \lim_{x \to 0} \frac{x^2+p^2-p^2}{x^2+q^2-q^2} \times \frac{\sqrt{x^2+p^2}+q}{\sqrt{x^2+q^2}+p}$$

$$= \lim_{x \to 0} \frac{2q}{2p} = \frac{q}{p}$$

Question-42

Find the indicated limit: $\lim_{x \to a} \frac{\sqrt[m]{x}-\sqrt[m]{a}}{x-a}$

Solution:

$$\lim_{x \to a} \frac{\sqrt[m]{x}-\sqrt[m]{a}}{x-a} = \lim_{x \to a} \frac{x^{1/m}-a^{1/m}}{x-a} = \frac{1}{m} a^{1/m - 1}$$ (na^{n-1} formula)

Question-43

Find the indicated limit: $\lim_{x \to 1} \dfrac{\sqrt[3]{x}-1}{\sqrt{x}-1}$

Solution:

$$\lim_{x \to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1} = \lim_{x \to 1} \frac{x^{1/3}-1}{x^{1/2}-1}$$

$$= \lim_{x \to 1} \frac{x^{1/3}-1}{x-1} \times \frac{x-1}{x^{1/2}-1} = \lim_{x \to 1} \frac{x^{1/3}-x^{1/3}}{x-1} \times \frac{x-1}{x^{1/2}-1^{1/2}}$$

$$= \frac{1}{3}(1)^{1/3-1} \times \frac{1}{1/2(1)^{1/2-1}} = \frac{1/3}{1/2} = \frac{2}{3}$$

Question-44

Find the indicated limit: $\lim_{x \to 0} \dfrac{\sqrt{1+x+x^2}-1}{x}$

Solution:

$$\lim_{x \to 0} \frac{\sqrt{1+x+x^2}-1}{x} = \lim_{x \to 0} \frac{\sqrt{1+x+x^2}-1}{x} \times \frac{\sqrt{1+x+x^2}+1}{\sqrt{1+x+x^2}+1} = \lim_{x \to 0} \frac{1+x+x^2-1}{x\left[\sqrt{1+x+x^2}+1\right]}$$

$$= \lim_{x \to 0} \frac{x(x+1)}{x(\sqrt{1+x+x^2}+1)}$$

$$= \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

Question-45

Find the indicated limit: $\lim_{x \to 0} \frac{\sin^2\left(\frac{x}{3}\right)}{x^2}$

Solution:

$$\lim_{x \to 0} \frac{\sin^2\left(\frac{x}{3}\right)}{x^2} = \lim_{x \to 0} \left(\frac{\sin x/3}{x/3}\right)^2 \frac{(x/3)^2}{x^2} = 1 \times \frac{1}{9} = \frac{1}{9}$$

Question-46

Find the indicated limit: $\lim_{x \to 0} \frac{\sin(a+x)-\sin(a-x)}{x}$

Solution:

$$\lim_{x \to 0} \frac{\sin(a+x)-\sin(a-x)}{x} = \lim_{x \to 0} \frac{2\cos a \sin x}{x} = \lim_{x \to 0} 2\cos a \left(\frac{\sin x}{x}\right) = 2\cos a$$

Question-47

Find the indicated limit: $\lim_{x \to 0} \frac{\log(1+\alpha x)}{x}$

Solution:

$$\lim_{x \to 0} \frac{\log(1+\alpha x)}{x} = \lim_{x \to 0} \frac{\alpha x - \frac{(\alpha x)^2}{2} + \frac{(\alpha x)^3}{3}}{x} \ldots\ldots = \lim_{x \to 0} \alpha - \frac{\alpha^2}{2}x + \frac{\alpha^3}{3}x^2 \ldots\ldots$$

Question-48

Find the indicated limit: $\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^{n+5}$

Solution:

$$\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^{n+5} = \lim_{n \to \infty} \left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)^{5} = e.\ (1)^5 = e$$

Question-49

Evaluate the left and right limits of f(x) = $\frac{x^3 - 27}{x - 3}$ at x = 3. Does the limit of f(x) = x › 3 exist? Justify your answer.

Solution:

$\lim_{x \to 3} \left(\frac{x^3 - 27}{x - 3}\right)$

Let x = 3 + h

Then $\lim_{h \to 0} \left(\frac{(3+h)^3 - 27}{(3+h) - 3}\right) = \left(\frac{9h^2 + 27h + h^3}{h}\right) = \lim_{h \to 0} 27 + 9h + h^2 = 27$

$\lim_{h \to 3} \frac{x^3 - 27}{x - 3}$

Let x = 3 – h

Then $\lim_{h \to 0} \frac{(3-h)^3 - 27}{(3 - h - 3)} = \lim_{h \to 0} \frac{-27h^2 + 9h^2 - h^3}{-h} = 27$

Also, $\lim_{x \to 3} \frac{x^3 - 27}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x^2 + 3x + 9)}{x - 3} = 9 + 9 + 9 = 27$

Question-50

Find the positive integer n such that $\lim_{x \to 3} \frac{x^n - 3^n}{x - 3} = 108$.

Solution:

$\lim_{x \to 3} \frac{x^n - 3^n}{x - 3} = 108$

$n\,3^{n-1} = 108$

Put n = 4, then $4.3^3 = 4 \times 27 = 108$

$\therefore n = 4$

Question-51

Evaluate $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$ [Hint: Take e^x or $e^{\sin x}$ as common factor in numerator]

Solution:

$$\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \lim_{x \to 0} \frac{e^x\left(1 - \frac{e^{\sin x}}{e^x}\right)}{(x - \sin x)}$$

$$= \frac{e^x(1 - e^{\sin x - x})}{(x - \sin x)} = \frac{e^x(1 - e^{-(x - \sin x)})}{(x - \sin x)}$$

$$= e^x \left[\frac{1 - (1 - (x - \sin x) + x - \sin x)^2 + \ldots\ldots}{(x - \sin x)}\right]$$

$$= e^x \left[\frac{(x - \sin x) - (x - \sin x)^2 + \ldots\ldots}{(x - \sin x)}\right]$$

$$= \lim_{x \to 0} e^x [1 - (x - \sin x) \ldots\ldots] = e^0 = 1$$

Question-52

If $f(x) = \frac{ax^2 + b}{x^2 - 1}$, $\lim_{x \to 0} f(x) = 1$ and $\lim_{x \to \infty} f(x) = 1$

Solution:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{ax^2 + b}{x^2 - 1} = b/-1 = -b = 1$$

$\therefore b = -1$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{a + \frac{b}{x^2}}{1 - \frac{1}{x^2}} = a = 1$$

$\therefore a = 1$

$$\therefore f(x) = \frac{x^2 - 1}{x^2 - 1} = 1$$

$f(2) = 1$ and $f(-2) = 1$

Question-53

Evaluate $\lim_{x \to 0} \frac{|x|}{x}$ and $\lim_{x \to 0} \frac{|x|}{x}$. What can you say about $\lim_{x \to 0} \frac{|x|}{x}$?

Solution:

Let $f(x) = \frac{x}{|x|}$ where $|x| = x \qquad x \geq 0$

$= -x \qquad x < 0$

Then,

$\underset{x \to 0}{Lt} f(x) = \underset{x \to 0}{Lt} \frac{x}{|x|} = \underset{x \to 0}{Lt} \frac{x}{x} = 1$

$\underset{x \to 0}{Lt} f(x) = \underset{x \to 0}{Lt} \frac{x}{|x|} = \underset{x \to 0}{Lt} \frac{x}{-x} = -1$

$\therefore \underset{x \to 0}{Lt} f(x) \neq \underset{x \to 0}{Lt} f(x)$

$\therefore \underset{x \to 0}{Lt} \frac{x}{|x|}$ does not exist.

Question-54

Compute $\lim_{x \to 0} \frac{a^x - b^x}{x}, a, b > 0$. Hence evaluate $\lim_{x \to 0} \frac{5^x - 6^x}{x}$.

Solution:

$\lim_{x \to 0} \frac{a^x - b^x}{x} = \lim_{x \to 0} \frac{(a^x - 1) - (b^x - 1)}{x} = \lim_{x \to 0} \left(\frac{a^x - 1}{x}\right) - \lim_{x \to 0}\left(\frac{b^x - 1}{x}\right) = \log a - \log b = \log \left(\frac{a}{b}\right)$

$\therefore \lim_{x \to 0} \frac{5^x - 6^x}{x} = \log\left(\frac{5}{6}\right)$

Question-55

Without using the series expansion of log(1 + x), prove that $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

Solution:

$\lim_{x \to 0} \frac{\log(1+x)}{x}$

Let $y = \log(1 + x)$ Then as $x \to 0$, $y \to 0$

$\lim_{y \to 0} \frac{y}{e^y - 1} = \lim_{y \to 0} \frac{y}{1 + y + \frac{y^2}{2} + \ldots\ldots - 1} = \lim_{y \to 0} \frac{y}{y\left(1 + \frac{y}{2} + \frac{y^2}{3} + \ldots\infty\right)} = \frac{1}{1} = 1$

Question-56

Differentiate the following with respect to x:

(i) $x^{7} + e^{x}$

(ii) $\log_7 x + 200$

(iii) $3 \sin x + 4 \cos x - e^{x}$

(iv) $e^{x} + 3 \tan x + \log x^{6}$

(v) $\sin 5 + \log_{10} x + 2 \sec x$

(vi) $x^{-3/2} + 8e + 7\tan x$

(vii) $\left(x + \frac{1}{x}\right)^{3}$

(viii) $\frac{(x-3)(2x^2-4)}{x}$

Solution:

(i) $y = x^{7} + e^{x}$

$$\frac{dy}{dx} = 7x^{6} + e^{x}$$

(ii) $y = \log_7 x + 200 = \log_e x \, . \, \log_{10} e + 200$

$$\frac{dy}{dx} = \log_{10} e\left(\frac{1}{x}\right)$$

(iii) $y = 3 \sin x + 4 \cos x - e^{x}$

$\frac{dy}{dx}$ = 3 cos x - 4 sin x – e^x

(iv) y = e^x + 3 tanx + 6 log x

$\frac{dy}{dx}$ = e^x + 3 $\sec^2 x$ + $\frac{6}{x}$

(v) y = sin 5 + $\log_{10} x$ + 2 sec x

= sin 5 + $\log_e x \log_{10} e$ + 2 sec x

$\frac{dy}{dx}$ = 0 + $\frac{\log_{10} e}{x} + 2 \sec x \tan x$

(vi) y = $x^{-3/2}$ + 8e + 7tanx

$\frac{dy}{dx} = -\frac{3}{2} x^{-5/2} + 7 \sec^2 x$

(vii) y = $\left(x + \frac{1}{x}\right)^3$ = x^3 + 3x + $\frac{3}{x} + \frac{1}{x^3}$ = x^3 + 3x + $3x^{-1}$ + x^{-3}

$\frac{dy}{dx}$ = $3x^2$ + 3 – $3x^{-2}$ - $3x^{-4}$

(viii) $\frac{(x-3)(2x^2-4)}{x} = \frac{2x^3 - 6x^2 - 4x + 12}{x}$

y = $2x^2$ – 6x – 4 + $\frac{12}{x}$

$\frac{dy}{dx}$ = 4x – 6 - $\frac{12}{x^2}$

Question-57

Differentiate the following functions with respect to x.

(i) $e^x \cos x$

(ii) $\sqrt[n]{x} \log \sqrt{x}, x > 0$

(iii) $6 \sin x \log_{10} x + e$

(iv) $(x^4 - 6x^3 + 7x^2 + 4x + 2)(x^3 - 1)$

(v) (a – b six (1 – 2 cos x)

(vi) cosec x . cotx

(vii) $\sin^2 x$

(viii) $\cos^2 x$

(ix) $(3x^2 + 1)^2$

(x) $(4x^2 - 1)(2x + 3)$

(xi) $(3 \sec x - 4 \operatorname{cosec} x)(2 \sin x + 5 \cos x)$

(xii) $x^2 e^x \sin x$

(xiii) $\sqrt{x}\, e^x \cos x$

Solution:

(i) $y = e^x \cos x$

$\frac{dy}{dx}$ = - e^x sin x + cos x e^x

(ii) y = $x^{1/n}$ log($\sqrt{x}$) = $\frac{1}{2}x^{1/n}$ log x

$\frac{dy}{dx} = \frac{1}{2}\left[x^{1/n}\frac{1}{x} + \log x\left(\frac{1}{n}x^{1/n - 1}\right)\right] = \frac{1}{2}\left[x^{1/n - 1} + \frac{1}{n}\log x . x^{\frac{1}{n}-1}\right]$

(iii) y = 6 sin x $\log_{10}$ x + e = 6 sin x $\log_e x$. $\log_{10}$ e + e

$\frac{dy}{dx}$ = 6 $\log_{10}$ e$\left(\sin x \frac{1}{x} + \log x \cos x\right)$

(iv) y = (x^4 – 6x^3 + 7x^2 + 4x + 2)(x^3 – 1)

$\frac{dy}{dx}$ = (x^4 – 6x^3 + 7x^2 + 4x + 2) (3x^2) + (x^3 – 1) (4x^3 – 18x^2 + 14x + 4)

(v) y = (a – b six (1 – 2 cos x)

$\frac{dy}{dx}$ = (a – b sinx) (2 sin x) + (1 – 2 cos x) (-b cosx)

$\frac{dy}{dx}$ = 2a sinx – 2b $\sin^2$ x – b cos x + 2b $\cos^2$x

(vi) y = cosec x. cot x

$\frac{dy}{dx}$ = -cosec x cosec^2x(-cosec x cotx) = -cosec^3 x – $\cot^2$x cosec x

Question-58

Differentiate the following function using quotient rule.

$\frac{5}{x^2}$

Solution:

Let y = $\frac{5}{x^2}$

$$\frac{dy}{dx} = \frac{(x^2).\frac{d}{dx}(5). - 5\frac{d}{dx}(x^2)}{(x^2)^2} = \frac{x^2.0 - 5(2x)}{x^4} = \frac{-10x}{x^4} = \frac{-10}{x^3}$$

Question-59

Differentiate the following function using quotient rule.

$\frac{2x-3}{4x+5}$

Solution:

Let $y = \frac{2x-3}{4x+5}$

$$\frac{dy}{dx} = \frac{(4x+5)\frac{d}{dx}(2x-3^2)-(2x-3)\frac{d}{dx}(4x+5)}{(4x+5)^2}$$

$$= \frac{(4x+5)(2)-(2x-3)(4)}{(4x+5)^2}$$

$$= \frac{8x+10-8x+12}{(4x+5)^2}$$

$$= \frac{22}{(4x+5)^2}$$

Question-60

Differentiate the following function using quotient rule.

$\frac{x^7-4^7}{x-4}$

Solution:

Let $y = \frac{x^7-4^7}{x-4}$

$$\frac{dy}{dx} = \frac{(x-4)7x^6-(x^7-4^7).1}{(x-4)^2} = \frac{6x^7-28x^6+4^7}{(x-4)^2}$$

Question-61

Differentiate the following function using quotient rule.

$$\frac{\cos x + \log x}{x^2 + e^x}$$

Solution:

Let y = $\frac{\cos x + \log x}{x^2 + e^x}$

$$\frac{dy}{dx} = \frac{(x^2 + e^x)\left(-\sin x + \frac{1}{x}\right) - (\cos x + \log x)(2x + e^x)}{(x^2 + e^x)^2}$$

$$\frac{dy}{dx} = \frac{e^x\left(\frac{1}{x} - \sin x - \cos x - \log x\right) - 2x(\cos x + \log x) + x - x^2 \sin x}{(x^2 + e^x)^2}$$

Question-62

Differentiate the following function using quotient rule.

$$\frac{\log x - 2x^2}{\log x + 2x^2}$$

Solution:

Let y = $\frac{\log x - 2x^2}{\log x + 2x^2}$

$$\frac{dy}{dx} = \frac{(\log x + 2x^2)\left(\frac{1}{x} - 4x\right) - (\log x - 2x^2)\left(\frac{1}{x} + 4x\right)}{(\log x + 2x^2)^2} = \frac{4x(1 - 2\log x)}{(\log x + 2x^2)^2}$$

Question-63

Differentiate the following function using quotient rule.

$$\frac{\log x}{\sin x}$$

Solution:

Let $y = \frac{\log x}{\sin x}$

$$\frac{dy}{dx} = \frac{\sin x\left(\frac{1}{x}\right) - \log x \cos x}{\sin^2 x}$$

Question-64

Differentiate the following function using quotient rule.

$$\frac{1}{ax^2 + bx + c}$$

Solution:

Let $y = \frac{1}{ax^2 + bx + c}$

$$\frac{dy}{dx} = \frac{(ax^2 + bx + c).0 - 1(2ax + b)}{(ax^2 + bx + c)^2} = -\frac{(2ax + b)}{(ax^2 + bx + c)^2}$$

Question-65

Differentiate the following function using quotient rule.

$$\frac{\tan x + 1}{\tan x - 1}$$

Solution:

Let $y = \frac{\tan x + 1}{\tan x - 1}$

$$\frac{dy}{dx} = \frac{(\tan x - 1)(\sec^2 x) - (\tan x + 1)\sec^2 x}{(\tan x - 1)^2} = \frac{-2\sec^2 x}{(\tan x - 1)^2}$$

Question-66

Differentiate the following function using quotient rule.

$$\frac{\sin x + x\cos x}{x\sin x - \cos x}$$

Solution:

Let $y = \frac{\sin x + x\cos x}{x\sin x - \cos x}$

$$\frac{dy}{dx} = \frac{(x\sin x - \cos x)(\cos x - x\sin x + \cos x) - (\sin x + x\cos x)(x\cos x + \sin x + \sin x)}{(x\sin x - \cos x)^2}$$

$$= \frac{x\sin x\cos x - \cos^2 x - x^2\sin^2 x + x\sin x\cos x + x\sin\cos x - \cos^2 x - x\sin x\cos x - x^2\cos^2 x - \sin^2 x + x\sin x\cos x - \sin^2 x - x\sin x\cos x)}{(x\sin x - \cos x)^2}$$

$$= \frac{-2\cos^2 x - 2\sin^2 x - x^2\sin^2 x + x^2\cos^2 x}{(x\sin x - \cos x)^2}$$

$$= \frac{-(2 + x^2)}{(x\sin x - \cos x)^2}$$

Question-67

Differentiate the following function using quotient rule.

Solution:

Let $y = \frac{\log x^2}{e^x} = \frac{2\log x}{e^x}$

$$\frac{dy}{dx} = \frac{e^x\frac{2}{x} - 2\log x . e^x}{(e^x)^2} = \frac{2e^x\left(\frac{1}{x} - \log x\right)}{(e^x)^2} = \frac{2\left(\frac{1}{x} - \log x\right)}{e^x} = e^{-x}\left(\frac{2}{x} - 2\log x\right)$$

Question-68

Differentiate the following function with respect to x.

Solution:

$y = \log(\sin x)$

Let $u = \sin x$

$\frac{du}{dx} = \cos x$

$y = \log u$

$\frac{dy}{du} = \frac{1}{u} = \frac{1}{\sin x}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u}.\cos x = \frac{1}{\sin x}\cos x = \cot x$

Question-69

Differentiate the following function with respect to x.

$e^{\sin x}$

Solution:

$y = e^{\sin x}$

Put $u = \sin x$

$\frac{du}{dx} = \cos x$

$y = e^{u}$

$\frac{dy}{du} = e^{u} = e^{\sin x}$

$\therefore \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = e^{\sin x}.\cos x$

Question-70

Differentiate the following function with respect to x. $\sqrt{1+\cot x}$

Solution:

$y = \sqrt{1 + \cot x}$

Put u = 1 + cotx

$\frac{du}{dx} = -\text{cosec}^2 x$

$y = u^{1/2}$

$\frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2}(1 + \cot x)^{-1/2}$

$\therefore \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{2}(1 + \cot x)^{1/2}(-\text{cosec}^2 x)$

Question-71

Differentiate the following function with respect to x. tan(logx)

Solution:

y = tan(logx)

Put u = logx

$\frac{du}{dx} = \frac{1}{x}$

y = tan u

$\frac{dy}{du} = \sec^2 u$

$\therefore \frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx} = \sec^2 u. \frac{1}{x} = \frac{\sec^2(\log x)}{x}$

Question-72

Differentiate the following function with respect to x.

$$\frac{e^{bx}}{\cos(a+b)}$$

Solution:

$$y = \frac{e^{bx}}{\cos(a+b)}$$

$$\frac{dy}{dx} = \frac{\cos(ax+b).\frac{d}{dx}(e^{bx}) - e^{bx}\frac{d}{dx}(\cos(ax+b))}{\cos^2(ax+b)}$$

$$= \frac{\cos(ax+b).e^{bx}.b + e^{bx}.\sin(ax+b)a}{\cos^2(ax+b)}$$

$$= \frac{e^{bx}(b\cos(ax+b) + a\sin(ax+b)}{\cos^2(ax+b)}$$

Question-73

Differentiate the following function with respect to x.

log sec $\left(\frac{\pi}{4}+\frac{x}{2}\right)$

Solution:

y = log sec $\left(\frac{\pi}{4}+\frac{x}{2}\right)$

Put u = $\frac{\pi}{4}+\frac{x}{2}$

$\frac{du}{dx} = \frac{1}{2}$

y = log sec u

y = log v

Put v = sec u

$\frac{dv}{du}$ = sec u tan u

$\frac{dy}{dv} = \frac{1}{v}$

$$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{v} \cdot \sec\left(\frac{\pi}{4} + \frac{x}{2}\right) \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \cdot \frac{1}{2}$$

$$= \frac{1}{\sec\left(\frac{\pi}{4} + \frac{x}{2}\right)} \sec\left(\frac{\pi}{4} + \frac{x}{2}\right) \cdot \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \cdot \frac{1}{2}$$

$$= \frac{1}{2} \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

Question-74

Differentiate the following function with respect to x.
$\log \sin(e^x + 4x + 5)$

Solution:

$y = \log \sin(e^x + 4x + 5)$

$$\frac{dy}{dx} = \frac{1}{\sin(e^x + 4x + 5)} \cos(e^x + 4x + 5)(e^x + 4)$$

$$= (e^x + 4) \frac{\cos(e^x + 4x + 5)}{\sin(e^x + 4x + 5)}$$

$$= (e^x + 4)\cot(e^x + 4x + 5)$$

Question-75

Differentiate the following function with respect to x.
$\sin(x^{3/2})$

Solution:
$y = \sin(x^{3/2})$

Put $u = x^{3/2}$

$y = \sin u$

$\frac{du}{dx} = \frac{3}{2} x^{1/2}$

$\frac{dy}{du} = \cos u$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u. \frac{3}{2} x^{1/2} = \cos x^{3/2} \left(\frac{3}{2} x^{1/2}\right)$

Question-76

Differentiate the following function with respect to x.
$\cos(\sqrt{x})$

Solution:
$y = \cos u$

Put $u = \sqrt{x}$

$\frac{dy}{du} = -\sin u$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = - \sin u \cdot \frac{1}{2\sqrt{x}} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$

Question-77

Differentiate the following function with respect to x.
$e^{\sin(\log x)}$

Solution:

$y = e^{\sin(\log x)}$

Put u = logx

$\frac{du}{dx} = \frac{1}{x}$

$y = e^{\sin u}$

Put v = sin u

$\frac{dv}{du} = \cos u = \cos(\log x)$

Put $y = e^{v}$

$\frac{dy}{dv} = e^{v}$

Hence $\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = e^{v} \cdot \cos(\log x) \cdot \frac{1}{x} = e^{\sin(\log x)} \cos^{(\log s)} \cdot \frac{1}{x}$

CBSE Class 11 Mathematics

Important Questions

Chapter 13

Limits and Derivatives

1 Marks Questions

1. Evaluate $\lim_{x\to 3}\left[\frac{x^2-9}{x-3}\right]$

Ans. $\lim_{x\to 3}\frac{x^2-9}{x-3}\ \frac{0}{0}\ form$

$$\lim_{x\to 3}\frac{(x+3)(x-3)}{(x-3)}=3+3=6$$

2. Evaluate $\lim_{x\to 0}\frac{\sin 3x}{5x}$

Ans. $\lim_{x\to 0}\frac{\sin 3x}{5x}$

$$=\lim_{3x\to 0}\frac{\sin 3x}{3x}\times\frac{3}{5}$$

$$=1\times\frac{3}{5}=\frac{3}{5}\left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

3. Find derivative of 2^x

Ans. Let $y=2^x$

$$\frac{dy}{dx}=\frac{d}{dx}2=2^x 10g2$$

4. Find derivative of $\sqrt{\sin 2x}$

Ans. $\frac{d}{dx}\sqrt{\sin 2x} = \frac{1}{2\sqrt{\sin 2x}}\frac{d}{dx}\sin 2x$

$$= \frac{1}{\cancel{2}\sqrt{\sin 2x}} \times \cancel{2}\cos 2x$$

$$= \frac{\cos 2x}{\sqrt{\cos 2x}}$$

5. Evaluate $\lim_{x\to 0}\frac{\sin^2 4x}{x^2}$

Ans.

$$\lim_{x\to 0}\frac{\sin^2 4x}{x^2 4^2}\times 4^2 = \lim_{4x\to 0}\left(\frac{\sin 4x}{4x}\right)^2 \times 16$$

$$= 1\times 16 = 16$$

6. What is the value of $\lim_{x\to a}\left(\frac{x^2 - a^n}{x-a}\right)$

Ans. $\lim_{x\to a}\frac{x^n - a^n}{x-a} = 1$

7. Differentiate $\frac{2^x}{x}$

Ans. $\frac{d}{dx}\frac{2^x}{x} = \frac{x\frac{d}{dx}2^x - 2^x\frac{d}{dx}x}{x^2}$

$$= \frac{x\times 2^x 10g2 - 2^x \times 1}{x^2}$$

$$= 2x\frac{[x+10g2-1]}{x^2}$$

8. If $y = e^{\sin x}$ **find** $\frac{dy}{dx}$

Ans. $y = e^{\sin x}$

$$\frac{dy}{dx} = \frac{d}{dx}e^{\sin x}$$

$$= e^{\sin x} \times \cos x = \cos x e^{\sin x}$$

9. Evaluate $\lim_{x \to 1} \frac{x^{15}-1}{x^{10}-1}$

Ans. $\lim_{x \to 1} \frac{x^{15}-1}{x^{10}-1}$

$$= \frac{\lim_{x \to 1} \frac{x^{15}-1^{15}}{x-1}}{\lim_{x \to 1} \frac{x^{10}-1^{10}}{x-1}} = \frac{15 \times 1^{14}}{10 \times 1^{9}}$$

$$= \frac{15}{10} = \frac{3}{2}$$

10. Differentiate $x \sin x$ **with respect to** x

Ans. $\frac{d}{dx} x \sin x = x \cos x + \sin x 1$

$= x \cos x + \sin x$

11. Evaluate $\lim_{x\to 1}\frac{x^2+1}{x+100}$

Ans. $\lim_{x\to 1}\frac{x^2+1}{x+100}=\frac{2}{101}$

12. Evaluate $\lim_{x\to 0}\left[\cos ecx-\cot x\right]$

Ans. $\lim_{x\to 0}\left[\cos ec-\cot x\right]$

$=\lim_{x\to 0}\left[\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right]$

$=\lim_{x\to 0}\frac{1-\cos x}{\sin x}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$

$=\lim_{x\to 0}\tan\frac{x}{2}=0$

13. Find $f^1(x)\ at\ x=100$

$if\ f(x)=99x$

Ans. $f(x)=99x$

$f^1(x)=99$ at x=100

$f^1(x)=99$

14. Evaluate $\lim_{x\to -2}\frac{\tan \pi x}{x+2}$

Ans. $\lim_{x\to -2} \frac{\tan \pi x}{x+2} \qquad \frac{1}{0}\ form$

$let\ x+2=y$

$x=y-2$

$$\lim_{y\to 0} \frac{\tan \pi(y-2)}{y}$$

$$\lim_{y\to 0} \frac{-\tan \pi(2-y)}{y} = \lim_{y\to 0} \frac{\tan[2\pi-2y]}{y}$$

$$= \lim_{2y\to 0} \frac{+\tan 2y}{2y}\times 2$$

$$=1\times 2=2$$

15. Find derivative of $\sin^n x$

Ans. $\frac{d}{dx}\sin^n x$

$$= n\sin^{n-1}x \quad \frac{d}{dx}\sin x$$

$$= n\sin^{n-1}x\ \cos x$$

16. Find derivative of $1+x+x^2+x^3+---+x^{50} at\ x=1$

Ans. $f(x)=1+x+x^2+x^3+---+x^{50}$

$f^1(x)=1+2x+3x^2+---+50x^{49}$

$at\ x=1$

$$f^1(1)=1+2+3+---+50=\frac{50(50+1)}{2}$$

$= 25 \times 51 = 1275$

17. The value of $\lim_{h \to 0} \frac{e^{2h} - 1}{h}$

Ans. $\lim_{2h \to 0} \frac{e^{2h} - 1}{2h} \times 2$

$= 1 \times 2 = 2$

18. Evaluate $\lim_{x \to 0} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$

Ans. $\lim_{x \to 0} \left[\frac{(1+x)^6 - 1}{(1+x)^2 - 1} \right]$

$let\ 1 + x = y$

$x \to 0, y \to 1$

$$\lim_{y \to 1} \frac{y^6 - 1}{y^2 - 1} = \lim_{y \to 1} \frac{\frac{y^6 - 1^6}{y - 1}}{\frac{y^2 - 1^2}{y - 1}}$$

$$= \frac{6 \times 1^5}{2 \times 1} = \frac{6}{2} = 3$$

19. $\lim_{x \to a} \frac{x^7 + a^7}{x + a} = 7$ **find the value of 'a**

Ans. $\lim_{x \to a} \frac{x^7 + a^7}{x + a} = 7$

$$= \frac{a^7 + a^7}{a + a} = 7$$

$$= \frac{2a^7}{2a} = 7$$

$$= a^6 = 7$$

$$= a = \sqrt[6]{7}$$

20. Differentiate $x^{-3}(5+3x)$

Ans. $\frac{d}{dx} x^{-3}(5+3x)$

$$= \frac{d}{dx}\left[5x^{-3} + 3x^{-2}\right]$$

$$= 5 \times -3x^{-4} + 3 \times -2x^{-3}$$

$$= \frac{-15}{x^4} - \frac{6}{x^3}$$

CBSE Class 12 Mathematics

Important Questions

Chapter 13

Limits and Derivatives

4 Marks Questions

1. Prove that $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)=1$

Ans. We have

$$\lim_{x\to 0}\frac{e^x-1}{x}$$

$$\lim_{x\to 0}\left\{\frac{\left[1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+---\right]-1}{x}\right\}\left[\because e^x=1+x+\frac{x^2}{2!}+---\right]$$

$$\lim_{x\to 0}\left\{\frac{x+\frac{x^2}{2!}+\frac{x^3}{3!}+---}{x}\right\}$$

$$\lim_{x\to 0}x\left\{\frac{1+\frac{x}{2!}+\frac{x^2}{3!}+---}{x}\right\}$$

$=1+0=1$

2. Evaluate $\lim_{x\to 1} \frac{(2x-3)(\sqrt{x}-1)}{(2x^2+x-3)}$

Ans. $\lim_{x\to 1} \frac{(2x-3)(\sqrt{x}-1)}{(2x^2+x-3)}$

$= \lim_{x\to 1} \frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)}$

$\lim_{x\to 1} \frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)} \times \frac{(\sqrt{x}+1)}{(\sqrt{x}+1)}$

$\lim_{x\to 1} \frac{(2x-3)(x-1)}{(2x+3)(x-1)(\sqrt{x}+1)}$

$\lim_{x\to 1} \frac{(2x-3)}{(2x+3)(\sqrt{x}+1)} = \frac{2\times 1-3}{(2\times 1+3)(\sqrt{1}+1)}$

$= \frac{-1}{10}$

3. Evaluate $\lim_{x\to 0} \frac{x\tan 4x}{1-cas4x}$

Ans. $\lim_{x\to 0} \frac{x\tan 4x}{1-cas4x}$

$= \lim_{x\to 0} \frac{x\sin 4x}{\cos 4x\left[2\sin^2 2x\right]}$

$$= \lim_{x\to 0} \frac{2x \sin 2x \cos 2x}{\cos 4x\left(2\sin^2 2x\right)}$$

$$= \lim_{x\to 0}\left(\frac{\cos 2x}{\cos 4x}.\frac{2x}{\sin 2x}\times\frac{1}{2}\right)$$

$$= \frac{1}{2}\frac{\lim\limits_{2x\to 0}\cos 2x}{\lim\limits_{4x\to 0}\cos 4x}\times\lim_{2x\to 0}\left(\frac{2x}{\sin 2x}\right) = \frac{1}{2}\times 1 = \frac{1}{2}$$

4. It $y = \frac{(1-\tan x)}{(1+\tan x)}$ **. Show that** $\frac{dy}{dx} = \frac{-2}{(1+\sin 2x)}$

Ans. $y = \frac{(1-\tan x)}{(1+\tan x)}$

$$\frac{dy}{dx} = \frac{(1+\tan x)\frac{d}{dx}(1-\tan x)-(1-\tan x)\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{(1+\tan x)\left(-\sec^2 x\right)-(1-\tan x)\sec^2 x}{(1+\tan x)^2}$$

$$= \frac{-\sec^2 x - \tan x \sec^2 x - \sec^2 + \tan x \sec^2 x}{(1+\tan x)^2}$$

$$= \frac{-2\sec^2 x}{(1+\tan x)^2} = \frac{-2}{\cos^2 x\left[1+\frac{sicx}{\cos x}\right]^2}$$

$$= \frac{-2}{\cos^2 x\left[\frac{\cos x+\sin x}{\cos^2 x}\right]^2}$$

$= \frac{-2}{\cos^2 x + \sin^2 x + 2\sin x \cos x} = \frac{-2}{1+\sin^2 x}$

$\therefore \frac{dy}{dx} = \frac{-2}{1+\sin 2x}$ Hence proved

5. Differentiate $e^{\sqrt{\cot x}}$

Ans. Let $y = e^{\sqrt{\cot x}}$

$\frac{dy}{dx} = \frac{d}{dx} e^{\sqrt{\cot x}} = e^{\sqrt{\cot x}} \frac{d}{dx} \sqrt{\cot x}$

$= e^{\sqrt{\cot x}} \times \frac{1}{2\sqrt{\cot x}} . \frac{d}{dx} \cot x$

$= \frac{e^{\sqrt{\cot x}}}{2\sqrt{\cot x}} - \cos ec^2 x$

$= \frac{-\cos cec^2 e^{\sqrt{\cot x}}}{2\sqrt{\cot x}}$

6. Let $f(x) \begin{cases} a+bx, x<1 \\ 4, x=1 \\ b-ax, x>1 \end{cases}$ **and if** $\lim_{x\to 1} f(x) = f(1)$ **What are the possible value of a and b ?**

Ans. Given f (1) =4

$\because \lim_{x\to 1} f(x) = f(1) = 4$

$\therefore \lim_{x\to 1} f(x) exist$

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 4 ----(1)$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (a + bx) \quad \begin{bmatrix} \because for x < 1 \\ f(x) = a + bx \end{bmatrix}$$

$= a + b$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (b - ax) \begin{bmatrix} \because for x > 1 \\ f(x) = b - ax \end{bmatrix}$$

$= b - a$

By eq. (1)

$a + b = b - a = 4$

$a + b = 4$

$b - a = 4$

$\therefore a = 0 \ \ and \ \ b = 4$

7. If $y = \dfrac{1}{\sqrt{a^2 - x^2}}$, **find** $\dfrac{dy}{dx}$

Ans. $y = \dfrac{1}{\sqrt{a^2 - x^2}}$

$put \ \left(a^2 - x^2\right) = t$

$y = \dfrac{1}{\sqrt{t}}$ and $t = a^2 - x^2$

$$\frac{dy}{dt} = \frac{d}{dt} t^{\frac{-1}{2}}$$

$$= \frac{-1}{2} t^{\frac{-1}{2} - 1}$$

$$= \frac{-1}{2} t^{\frac{-3}{2}}$$

$$\frac{dt}{dx} = -2x$$

so.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{-1}{\cancel{2}} t^{\frac{-3}{2}} \times (-\cancel{2}x) = x\, t^{\frac{-3}{2}}$$

$$= x\left(a^2 - x^2\right)^{\frac{-3}{2}}$$

8. Differentiate $\sqrt{\frac{1-\tan x}{1+\tan x}}$

Ans. let $y = \sqrt{\frac{1-\tan x}{1+\tan x}}$

put $\frac{1-\tan x}{1+\tan x} = t$

$y = \sqrt{t}$ and $t = \frac{1-\tan x}{1+\tan x}$

$$\frac{dy}{dt} = \frac{d}{dt} t^{\frac{1}{2}}$$

$$= \frac{1}{2} t^{\frac{1}{2}-1} = \frac{1}{2} t^{\frac{-1}{2}}$$

$$= \frac{1}{2\sqrt{t}} = \frac{1}{2\sqrt{\frac{1-\tan x}{1+\tan x}}} = \frac{1}{2}\sqrt{\frac{1+\tan x}{1-\tan x}}$$

$$\frac{dt}{dx}=\frac{(1+\tan x)\frac{d}{dx}(1-\tan x)-(1-\tan x)\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$=\frac{(1+\tan x)(0-\sec^2 x)-(1-\tan x)(0+\sec^2 x)}{(1+\tan x)^2}$$

$$=\frac{\sec^2 x[-1-\tan x-1+\tan x]}{(1+\tan x)^2}$$

$$=\frac{-2\sec^2 x}{(1+\tan x)^2}$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dt}{dx}}=\frac{1}{2}\sqrt{\frac{1+\tan x}{1-\tan x}}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$$

$$=\frac{-2\sec^2 x}{(1+\tan x)^2}\times\frac{1}{2}\sqrt{\frac{1+\tan x}{1-\tan x}}$$

$$=\frac{-\sec^2 x}{(1+\tan x)^{\frac{3}{2}}(1-\tan x)^{\frac{1}{2}}}$$

9. Differentiate $(i)\ \left(\frac{\sin x+\cos x}{\sin x-\cos x}\right)$ $(ii)\ \left(\frac{\sin x-1}{\sec x+1}\right)$

Ans. (i) $\frac{d}{dx}\left(\frac{\sin x+\cos x}{\sin x-\cos x}\right)$

$$=\frac{(\sin x-\cos x).\frac{d}{dx}(\sin x+\cos x)-(\sin x+\cos x).\frac{d}{dx}(\sin x-\cos x)}{(\sin x-\cos x)^2}$$

$$=\frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$=\frac{-\left[(\sin x-\cos x)^2+(\sin x+\cos x)^2\right]}{(\sin x-\cos x)^2}$$

$$=\frac{-2(\sin^2 x+\cos^2 x)}{(\sin^2 x+\cos^2 x-2\sin x\cos x)}$$

$$=\frac{-2}{(1-\sin 2x)}$$

(ii) $\frac{d}{dx}\left[\frac{\sec x-1}{\sec x+1}\right]$

$$=\frac{(\sec x+1).\frac{d}{dx}(\sec x-1)-(\sec x-1).\frac{d}{dx}(\sec x+1)}{(\sec x+1)^2}$$

$$=\frac{(\sec x+1)\sec x\tan x-(\sec x-1)\sec x\tan x}{(\sec x+1)^2}$$

$$=\frac{2\sec x\tan x}{(\sec x+1)^2}$$

10. Evaluate $\lim_{x\to\frac{\pi}{4}}\frac{\sin x-cox}{\left(x-\frac{\pi}{4}\right)}$

Ans. put $\left(x-\frac{\pi}{4}\right)=y$, so that when $x\to\frac{\pi}{4}$ then $y\to 0$

$$\therefore \lim_{x\to\frac{\pi}{4}} \frac{\sin x-\cos x}{\left(x-\frac{\pi}{4}\right)}$$

$$=\lim_{y\to 0}\frac{\left[\sin\left(\frac{\pi}{4}+y\right)-\cos\left(\frac{\pi}{4}+y\right)\right]}{y}\left[\text{puthing }\left(x-\frac{\pi}{4}=y\right)\right]$$

$$=\lim_{y\to 0}\frac{\left[\left(\sin\frac{\pi}{4}\cos y+\cos\frac{\pi}{4}\sin y-\sin\frac{\pi}{4}\sin y\right)\right]}{y}$$

$$=\frac{2}{\sqrt{2}}\times\lim_{y\to 0}\left(\frac{\sin y}{y}\right)=\left(\sqrt{2}\times 1=\sqrt{2}.\right)$$

11. Evaluate $\lim_{x\to 0}\frac{(1+x)^6-1}{(1+x)^5-1}$

Ans. put $(1+x)=y$, so that when $x\to 0$ then $y\to 1$

$$\therefore \lim_{x\to 0}\frac{(1+x)^6-1}{(1+x)^5-1}$$

$$=\lim_{y\to 1}\left[\frac{y^6-1}{y^5-1}\right]=\frac{\lim_{y\to 1}\left(\frac{y^6-1}{y-1}\right)}{\lim_{y\to 1}\left(\frac{y^5-1}{y-1}\right)}$$

$$= \frac{\lim_{y\to 1}\left(\frac{y^6-1}{y-1}\right)}{\lim_{y\to 1}\left(\frac{y^5-1^5}{y-1}\right)} = \frac{6\times 1^{(6-1)}}{5\times 1^{(5-1)}}$$

$$= \frac{6\times 1^5}{5\times 1^4} = \frac{6}{5}\left[\because \lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right) = na^{n-1}\right]$$

12. Evaluate $\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

Ans. $\lim_{x\to a}\frac{\left(\sqrt{a+2x}-\sqrt{3x}\right)}{\left(\sqrt{3a+x}-2\sqrt{x}\right)}$

$$= \lim_{x\to a}\frac{\left(\sqrt{a+2x}-\sqrt{3x}\right)}{\left(\sqrt{3a+x}-2\sqrt{x}\right)}\times\frac{\left(\sqrt{3a+x}+2\sqrt{x}\right)}{\left(\sqrt{3a+x}+2\sqrt{x}\right)}\times\frac{\left(\sqrt{a+2x}+\sqrt{3x}\right)}{\left(\sqrt{a+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x\to a}\frac{\left[(a+2x)-3x\right]\times\left(\sqrt{3a+x}+2\sqrt{x}\right)}{\left[(3a+x)-4x\right]\times\left(\sqrt{a+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x\to a}\frac{\left(\cancel{a-x}\right)\times\left(\sqrt{3a}+x+2\sqrt{x}\right)}{3\left(\cancel{a-x}\right)\times\left(\sqrt{a+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x\to a}\frac{\left(\sqrt{3a+x}+2\sqrt{x}\right)}{3\left[\sqrt{a+2x}+\sqrt{3x}\right]}$$

$$= \frac{\left(\sqrt{4a}+2\sqrt{a}\right)}{3\left(\sqrt{3a}+\sqrt{3a}\right)} = \frac{4\sqrt{a}}{6\sqrt{3}\sqrt{a}} = \frac{\sqrt{2}}{3\sqrt{3}}$$

13. Find the derivative of $f(x) = 1 + x + x^2 + - - + x^{50}$ at $x = 1$

Ans. $f(x) = 1 + x + x^2 + - - - + x^{50}$

$$f1(x) = \frac{d}{dx}\left(1 + x + x^2 + - - - + x^{50}\right)$$

$$= 0 + 1 + 2x + 3x^2 + - - + 50x^{49}$$

At $x = 1$

$$f^1(1) = 1 + 2 + 3 + - - 50$$

$$= \frac{\cancel{50}^{25}(50+1)}{\cancel{2}} = 25 \times 51 \begin{bmatrix} 1 + 2 + 3 - - + n \\ = \frac{n(n+1)}{2} \end{bmatrix}$$

$$= 1305$$

14. Find the derivative of $\sin^2 x$ with respect to x using product rule

Ans. let

$$y = \sin^2 x$$
$$y = \sin x \times \sin x$$

$$\frac{dy}{dx} = \frac{d}{dx}.\sin x \times \sin x$$

$$= \sin x \frac{d}{dx} \sin x + \sin x \frac{d}{dx} \sin x$$

$$= \sin x.\cos x + \sin x.\cos x$$

$$= 2\sin x.\cos x = \sin 2x$$

15. Find the derivative of $\frac{x^5 - \cos x}{\sin x}$ **with respect to** x

Ans. let

$$y = \frac{x^5 - \cos x}{\sin x}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x5 - \cos x}{\sin x}\right)$$

$$= \frac{\sin x \frac{d}{dx}\left(x^5 - \cos x\right) - \left(x^5 - \cos x\right)\frac{d}{dx}\sin x}{\sin^2 x}$$

$$= \frac{\sin x\left[5x^4 + \sin x\right] - \left(x^5 - \cos x\right)\cos x}{\sin^2 x}$$

$$= \frac{5x^4 \sin x + \sin^2 x - x^5 \cos x + \cos^2 x}{\sin^2 x}$$

$$= \frac{5x^4 \sin x - x^5 \cos x + 1}{\sin^2 x}$$

16. Find $\lim_{x \to 0} f(x)$.

when $f(x) = \begin{cases} \frac{|x|}{x}; x \neq 0 \\ 0; x = 0 \end{cases}$

Ans. $f(x) = \begin{cases} \frac{|x|}{x}; x \neq 0 \\ 0; x = 0 \end{cases}$

We know that $|x| = \begin{cases} x; x \geq 0 \\ -x; x < 0 \end{cases}$

$$\therefore f(x) = \begin{cases} \frac{x}{x} = 1. x > 0 \\ \frac{-x}{x} = -1. x < 0 \\ 0. x = 0 \end{cases}$$

L. H. L. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0} -1 = -1$

R. H. L. $\lim_{x \to 0^+} d(x) = \lim_{x \to 0} 1 = 1$

L. H. L. $\neq$ R. H. L $\therefore \lim_{x \to 0} f(x)$ does not exist

17. Find the derivative of the function $f(x) = 2x^2 + 3x - 5$ at $x = -1$. Also show that $f'(0) + 3f'(-1) = 0$

Ans. $f(x) = 2x^2 + 3x - 5$

$f^1(x) = \frac{d}{dx}(2x^2 + 3x - 5)$

$= 4x + 3$

At $x = -1$

$f^1(-1) 4 \times -1 + 3 = -4 + 3 = -1$

$f^1(0) = 4 \times 0 + 3 = 3$

$f^1(0) + 3f^1(-1) = 3 + 3 \times -1$

$= 3 - 3 = 0$ Hence proved

18. Evaluate $\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$

Ans. $\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$

$=\lim_{x\to 0}\frac{x[a+\cos x]}{xb\frac{\sin x}{x}}$

$\frac{\lim_{x\to 0}(a+\cos x)}{\lim_{x\to 0}b\frac{\sin x}{x}}$

$\frac{a+1}{b\times 1}=\frac{a+1}{b}\begin{bmatrix}\because \lim_{x\to 0}\cos x=1 \\ \lim_{x\to 0}\frac{sicx}{x}=1\end{bmatrix}$

19. Find derivative of $\tan x$ **by first principle**

Ans. $let\, f(x)=\tan x$

$f(x+h)=\tan(x+h)$

$f^{1}(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{\tan(x+h-\tan x)}{h}$

$=\lim_{h\to 0}\frac{\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}}{h}$

$$= \lim_{h\to 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{\cos(x+h)\cos x\, h}$$

$$= \lim_{h\to 0} \frac{\sin[x+h-x]}{\cos(x+h)\cos x\, h}$$

$$= \frac{\lim_{h\to 0} \frac{\sinh}{h}}{\lim_{h\to 0} \cos(x+h)\cos x} = \frac{1}{\cos(x+0)\cos x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

20. Evaluate $\lim_{x\to 1} \frac{x+x^2+x^3+---+x^n-n}{(x-1)}$

Ans. $\lim_{x\to 1} \frac{x+x^2+x^3+---+x^n-1}{(x-1)}$

$$= \lim_{x\to 1} \frac{(x-1)+(x^2-1)+(x^3-1+--+(x^n-1))}{(x-1)}$$

$$= \lim_{x\to 1} \frac{(x-1)[1+(x+1)+(x^2+x+1)+---+x^{n-1}+x^{n-2}+---+1]}{(x-1)}$$

$$= 1+2+3+---+n = \frac{n(n+1)}{2}$$

21. Evaluate $\underset{x\to 4}{Lt} \frac{|4-x|}{x-4}$ **(if it exist)**

Ans. $\lim_{x\to 4} \frac{|4-x|}{x-4}$

$L.H.L.\quad \lim_{x\to 4^-} \frac{-(4-x)}{x-4} = \lim_{x\to 4} \frac{-(4-x)}{-(4-x)} = 1$

$R.H.L.\quad \lim_{x\to 4^+} \frac{4-x}{x-4} = \lim_{x\to 4} \frac{-(x-4)}{(x-4)} = -1$

$L.H.L. \neq R.H.L.$

$\therefore \lim_{x\to 4} \frac{|4-x|}{x-4}$ does not exist

22. For what integers m and n does both

$\underset{x\to 4}{Lt} f(x)$ *and* $\underset{x\to 1}{Lt} f(x)$ exist it

$$f(x) = \begin{cases} mx^2 + n;\ x < 0 \\ nx + m;\ 0 \leq x \leq 1 \\ nx^3 + m;\ x > 1 \end{cases}$$

Ans. *for* $x = 0$

$L.H.L.\quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0} mx^2 + n$

$= n$

$R.H.L.\quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} nx + m$

$= m$

$\because \lim_{x\to 0} f(x)$ exist

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

$n = m$

For all real number $m = n$ $\lim_{x\to 0} f(x)$ exist

For $x = 1$

$L.H.L.\ \lim_{x\to1^-} f(x) = \lim_{x\to1} nx + m$

$= n + m$

$R.H.L.\ \lim_{x\to1} f(x) = \lim_{x\to1} nx^3 + m$

$= n + m$

$\therefore \lim_{x\to1} f(x) = \lim_{x\to1^-} f(x)$

$m + n = m + n$

$\therefore$ all integral values of $m + n \lim_{x\to1} f(x)$ exist

23. If $y = \sqrt{x+}\dfrac{1}{\sqrt{x}}$. prove that $2x\dfrac{dy}{dx+y} + y = 2\sqrt{x}$

Ans. $y = \sqrt{x} + \dfrac{1}{\sqrt{x}} = x^{\frac{1}{2}} + x^{\frac{-1}{2}}$

Differentiating w. r. t. x we gill

$$\frac{dy}{dx} = \frac{1}{2}x^{\frac{-1}{2}} + \left(\frac{-1}{2}\right)x^{\frac{-3}{2}}$$

$$= \frac{1}{2\sqrt{x}} - \frac{1}{2x^{\frac{3}{2}}}$$

$$2x\frac{dy}{dx} = \sqrt{x} - \frac{1}{\sqrt{x}}$$

$$2x\frac{dy}{dx} + y = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) + \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$$

$$2x\frac{dy}{dx}+y=2\sqrt{x}\text{ Hence proved}$$

24. Evaluate $\lim_{x\to\frac{\pi}{2}}\frac{1+\cos 2x}{(\pi-2x)^2}$

Ans. $\lim_{x\to\frac{\pi}{2}}\frac{1+\cos 2x}{(\pi-2x)^2}$

$$let\ \pi-2x=y$$

$$2x=\pi-y$$

$$x\to\frac{\pi}{2},y\to 0$$

$$\lim_{y\to 0}\frac{1+\cos(\pi-y)}{y^2}=\lim_{y\to 0}\frac{1-\cos y}{y^2}\frac{1}{2}$$

$$=\lim_{y\to 0}\frac{2\sin^2\frac{y}{2}}{4\times\frac{y^2}{4}}$$

$$=\lim_{y\to 0}\frac{1}{2}\times\frac{\sin^2\frac{y}{2}^2}{\left(\frac{y}{2}\right)^2}\quad=$$

$$=\frac{1}{2}\lim_{\frac{y}{2}\to 0}\left[\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right]^2$$

$$=\frac{1}{2}\times 1$$

25. Differentiate the function $y=\dfrac{(x+2)(3x-1)}{(2x+5)}$ **with respect to** x

Ans. $y=\dfrac{(x+2)(3x-1)}{(2x+5)}$

$$\frac{dy}{dx}=\frac{d}{dx}\frac{(x+2)(3x-1)}{(2x+5)}$$

$$=\frac{(2x+5)\frac{d}{dx}(x+2)(3x-1)-(x+2)(3x-1)\frac{d}{dx}(2x+5)}{(2x+5)^2}$$

$$=\frac{(2x+5)\left[(x+2)\frac{d}{dx}(3x-1)+(3x-1)\frac{d}{dx}(x+2)\right]-(x+2)(3x-1)[2+0]}{(2x+5)^2}$$

$$=\frac{(2x+5)[(x+2)\times 3+(3x-1)\times 1]-2\left[3x^2+6x-x-2\right]}{(2x+5)^2}$$

$$=\frac{(2x+5)[3x+6+3x-1]-6x^2-12x+2x+4}{(2x+5)^2}$$

$$=\frac{12x^2+30x+10x+25-6x^2-10x+4}{(2x+5)^2}$$

$$=\frac{6x^2+30x+29}{(2x+5)^2}$$

26. Find $\lim\limits_{x\to 5}|x|-5$

Ans. $L.H.S.\lim\limits_{x\to 5^-}f(x)$

$x=5-h$

$x\to 5, h\to 0$

$\lim_{h\to 0} f(5-h)$

$\lim_{h\to 0} |5-h|-5$

$=0$

$R.H.S. \lim_{x\to 5^+} f(x)$

$put\ x=5+h$

$x\to 5,\ h\to 0$

$\lim_{h\to 0} f(5+h)=\lim_{h\to 0} |5+h|-5$

$=0$

$R.H.S.=R.H.S.$

$\therefore \lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)$

$\therefore \lim_{x\to 5} f(x)\ exist$

$\therefore \lim_{x\to 5} f(x)=0$

27. Find $\lim_{x\to 0} f(x)\ and\ \lim_{x\to 1} f(x)\ where\ \ f(x)=\begin{Bmatrix} 2x+3; x\le 0 \\ 3(x+1); (x>0) \end{Bmatrix}$

Ans. $given\ f(x)=\begin{Bmatrix} 2x+3, x\le 0 \\ 3(x+1), x>0 \end{Bmatrix}$

$for\ x=0$

$L.H.S. \lim_{x\to 0^-} f(x)=\lim_{x\to 0} 2x+3=3$

$R.H.S. \lim_{x\to 0^+} f(x) = \lim_{x\to 0} 3(x+1) = 3$

$L.H.S. = R.H.S.$

$\therefore \lim_{x\to 0} f(x)$ *exist*

$\therefore \lim_{x\to 0} f(x) = 3$

for $x = 1$

$L.H.S.$

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1} 3(x+1)$

$= 3(1+1) = 6$

$R.H.S.$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} 3(x+1)$

$= 3(1+1) = 6$

$L.H.S. = R.H.S.$

$\therefore \lim_{x\to 1} f(x)$ *exist*

$\lim_{x\to 1} f(x) = 6$

28. Find derivative of $\sec x$ **by first principle**

Ans. *let* $f(x) = \sec x$

$f(x+h) = \sec(x+h)$

$$f^1(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sec(x+h) - \sec x}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{\cos(s+h)} - \frac{1}{\cos x}}{h}$$

$$= \lim_{h \to 0} \frac{\cos x - \cos(x+h)}{\cos(x+h)\cos xh}$$

$$= \lim_{h \to 0} \frac{-2\sin\left[\frac{2x+h}{2}\right]\sin\left[\frac{-h}{2}\right]}{\cos(x+h)\cos x.h}$$

$$= \lim_{h \to 0} \frac{2\sin\left[\frac{2x+h}{2}\right]\sin\frac{h}{2}}{\cos(x+h)\cos xh} \qquad [\sin(-\theta = -\sin\theta)]$$

$$= \lim_{h \to 0} \frac{2\sin\frac{h}{2}}{2\frac{h}{2}} \times \frac{\lim_{h \to 0} \sin\frac{(2x+h)}{2}}{\lim_{h \to 0} \cos(x+h)\cos x}$$

$$= 1 \times \frac{\sin\left(\frac{2x+0}{2}\right)}{\cos(x+0)\cos x} = \frac{\sin x}{\cos x \ \cos x}$$

$$= \tan x \ \sec x$$

29. Find derivative of $f(x) = \frac{4x+5\sin x}{3x+7\cos x}$

Ans. $f(x) = \frac{4x+5\sin x}{3x+7\cos x}$

$$f^1(x)=\frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x)-(4x+5\sin x)\times\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)(4+5\cos x)-(4x+5\sin x)(3-7\sin x)}{(3x+7\cos x)^2}$$

$$=\frac{12x+15x\cos x+28\cos x+35\cos^2 x-12x+28\sin x+15\sin x+35\sin^2 x}{(3x+7\cos x)^2}$$

$$=\frac{15x\cos x+35\left[\sin^2 x+\cos^2 x\right]+28+\cos x+43\sin x}{(3x+7\cos x)^2}$$

$$=\frac{15x\cos x+35+28\cos x+45\sin x}{(3x+7\cos x)^2}$$

30. Find derivative of $\frac{x^n-a^n}{x-a}$

Ans. $\frac{d}{dx}\ \frac{x^n-a^n}{x-a}$

$$=\frac{(x-a)\frac{d}{dx}(x^n-a^n)-(x^n-a^n)\frac{d}{dx}(x-a)}{(x-a)^2}$$

$$=\frac{(x-a)\left[nx^{n-1}-0\right]-(x^n-a^n)[1-0]}{(x-a)^2}$$

$$=\frac{nx^{n-1}(x-a)-x^n+a^n}{(x-a)^2}$$

$$=\frac{nx^n-nax^{n-1}-x^n+a^n}{(x-a)^2}=\frac{x^n(n-1)-nax^{n-1}+a^n}{(x-a)^2}$$

CBSE Class 12 Mathematics

Important Questions

Chapter 13

Limits and Derivatives

6 Marks Questions

1. Differentiate tan x from first principle.

Ans. $f(x) = \tan x$

$f(x+h) = \tan(x+h)$

$f^1(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h}$

$= \lim_{h \to 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h}$

$= \lim_{h \to 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos(x+h)\cos x}$

$= \lim_{h \to 0} \frac{\sin[x+h-x]}{h\cos(x+h)\cos x} \left[\because \sin(A-B) = \sin A\cos B - \cos A\sin B\right]$

$= \lim_{h \to 0} \frac{\sin h}{h\cos(x+h)\cos x}$

$$= \frac{\lim_{h\to 0} \frac{\sin h}{h}}{\lim_{h\to 0} \cos(x+h)\cos x} = \frac{1}{\cos x.\cos x} \left[\because \lim_{h\to 0} \frac{\sin h}{h} = 1\right]$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

2. Differentiate $(x+4)^6$ From first principle.

Ans. let $f(x) = (x+4)^6$

$$f(x+h) = (x+h+4)^6$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h+4)^6 - (x+4)^6}{h}$$

$$= \lim_{(x+h+4)\to(x+4)} \frac{(x+h+4)^6 - (x+4)^6}{(x+h+4) - (x-4)}$$

$$= 6(x+4)^{(6-1)} \left[\because \lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

$$= 6(x+4)^5$$

3. Find derivative of cosec x by first principle

Ans. proof let $f(x) = \operatorname{cosec} x$

$$\text{By def.} f(x) = \underset{h\to 0}{Lt} \frac{f(x+h) - f(h)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{\text{cosec}(x+h) - \text{cosec}\, x}{h}$$

$$= \underset{h\to 0}{Lt} \frac{\frac{1}{\sin(x+h)} - \frac{1}{\sin x}}{h} = \underset{h\to 0}{Lt} \frac{\sin x - \sin(x+h)}{h\sin(x+h)\sin x}$$

$$= \underset{h\to 0}{Lt} \frac{2\cos\frac{x+x+h}{2}\sin\frac{x-x+h}{2}}{h\sin(x+h)\sin x}$$

$$= \underset{h\to 0}{Lt} \frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(-\frac{h}{2}\right)}{h\sin(x+h)\sin x}$$

$$= \frac{\underset{\frac{h}{2}\to 0}{Lt}\cos\left(x+\frac{h}{2}\right)}{\cos x.\underset{h\to 0}{L} + \sin(x+h)} \,.\, \underset{\frac{h}{2}\to 0}{Lt} \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= -\frac{\cos x}{\sin x.\sin x}.1 = -\cos ecx\cot x$$

4. Find the derivatives of the following fuchsias:

$(i)\ \left(x - \frac{1}{x}\right)^3 \quad (ii)\ \frac{(3x+1)\left(2\sqrt{x-1}\right)}{\sqrt{x}}$

Ans. (i) let $f(x) = \left(x - \frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3x + \frac{1}{x}\left(x - \frac{1}{x}\right)$

$= x^3 - x^{-3} - 3x + 3x^{-1}.d.ff\ wr.t4, we$ get

$\text{f}(x) = 3\times x^2 - (-3)x^{-4} - 3\times 1 + 3\times(-1)x^{-2}$

$= 3x^2 + \frac{3}{x^4} - 3 - \frac{3}{x^2}.$

(ii) let $f(x) = \frac{(3x+1)(2\sqrt{x}-1)}{\sqrt{x}} = \frac{6x^{\frac{3}{2}} - 3x + 2\sqrt{x} - 1}{\sqrt{x}}$

$= 6x - 3x^{\frac{1}{2}} + 2 - x^{-\frac{1}{2}}, d : ff$ w.r.t. x.weget

$f(x) = 6\times1 - 3\times\frac{1}{2}\times x^{-\frac{1}{2}} + 0 - \left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$

$= 6 - \frac{3}{2\sqrt{x}} + \frac{1}{2.x^{\frac{3}{2}}}.$

5. If $f(x) = \begin{Bmatrix} |x| + a; x < 0 \\ 0; x = 0 \\ |x| - a; x > 0 \end{Bmatrix}$ **for what Values of 'a' does** $\lim_{x\to0} f(x)$ **exist**

Ans.given $f(x) = \begin{Bmatrix} |x| + a; x < 0 \\ 0; x = 0 \\ |x| - a; x > 0 \end{Bmatrix}$

a=0

L.H.L. $\lim_{x\to0^-} f(x) = \lim_{x\to0^-} |x| + a$

$= \lim_{x\to0} -x + a = a$

R.H.L. $\lim_{x\to0^+} f(x) = \lim_{x\to0} |x| - a$

$\because \lim_{x\to0} f(x)$ exist

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

$a = -a$

$2a = 0$

$a = 0$

At $a = 0 \lim_{x\to 0} f(x)$ exist

6. Find the derivative of sin $(x+1)$. with respect to x, from first principle.

Ans. let $f(x) = \sin(x+1)$

$f(x+h) = \sin(x+h+1)$

$$f^1(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h+1) - \sin(x+1)}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left[\frac{x+h+1+x+1}{2}\right]\sin\left[\frac{x+h+1-x-1}{2}\right]}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left[x+1+\frac{h}{2}\right]\sin\frac{h}{2}}{h}$$

$$= \lim_{h\to 0} \cancel{2}\cos\left(x+1+\frac{h}{2}\right) \times \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\cancel{2}\frac{h}{2}}$$

$$= \cos(x+1) \times 1 = \cos(x+1)$$

7. Find the derivative of $\sin x+\cos x$ **from first principle**

Ans. $let\ f(x)=\sin x+\cos x$

$$f(x+h)=\sin(x+h)+\cos(x+h)$$

$$f^1(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left[\sin(x+h)\cos(x+h)\right]-\left[\sin x+\cos x\right]}{h}$$

$$=\lim_{h\to 0}\frac{\left[\sin(x+h)-\sin x\right]+\left[\cos(x+h)-\cos x\right]}{h}$$

$$=\lim_{h\to 0}\frac{2\cos\left[\frac{x+h+x}{2}\right]\sin\frac{(x+h-x)}{2}-2\sin\frac{(x+h+x)}{2}\times\sin\left[\frac{x+h-x}{2}\right]}{h}$$

$$=\lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}+\lim_{h\to 0}\frac{-2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}$$

$$=\lim_{h\to 0}\not{2}\cos\left(x+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{\not{2}\frac{h}{2}}+\lim_{h\to 0}-\not{2}\sin\left(x+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{\not{2}\frac{h}{2}}$$

$$=\cos(x+0)\times 1-\sin(0+x)\times 1$$

$$=\cos x-\sin x$$

8. Find derivative of

$$(i)\frac{x\sin x}{1+\cos x}(ii)(ax+b)(x+d)^2$$

Ans. (i) $\frac{d}{dx}\frac{x\sin x}{1+\cos x}$

$$=\frac{(1+\cos x)\frac{d}{dx}(x\sin x)-x\sin x\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)\left[x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(x)\right]-x\sin x[0-\sin x]}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)[x\cos x+\sin x\times 1]+x\sin^2 x}{(1+\cos x)^2}$$

$$=\frac{x\cos x+x\cos^2 x+\sin x+\sin x\cos x+x\sin^2 x}{(1+\cos x)^2}$$

$$=\frac{x(\cos^2 x+\sin^2 x)+x\cos x+\sin x+\sin x\cos x}{(1+\cos x)^2}$$

$$=\frac{x+x\cos x+\sin x+\sin x\cos x}{(1+\cos x)^2}$$

(ii) $\frac{d}{dx}(ax+b)(cx+d)^2$

$$=(ax+b)\frac{d}{dx}(cx+d)^2+(cx+d)^2\frac{d}{dx}(ax+b)$$

$$=(ax+b)2(cx+d)\frac{d}{dx}(cx+d)+(cx+d)^2\times a$$

$$=2(ax+b)(cx+d)\times c+a(cx+d)^2$$

$$=(cx+d)\left[2c(ax+b)+a(cx+d)\right]$$

$$=(cx+d)[2acx+2bc+acx+ad]$$

$$=(cx+d)[3acx+2abc+ad]$$

9. Evaluate $\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$

Ans. $\lim_{h\to 0}\frac{(a+h)^2\sin(a+x)-a^2\sin a}{h}$

$$=\lim_{h\to 0}\frac{(a^2+2ah+h^2)\sin(a+h)-a^2\sin a}{h}$$

$$=\lim_{h\to 0}\frac{a^2\sin(a+h)+2ah\sin(a+h)+h^2\sin(a+h)-a^2\sin a}{h}$$

$$=\lim_{h\to 0}\frac{a^2[\sin(a+h)-\sin a]+2ah\sin(a+h)+h^2\sin(a+h)}{h}$$

$$=\lim_{h\to 0}\frac{a^2 2\cos\left[\frac{2a+h}{2}\right]\sin\frac{h}{2}}{2\frac{h}{2}}+\lim_{h\to 0}2a\sin(a+h)+\lim_{h\to 0}h\sin(a+h)$$

$$=a^2\cos\left[\frac{2a+0}{2}\right]\times 1+2a\sin[a+0]+0\times\sin a$$

$$=a^2\cos a+2a\sin a$$

10. Differentiate

$(i)\left(\frac{a}{x^4}\right)-\frac{b}{x^2}+\cos x\quad(ii)(x+\cos x)(x-\tan x)$

Ans. (i) $\frac{d}{dx}\left[\frac{a}{x^4}-\frac{b}{x^2}+\cos x\right]$

$=\frac{d}{dx}ax^{-4}-\frac{d}{dx}bx^{-2}+\frac{d}{dx}\cos x$

$=a\left(-4x^{-5}\right)-b\left(-2x^{-3}\right)-\sin x$

$=\frac{-4a}{x^5}+\frac{2b}{x^3}-\sin x$

(ii) $\frac{d}{dx}(x+\cos x)(x-\tan x)$

$=(x+\cos x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\cos x)$

$=(x+\cos x)\left(1-\sec^2 x\right)+(x-\tan x)(1-\sin x)$

$=x-x\sec^2 x+\cos x-\cos x\sec^2 x+x-x\sin x-tacx+\tan x\sin x in x$

$=2x-x\sec^2 x+\cancel{\cos x}-\cancel{\cos x}-x\sin x-\tan x+\tan x\sin x$

$=2x-x\sec^2 x-x\sin x-\tan x+\tan x s$

Limits & Derivatives

Evaluate :

1. $\lim_{x\to 3}\frac{x^2-9}{x-3}$

2. $\lim_{x\to \frac{1}{2}}\frac{4x^2-1}{2x-1}$

3. $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$

4. $\lim_{x\to 0}\frac{(x+2)^{\frac{1}{3}}-2^{\frac{1}{3}}}{x}$

5. $\lim_{x\to 1}\frac{(1+x)^6-1}{(1+x)^2-1}$

6. $\lim_{x\to a}\frac{(2+x)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{x-a}$

7. $\lim_{x\to 1}\frac{x^4-\sqrt{x}}{\sqrt{x}-1}$

8. $\lim_{x\to 2}\frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$

9. $\lim_{x\to\sqrt{2}}\frac{x^4-4}{x^2+3\sqrt{2}x-8}$

10. $\lim_{x\to 1}\frac{x^7-2x^5+1}{x^3-3x^2+2}$

11. $\lim_{x\to 0}\frac{\sqrt{1+x^3}-\sqrt{1-x^3}}{x^2}$

12. $\lim_{x\to -3}\frac{x^3+27}{x^5+243}$

13. $\lim_{x\to \frac{1}{2}}\left(\frac{8x-3}{2x-1}-\frac{4x^2+1}{4x^2-1}\right)$

14. Find 'n', if $\lim_{x\to 2}\frac{x^n-2^n}{x-2}=80$, $n\in \mathbf{N}$

15. $\lim_{x\to a}\frac{\sin 3x}{\sin 7x}$

16. $\lim_{x\to 0}\frac{\sin^2 2x}{\sin^2 4x}$

17. $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$

18. $\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3}$

19. $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}$

20. $\lim_{x\to \frac{\pi}{3}}\frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)}$

21. $\lim_{x\to \frac{\pi}{4}}\frac{\sin x-\cos x}{x-\frac{\pi}{4}}$

22. $\lim_{x\to \frac{\pi}{6}}\frac{\sqrt{3}\sin x-\cos x}{x-\frac{\pi}{6}}$

23. $\lim_{x\to 0}\frac{\sin 2x+3x}{2x+\tan 3x}$

24. $\lim_{x\to a}\frac{\sin x-\sin a}{\sqrt{x}-\sqrt{a}}$

25. $\lim_{x\to \frac{\pi}{6}}\frac{\cot^2 x-3}{\operatorname{cosec} x-2}$

26. $\lim_{x\to 0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$

27. $\lim_{x \to 0} \frac{\sin x - 2\sin 3x + \sin 5x}{x}$

28. If $\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim_{x \to k} \frac{x^3 - k^3}{x^2 - k^2}$, then find the value of k.

Differentiate each of the functions w. r. to x in Exercises 29 to 42.

29. $\frac{x^4 + x^3 + x^2 + 1}{x}$

30. $\left(x + \frac{1}{x}\right)^3$

31. $(3x + 5)(1 + \tan x)$

32. $(\sec x - 1)(\sec x + 1)$

33. $\frac{3x + 4}{5x^2 - 7x + 9}$

34. $\frac{x^5 - \cos x}{\sin x}$

35. $\frac{x^2 \cos \frac{\pi}{4}}{\sin x}$

36. $(ax^2 + \cot x)(p + q \cos x)$

37. $\frac{a + b \sin x}{c + d \cos x}$

38. $(\sin x + \cos x)^2$

39. $(2x - 7)^2 (3x + 5)^3$

40. $x^2 \sin x + \cos 2x$

41. $\sin^3 x \cos^3 x$

42. $\frac{1}{ax^2 + bx + c}$

Differentiate each of the functions with respect to 'x' in Exercises 43 to 46 using first principle.

43. $\cos(x^2 + 1)$

44. $\frac{ax + b}{cx + d}$

45. $x^{\frac{2}{3}}$

46. $x \cos x$

Evaluate each of the following limits in Exercises 47 to 53.

47. $\lim_{y \to 0} \frac{(x + y)\sec(x + y) - x\sec x}{y}$

48. $\lim_{x\to 0} \frac{(\sin(\alpha+\beta)\,x + \sin(\alpha-\beta)x + \sin 2\alpha\, x)}{\cos 2\beta x - \cos 2\alpha x} \cdot x$

49. $\lim_{x\to \frac{\pi}{4}} \frac{\tan^3 x - \tan x}{\cos\left(x+\frac{\pi}{4}\right)}$ 50. $\lim_{x\to \pi} \frac{1-\sin\frac{x}{2}}{\cos\frac{x}{2}\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)}$

51. Show that $\lim_{x\to 4} \frac{|x-4|}{x-4}$ does not exists

52. Let $f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x} & \text{when } x \neq \frac{\pi}{2} \\ 3 & x = \frac{\pi}{2} \end{cases}$ and if $\lim_{x\to\frac{\pi}{2}} f(x) = f(\frac{\pi}{2})$,

find the value of k.

53. Let $f(x) = \begin{cases} x+2 & x \le -1 \\ cx^2 & x > -1 \end{cases}$, find '$c$' if $\lim_{x\to -1} f(x)$ exists.

Objective Type Questions

Choose the correct answer out of 4 options given against each Exercise 54 to 76 (M.C.Q).

54. $\lim_{x\to\pi} \frac{\sin x}{x-\pi}$ is

(A) 1   (B) 2   (C) –1   (D) –2

55. $\lim_{x\to 0} \frac{x^2 \cos x}{1-\cos x}$ is

(A) 2   (B) $\frac{3}{2}$ (C) $\frac{-3}{2}$ (D) 1

56. $\lim_{x\to 0}\frac{(1+x)^n-1}{x}$ is

(A) n (B) 1 (C) $-n$ (D) 0

57. $\lim_{x\to 1}\frac{x^m-1}{x^n-1}$ is

(A) 1 (B) $\frac{m}{n}$ (C) $-\frac{m}{n}$ (D) $\frac{m^2}{n^2}$

58. $\lim_{x\to 0}\frac{1-\cos 4\theta}{1-\cos 6\theta}$ is

(A) $\frac{4}{9}$ (B) $\frac{1}{2}$ (C) $\frac{-1}{2}$ (D) -1

59. $\lim_{x\to 0}\frac{\operatorname{cosec}x-\cot x}{x}$ is

(A) $\frac{-1}{2}$ (B) 1 (C) $\frac{1}{2}$ (D) 1

60. $\lim_{x\to 0}\frac{\sin x}{\sqrt{x+1}-\sqrt{1-x}}$ is

(A) 2 (B) 0 (C) 1 (D) -1

61. $\lim_{x\to \frac{\pi}{4}}\frac{\sec^2 x-2}{\tan x-1}$ is

(A) 3 (B) 1 (C) 0 (D) $\sqrt{2}$

62. $\lim_{x\to 1}\frac{(\sqrt{x}-1)(2x-3)}{2x^2+x-3}$ is

(A) $\frac{1}{10}$ (B) $\frac{-1}{10}$ (C) 1 (D) None of these

63. If $f(x)=\begin{cases}\frac{\sin[x]}{[x]}, & [x]\neq 0\\ 0 & ,[x]=0\end{cases}$, where [.] denotes the greatest integer function,

then $\lim_{x\to 0} f(x)$ is equal to

(A) 1 (B) 0 (C) –1 (D) None of these

64. $\lim_{x\to 0}\frac{|\sin x|}{x}$ is

(A) 1 (B) –1 (C) does not exist (D) None of these

65. Let $f(x)=\begin{cases}x^2-1, 0<x<2\\ 2x+3, 2\le x<3\end{cases}$, the quadratic equation whose roots are $\lim_{x\to 2^-} f(x)$ and

$\lim_{x\to 2^+} f(x)$ is

(A) $x^2-6x+9=0$ (B) $x^2-7x+8=0$

(C) $x^2-14x+49=0$ (D) $x^2-10x+21=0$

66. $\lim_{x\to 0}\frac{\tan 2x - x}{3x-\sin x}$ is

(A) 2 (B) $\frac{1}{2}$ (C) $\frac{-1}{2}$ (D) $\frac{1}{4}$

67. Let $f(x)=x-[x]; \in \mathbf{R}$, then $f'\left(\frac{1}{2}\right)$ is

(A) $\frac{3}{2}$ (B) 1 (C) 0 (D) –1

68. If $y=\sqrt{x}+\frac{1}{\sqrt{x}}$, then $\frac{dy}{dx}$ at $x=1$ is

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) 0

69. If $f(x) = \frac{x-4}{2\sqrt{x}}$, then $f'(1)$ is

(A) $\frac{5}{4}$ (B) $\frac{4}{5}$ (C) 1 (D) 0

70. If $y = \frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}$, then $\frac{dy}{dx}$ is

(A) $\frac{-4x}{(x^2-1)^2}$ (B) $\frac{-4x}{x^2-1}$ (C) $\frac{1-x^2}{4x}$ (D) $\frac{4x}{x^2-1}$

71. If $y = \frac{\sin x + \cos x}{\sin x - \cos x}$, then $\frac{dy}{dx}$ at $x = 0$ is

(A) -2 (B) 0 (C) $\frac{1}{2}$ (D) does not exist

72. If $y = \frac{\sin(x+9)}{\cos x}$ then $\frac{dy}{dx}$ at $x = 0$ is

(A) $\cos 9$ (B) $\sin 9$ (C) 0 (D) 1

73. If $f(x) = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^{100}}{100}$, then $f'(1)$ is equal to

(A) $\frac{1}{100}$ (B) 100 (C) does not exist (D) 0

74. If $f(x) = \frac{x^n - a^n}{x - a}$ for some constant 'a', then $f'(a)$ is

(A) 1 (B) 0 (C) does not exist (D) $\frac{1}{2}$

75. If $f(x) = x^{100} + x^{99} + \ldots + x + 1$, then $f'(1)$ is equal to

(A) 5050 (B) 5049 (C) 5051 (D) 50051

76. If $f(x) = 1 - x + x^2 - x^3 \ldots - x^{99} + x^{100}$, then $f'(1)$ is euqal to

(A) 150 (B) -50 (C) -150 (D) 50