

# PRINCIPLE OF MATHEMATICAL INDUCTION

## 4.1 Overview

Mathematical induction is one of the techniques which can be used to prove variety of mathematical statements which are formulated in terms of  $n$ , where  $n$  is a positive integer.

### 4.1.1 The principle of mathematical induction

Let  $P(n)$  be a given statement involving the natural number  $n$  such that

- (i) The statement is true for  $n = 1$ , i.e.,  $P(1)$  is true (or true for any fixed natural number) and
- (ii) If the statement is true for  $n = k$  (where  $k$  is a particular but arbitrary natural number), then the statement is also true for  $n = k + 1$ , i.e, truth of  $P(k)$  implies the truth of  $P(k + 1)$ . Then  $P(n)$  is true for all natural numbers  $n$ .

## 4.2 Solved Examples

### Short Answer Type

Prove statements in Examples 1 to 5, by using the Principle of Mathematical Induction for all  $n \in \mathbf{N}$ , that :

**Example 1**  $1 + 3 + 5 + \dots + (2n - 1) = n^2$

**Solution** Let the given statement  $P(n)$  be defined as  $P(n) : 1 + 3 + 5 + \dots + (2n - 1) = n^2$ , for  $n \in \mathbf{N}$ . Note that  $P(1)$  is true, since

$$P(1) : 1 = 1^2$$

Assume that  $P(k)$  is true for some  $k \in \mathbf{N}$ , i.e.,

$$P(k) : 1 + 3 + 5 + \dots + (2k - 1) = k^2$$

Now, to prove that  $P(k + 1)$  is true, we have

$$\begin{aligned} 1 + 3 + 5 + \dots + (2k - 1) + (2k + 1) \\ &= k^2 + (2k + 1) && \text{(Why?)} \\ &= k^2 + 2k + 1 = (k + 1)^2 \end{aligned}$$

Thus,  $P(k + 1)$  is true, whenever  $P(k)$  is true.

Hence, by the Principle of Mathematical Induction,  $P(n)$  is true for all  $n \in \mathbf{N}$ .

**Example 2**  $\sum_{t=1}^{n-1} t(t+1) = \frac{n(n-1)(n+1)}{3}$ , for all natural numbers  $n \geq 2$ .

**Solution** Let the given statement  $P(n)$ , be given as

$$P(n) : \sum_{t=1}^{n-1} t(t+1) = \frac{n(n-1)(n+1)}{3}, \text{ for all natural numbers } n \geq 2.$$

We observe that

$$\begin{aligned} P(2) : \sum_{t=1}^{2-1} t(t+1) &= \sum_{t=1}^1 t(t+1) = 1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3} \\ &= \frac{2 \cdot (2-1) \cdot (2+1)}{3} \end{aligned}$$

Thus,  $P(n)$  is true for  $n = 2$ .

Assume that  $P(n)$  is true for  $n = k \in \mathbf{N}$ .

i.e., 
$$P(k) : \sum_{t=1}^{k-1} t(t+1) = \frac{k(k-1)(k+1)}{3}$$

To prove that  $P(k + 1)$  is true, we have

$$\begin{aligned} \sum_{t=1}^{(k+1)-1} t(t+1) &= \sum_{t=1}^k t(t+1) \\ &= \sum_{t=1}^{k-1} t(t+1) + k(k+1) = \frac{k(k-1)(k+1)}{3} + k(k+1) \\ &= k(k+1) \left[ \frac{k-1+3}{3} \right] = \frac{k(k+1)(k+2)}{3} \\ &= \frac{(k+1)((k+1)-1)((k+1)+1)}{3} \end{aligned}$$

Thus,  $P(k + 1)$  is true, whenever  $P(k)$  is true.

Hence, by the Principle of Mathematical Induction,  $P(n)$  is true for all natural numbers  $n \geq 2$ .

**Example 3**  $\left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$ , for all natural numbers,  $n \geq 2$ .

**Solution** Let the given statement be  $P(n)$ , i.e.,

$$P(n) : \left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}, \text{ for all natural numbers, } n \geq 2$$

We, observe that  $P(2)$  is true, since

$$\left(1 - \frac{1}{2^2}\right) = 1 - \frac{1}{4} = \frac{4-1}{4} = \frac{3}{4} = \frac{2+1}{2 \times 2}$$

Assume that  $P(n)$  is true for some  $k \in \mathbf{N}$ , i.e.,

$$P(k) : \left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{k^2}\right) = \frac{k+1}{2k}$$

Now, to prove that  $P(k+1)$  is true, we have

$$\begin{aligned} & \left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{k^2}\right) \cdot \left(1 - \frac{1}{(k+1)^2}\right) \\ &= \frac{k+1}{2k} \left(1 - \frac{1}{(k+1)^2}\right) = \frac{k^2 + 2k}{2k(k+1)} = \frac{(k+1)+1}{2(k+1)} \end{aligned}$$

Thus,  $P(k+1)$  is true, whenever  $P(k)$  is true.

Hence, by the Principle of Mathematical Induction,  $P(n)$  is true for all natural numbers,  $n \geq 2$ .

**Example 4**  $2^{2n} - 1$  is divisible by 3.

**Solution** Let the statement  $P(n)$  given as

$P(n) : 2^{2n} - 1$  is divisible by 3, for every natural number  $n$ .

We observe that  $P(1)$  is true, since

$$2^2 - 1 = 4 - 1 = 3.1 \text{ is divisible by } 3.$$

Assume that  $P(n)$  is true for some natural number  $k$ , i.e.,

$P(k) : 2^{2k} - 1$  is divisible by 3, i.e.,  $2^{2k} - 1 = 3q$ , where  $q \in \mathbf{N}$

Now, to prove that  $P(k+1)$  is true, we have

$$\begin{aligned} P(k+1) : 2^{2(k+1)} - 1 &= 2^{2k+2} - 1 = 2^{2k} \cdot 2^2 - 1 \\ &= 2^{2k} \cdot 4 - 1 = 3 \cdot 2^{2k} + (2^{2k} - 1) \end{aligned}$$

$$\begin{aligned}
 &= 3 \cdot 2^{2k} + 3q \\
 &= 3(2^{2k} + q) = 3m, \text{ where } m \in \mathbf{N}
 \end{aligned}$$

Thus  $P(k + 1)$  is true, whenever  $P(k)$  is true.

Hence, by the Principle of Mathematical Induction  $P(n)$  is true for all natural numbers  $n$ .

**Example 5**  $2n + 1 < 2^n$ , for all natural numbers  $n \geq 3$ .

**Solution** Let  $P(n)$  be the given statement, i.e.,  $P(n) : (2n + 1) < 2^n$  for all natural numbers,  $n \geq 3$ . We observe that  $P(3)$  is true, since

$$2 \cdot 3 + 1 = 7 < 8 = 2^3$$

Assume that  $P(n)$  is true for some natural number  $k$ , i.e.,  $2k + 1 < 2^k$

To prove  $P(k + 1)$  is true, we have to show that  $2(k + 1) + 1 < 2^{k+1}$ . Now, we have

$$\begin{aligned}
 2(k + 1) + 1 &= 2k + 3 \\
 &= 2k + 1 + 2 < 2^k + 2 < 2^k + 2 \cdot 2 = 2^{k+1}.
 \end{aligned}$$

Thus  $P(k + 1)$  is true, whenever  $P(k)$  is true.

Hence, by the Principle of Mathematical Induction  $P(n)$  is true for all natural numbers,  $n \geq 3$ .

### Long Answer Type

**Example 6** Define the sequence  $a_1, a_2, a_3, \dots$  as follows :

$a_1 = 2, a_n = 5 a_{n-1}$ , for all natural numbers  $n \geq 2$ .

- Write the first four terms of the sequence.
- Use the Principle of Mathematical Induction to show that the terms of the sequence satisfy the formula  $a_n = 2 \cdot 5^{n-1}$  for all natural numbers.

### Solution

- We have  $a_1 = 2$   
 $a_2 = 5a_{2-1} = 5a_1 = 5 \cdot 2 = 10$   
 $a_3 = 5a_{3-1} = 5a_2 = 5 \cdot 10 = 50$   
 $a_4 = 5a_{4-1} = 5a_3 = 5 \cdot 50 = 250$
- Let  $P(n)$  be the statement, i.e.,

$P(n) : a_n = 2 \cdot 5^{n-1}$  for all natural numbers. We observe that  $P(1)$  is true

Assume that  $P(n)$  is true for some natural number  $k$ , i.e.,  $P(k) : a_k = 2 \cdot 5^{k-1}$ .

Now to prove that  $P(k + 1)$  is true, we have

$$\begin{aligned} P(k + 1) : a_{k+1} &= 5 \cdot a_k = 5 \cdot (2 \cdot 5^{k-1}) \\ &= 2 \cdot 5^k = 2 \cdot 5^{(k+1)-1} \end{aligned}$$

Thus  $P(k + 1)$  is true whenever  $P(k)$  is true.

Hence, by the Principle of Mathematical Induction,  $P(n)$  is true for all natural numbers.

**Example 7** The distributive law from algebra says that for all real numbers  $c, a_1$  and  $a_2$ , we have  $c(a_1 + a_2) = ca_1 + ca_2$ .

Use this law and mathematical induction to prove that, for all natural numbers,  $n \geq 2$ , if  $c, a_1, a_2, \dots, a_n$  are any real numbers, then

$$c(a_1 + a_2 + \dots + a_n) = ca_1 + ca_2 + \dots + ca_n$$

**Solution** Let  $P(n)$  be the given statement, i.e.,

$P(n) : c(a_1 + a_2 + \dots + a_n) = ca_1 + ca_2 + \dots + ca_n$  for all natural numbers  $n \geq 2$ , for  $c, a_1, a_2, \dots, a_n \in \mathbf{R}$ .

We observe that  $P(2)$  is true since

$$c(a_1 + a_2) = ca_1 + ca_2 \quad (\text{by distributive law})$$

Assume that  $P(n)$  is true for some natural number  $k$ , where  $k > 2$ , i.e.,

$$P(k) : c(a_1 + a_2 + \dots + a_k) = ca_1 + ca_2 + \dots + ca_k$$

Now to prove  $P(k + 1)$  is true, we have

$$\begin{aligned} P(k + 1) : c(a_1 + a_2 + \dots + a_k + a_{k+1}) \\ &= c((a_1 + a_2 + \dots + a_k) + a_{k+1}) \\ &= c(a_1 + a_2 + \dots + a_k) + ca_{k+1} \quad (\text{by distributive law}) \\ &= ca_1 + ca_2 + \dots + ca_k + ca_{k+1} \end{aligned}$$

Thus  $P(k + 1)$  is true, whenever  $P(k)$  is true.

Hence, by the principle of Mathematical Induction,  $P(n)$  is true for all natural numbers  $n \geq 2$ .

**Example 8** Prove by induction that for all natural number  $n$   
 $\sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin(\alpha + (n - 1)\beta)$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

**Solution** Consider  $P(n) : \sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin(\alpha + (n - 1)\beta)$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}, \text{ for all natural number } n.$$

We observe that

P (1) is true, since

$$P(1) : \sin \alpha = \frac{\sin(\alpha+0)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

Assume that P(n) is true for some natural numbers k, i.e.,

P (k) :  $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \dots + \sin (\alpha + (k - 1)\beta)$

$$= \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Now, to prove that P (k + 1) is true, we have

P (k + 1) :  $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \dots + \sin (\alpha + (k - 1)\beta) + \sin (\alpha + k\beta)$

$$= \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} + \sin(\alpha + k\beta)$$

$$= \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\frac{k\beta}{2} + \sin(\alpha + k\beta)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta - \frac{\beta}{2}\right) + \cos\left(\alpha + k\beta - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$\begin{aligned}
 &= \frac{\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta + \frac{\beta}{2}\right)}{2 \sin \frac{\beta}{2}} \\
 &= \frac{\sin\left(\alpha + \frac{k\beta}{2}\right) \sin\left(\frac{k\beta + \beta}{2}\right)}{\sin \frac{\beta}{2}} \\
 &= \frac{\sin\left(\alpha + \frac{k\beta}{2}\right) \sin(k+1)\left(\frac{\beta}{2}\right)}{\sin \frac{\beta}{2}}
 \end{aligned}$$

Thus  $P(k+1)$  is true whenever  $P(k)$  is true.

Hence, by the Principle of Mathematical Induction  $P(n)$  is true for all natural number  $n$ .

**Example 9** Prove by the Principle of Mathematical Induction that

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = (n+1)! - 1 \text{ for all natural numbers } n.$$

**Solution** Let  $P(n)$  be the given statement, that is,

$$P(n) : 1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = (n+1)! - 1 \text{ for all natural numbers } n.$$

Note that  $P(1)$  is true, since

$$P(1) : 1 \times 1! = 1 = 2 - 1 = 2! - 1.$$

Assume that  $P(n)$  is true for some natural number  $k$ , i.e.,

$$P(k) : 1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + k \times k! = (k+1)! - 1$$

To prove  $P(k+1)$  is true, we have

$$\begin{aligned}
 P(k+1) &: 1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + k \times k! + (k+1) \times (k+1)! \\
 &= (k+1)! - 1 + (k+1)! \times (k+1) \\
 &= (k+1+1) (k+1)! - 1 \\
 &= (k+2) (k+1)! - 1 = ((k+2)! - 1
 \end{aligned}$$

Thus  $P(k+1)$  is true, whenever  $P(k)$  is true. Therefore, by the Principle of Mathematical Induction,  $P(n)$  is true for all natural number  $n$ .

**Example 10** Show by the Principle of Mathematical Induction that the sum  $S_n$  of the  $n$  term of the series  $1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 \dots$  is given by

$$S_n = \begin{cases} \frac{n(n+1)^2}{2}, & \text{if } n \text{ is even} \\ \frac{n^2(n+1)}{2}, & \text{if } n \text{ is odd} \end{cases}$$

**Solution** Here  $P(n) : S_n = \begin{cases} \frac{n(n+1)^2}{2}, & \text{when } n \text{ is even} \\ \frac{n^2(n+1)}{2}, & \text{when } n \text{ is odd} \end{cases}$

Also, note that any term  $T_n$  of the series is given by

$$T_n = \begin{cases} n^2 & \text{if } n \text{ is odd} \\ 2n^2 & \text{if } n \text{ is even} \end{cases}$$

We observe that  $P(1)$  is true since

$$P(1) : S_1 = 1^2 = 1 = \frac{1 \cdot 2}{2} = \frac{1^2 \cdot (1+1)}{2}$$

Assume that  $P(k)$  is true for some natural number  $k$ , i.e.

**Case 1** When  $k$  is odd, then  $k + 1$  is even. We have

$$\begin{aligned} P(k+1) : S_{k+1} &= 1^2 + 2 \times 2^2 + \dots + k^2 + 2 \times (k+1)^2 \\ &= \frac{k^2(k+1)}{2} + 2 \times (k+1)^2 \\ &= \frac{(k+1)}{2} [k^2 + 4(k+1)] \quad (\text{as } k \text{ is odd, } 1^2 + 2 \times 2^2 + \dots + k^2 = k^2 \frac{(k+1)}{2}) \\ &= \frac{k+1}{2} [k^2 + 4k + 4] \\ &= \frac{k+1}{2} (k+2)^2 = (k+1) \frac{[(k+1)+1]^2}{2} \end{aligned}$$

So  $P(k+1)$  is true, whenever  $P(k)$  is true in the case when  $k$  is odd.

**Case 2** When  $k$  is even, then  $k + 1$  is odd.



$$\begin{aligned} \text{Now, } P(k+1) &: 1^2 + 2 \times 2^2 + \dots + 2 \cdot k^2 + (k+1)^2 \\ &= \frac{k(k+1)^2}{2} + (k+1)^2 \quad (\text{as } k \text{ is even, } 1^2 + 2 \times 2^2 + \dots + 2k^2 = k \frac{(k+1)^2}{2}) \\ &= \frac{(k+1)^2(k+2)}{2} = \frac{(k+1)^2((k+1)+1)}{2} \end{aligned}$$

Therefore,  $P(k+1)$  is true, whenever  $P(k)$  is true for the case when  $k$  is even. Thus  $P(k+1)$  is true whenever  $P(k)$  is true for any natural numbers  $k$ . Hence,  $P(n)$  true for all natural numbers.

### Objective Type Questions

Choose the correct answer in Examples 11 and 12 (M.C.Q.)

**Example 11** Let  $P(n) : "2^n < (1 \times 2 \times 3 \times \dots \times n)"$ . Then the smallest positive integer for which  $P(n)$  is true is

- (A) 1                      (B) 2                      (C) 3                      (D) 4

**Solution** Answer is D, since

$$P(1) : 2 < 1 \text{ is false}$$

$$P(2) : 2^2 < 1 \times 2 \text{ is false}$$

$$P(3) : 2^3 < 1 \times 2 \times 3 \text{ is false}$$

But  $P(4) : 2^4 < 1 \times 2 \times 3 \times 4$  is true

**Example 12** A student was asked to prove a statement  $P(n)$  by induction. He proved that  $P(k+1)$  is true whenever  $P(k)$  is true for all  $k > 5 \in \mathbf{N}$  and also that  $P(5)$  is true. On the basis of this he could conclude that  $P(n)$  is true

- (A) for all  $n \in \mathbf{N}$                       (B) for all  $n > 5$   
 (C) for all  $n \geq 5$                       (D) for all  $n < 5$

**Solution** Answer is (C), since  $P(5)$  is true and  $P(k+1)$  is true, whenever  $P(k)$  is true.

Fill in the blanks in Example 13 and 14.

**Example 13** If  $P(n) : "2 \cdot 4^{2n+1} + 3^{3n+1}$  is divisible by  $\lambda$  for all  $n \in \mathbf{N}$ " is true, then the value of  $\lambda$  is \_\_\_\_\_

**Solution** Now, for  $n = 1$ ,

$$2 \cdot 4^{2+1} + 3^{3+1} = 2 \cdot 4^3 + 3^4 = 2 \cdot 64 + 81 = 128 + 81 = 209,$$

for  $n = 2, 2 \cdot 4^5 + 3^7 = 8 \cdot 256 + 2187 = 2048 + 2187 = 4235$

Note that the H.C.F. of 209 and 4235 is 11. So  $2 \cdot 4^{2n+1} + 3^{3n+1}$  is divisible by 11. Hence,  $\lambda$  is 11

**Example 14** If  $P(n) : "49^n + 16^n + k$  is divisible by 64 for  $n \in \mathbf{N}"$  is true, then the least negative integral value of  $k$  is \_\_\_\_\_.

**Solution** For  $n = 1$ ,  $P(1) : 65 + k$  is divisible by 64.

Thus  $k$ , should be  $-1$  since,  $65 - 1 = 64$  is divisible by 64.

**Example 15** State whether the following proof (by mathematical induction) is true or false for the statement.

$$P(n) : 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

**Proof** By the Principle of Mathematical induction,  $P(n)$  is true for  $n = 1$ ,

$1^2 = 1 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}$ . Again for some  $k \geq 1$ ,  $k^2 = \frac{k(k+1)(2k+1)}{6}$ . Now we prove that

$$(k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

**Solution** False

Since in the inductive step both the inductive hypothesis and what is to be proved are wrong.

### 4.3 EXERCISE

#### Short Answer Type

1. Give an example of a statement  $P(n)$  which is true for all  $n \geq 4$  but  $P(1)$ ,  $P(2)$  and  $P(3)$  are not true. Justify your answer.
2. Give an example of a statement  $P(n)$  which is true for all  $n$ . Justify your answer. Prove each of the statements in Exercises 3 - 16 by the Principle of Mathematical Induction :
3.  $4^n - 1$  is divisible by 3, for each natural number  $n$ .
4.  $2^{3n} - 1$  is divisible by 7, for all natural numbers  $n$ .
5.  $n^3 - 7n + 3$  is divisible by 3, for all natural numbers  $n$ .
6.  $3^{2n} - 1$  is divisible by 8, for all natural numbers  $n$ .

7. For any natural number  $n$ ,  $7^n - 2^n$  is divisible by 5.
8. For any natural number  $n$ ,  $x^n - y^n$  is divisible by  $x - y$ , where  $x$  and  $y$  are any integers with  $x \neq y$ .
9.  $n^3 - n$  is divisible by 6, for each natural number  $n \geq 2$ .
10.  $n(n^2 + 5)$  is divisible by 6, for each natural number  $n$ .
11.  $n^2 < 2^n$  for all natural numbers  $n \geq 5$ .
12.  $2n < (n + 2)!$  for all natural number  $n$ .
13.  $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}$ , for all natural numbers  $n \geq 2$ .
14.  $2 + 4 + 6 + \dots + 2n = n^2 + n$  for all natural numbers  $n$ .
15.  $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$  for all natural numbers  $n$ .
16.  $1 + 5 + 9 + \dots + (4n - 3) = n(2n - 1)$  for all natural numbers  $n$ .

### Long Answer Type

Use the Principle of Mathematical Induction in the following Exercises.

17. A sequence  $a_1, a_2, a_3 \dots$  is defined by letting  $a_1 = 3$  and  $a_k = 7a_{k-1}$  for all natural numbers  $k \geq 2$ . Show that  $a_n = 3 \cdot 7^{n-1}$  for all natural numbers.
18. A sequence  $b_0, b_1, b_2 \dots$  is defined by letting  $b_0 = 5$  and  $b_k = 4 + b_{k-1}$  for all natural numbers  $k$ . Show that  $b_n = 5 + 4n$  for all natural number  $n$  using mathematical induction.
19. A sequence  $d_1, d_2, d_3 \dots$  is defined by letting  $d_1 = 2$  and  $d_k = \frac{d_{k-1}}{k}$  for all natural numbers,  $k \geq 2$ . Show that  $d_n = \frac{2}{n!}$  for all  $n \in \mathbf{N}$ .
20. Prove that for all  $n \in \mathbf{N}$

$$\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \dots + \cos (\alpha + (n - 1) \beta)$$

$$= \frac{\cos \left( \alpha + \left( \frac{n-1}{2} \right) \beta \right) \sin \left( \frac{n\beta}{2} \right)}{\sin \frac{\beta}{2}}$$

21. Prove that,  $\cos \theta \cos 2\theta \cos 2^2\theta \dots \cos 2^{n-1}\theta = \frac{\sin 2^n \theta}{2^n \sin \theta}$ , for all  $n \in \mathbf{N}$ .

22. Prove that,  $\sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin n\theta = \frac{\sin n\theta \sin \frac{(n+1)\theta}{2}}{2 \sin \frac{\theta}{2}}$ , for all  $n \in \mathbf{N}$ .

23. Show that  $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$  is a natural number for all  $n \in \mathbf{N}$ .
24. Prove that  $\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{13}{24}$ , for all natural numbers  $n > 1$ .
25. Prove that number of subsets of a set containing  $n$  distinct elements is  $2^n$ , for all  $n \in \mathbf{N}$ .

### Objective Type Questions

Choose the correct answers in Exercises 26 to 30 (M.C.Q.).

26. If  $10^n + 3 \cdot 4^{n+2} + k$  is divisible by 9 for all  $n \in \mathbf{N}$ , then the least positive integral value of  $k$  is  
 (A) 5 (B) 3 (C) 7 (D) 1
27. For all  $n \in \mathbf{N}$ ,  $3 \cdot 5^{2n+1} + 2^{3n+1}$  is divisible by  
 (A) 19 (B) 17 (C) 23 (D) 25
28. If  $x^n - 1$  is divisible by  $x - k$ , then the least positive integral value of  $k$  is  
 (A) 1 (B) 2 (C) 3 (D) 4

Fill in the blanks in the following :

29. If  $P(n) : 2n < n!, n \in \mathbf{N}$ , then  $P(n)$  is true for all  $n \geq$  \_\_\_\_\_.

State whether the following statement is true or false. Justify.

30. Let  $P(n)$  be a statement and let  $P(k) \Rightarrow P(k + 1)$ , for some natural number  $k$ , then  $P(n)$  is true for all  $n \in \mathbf{N}$ .

