Chapter 4 (Motion In a Plane)

Multiple Choice Questions

Single Correct Answer Type

1. The angle between $\vec{A} = \hat{i} + \hat{j}$ and $\vec{B} = \hat{i} - \hat{j}$ is

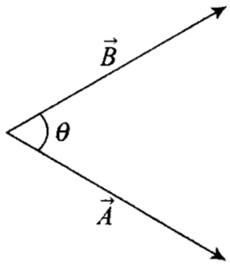
(a) $45°$ (b) $90°$ (c) $-45°$ (d) $180°$

Sol: (b)

Key concept: Scalar Product of Two Vectors:

(1) Definition : The scalar product (or dot product) of two vectors is defined as the product of the magnitude of two vectors with cosine of angle between them.

Thus if there are two vectors $\vec{A}$ and $\vec{B}$ having angle θ between them, then their scalar product written as $\vec{A} \cdot \vec{B}$ is defined as $\vec{A} \cdot \vec{B} = AB \cos\theta$



(2) *Properties*: (i) The angle between the vectors

$$\theta = \cos^{-1}\left[\frac{\vec{A} \cdot \vec{B}}{AB}\right]$$

(ii) Scalar product of two vectors will be maximum when $\cos\theta = \max = 1$, *i.e.* $\theta = 0°$, i.e., vectors are parallel

$$(\vec{A} \cdot \vec{B})_{\max} = AB$$

(iii) Scalar product of two vectors will be minimum when $|\cos\theta| = \min = 0$, *i.e.* $\theta = 90°$

$$(\vec{A} \cdot \vec{B})_{\min} = 0$$

i.e., if the scalar product of two non-zero vectors vanishes, the vectors are orthogonal.

(iv) In terms of components

$$\vec{A}\cdot\vec{B} = (\vec{i}A_x + \vec{j}A_y + \vec{k}A_z)\cdot(\vec{i}B_x + \vec{j}B_y + \vec{k}B_z)$$
$$= [A_xB_x + A_yB_y + A_ZB_z]$$

When we are dealing these types of problems, we always have to go for dot product or cross product.

According to the problem, $\vec{A} = \hat{i} + \hat{j}$

$\vec{B} = \hat{i} - \hat{j}$

We know that

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

and $\vec{A}\cdot\vec{B} = (\vec{i}A_x + \vec{j}A_y + \vec{k}A_z)\cdot(\vec{i}B_x + \vec{j}B_y + \vec{k}B_z)$

$$= [A_xB_x + A_yB_y + A_zB_z]$$

So by using properties of dot product, we get

$$(\hat{i} + \hat{j})\cdot(\hat{i} - \hat{j}) = (\sqrt{1+1})(\sqrt{1+1})\cos\theta$$

where θ is the angle between $\vec{A}$ and $\vec{B}$.

$$\Rightarrow \quad \cos\theta = \frac{1-0+0-1}{\sqrt{2}\cdot\sqrt{2}} = \frac{0}{2} = 0$$

$$\Rightarrow \quad \theta = 90°$$

Q2. Which one of the following statements is true?

(a) A scalar quantity is the one that is conserved in a process.

(b) A scalar quantity is the one that can never take negative values.

(c) A scalar quantity is the one that does not vary from one point to another in space.

(d) A scalar quantity has the same value for observers with different orientation of the axes.

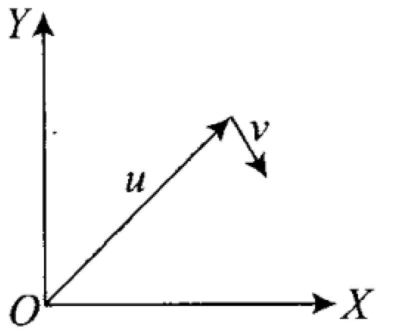
Sol: (d) A scalar quantity is independent of direction hence has the same value for observers with different orientations of the axes.

For example, a car is traveling along +x axis, it travels 10 m. If the same car is moving with the same speed for the same time interval along -x axis, then the distance meter of car shows the same travelled distance. The path length is same in both the cases.

Q3. Figure **shows the orientation of two vectors u and v in the TY-plane.**

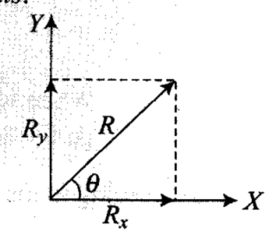
If $u = a\hat{i} + b\hat{j}$ and $v = p\hat{i} + q\hat{j}$,

which of the following is correct?
(a) a and p are positive while b and q are negative
(b) a, p and b are positive while q is negative
(c) a, q and b are positive while p is negative
(d) a, b, p and q are all positive



Sol. (b)

Key concept: *Resolution of Vector into Components:*
Consider a vector $\vec{R}$ in X-Y plane as shown in figure. If we draw orthogonal vectors $\vec{R}_x$ and $\vec{R}_y$ along x and y axes respectively, by law of vector addition, $\vec{R} = \vec{R}_x + \vec{R}_y$



Now as for any vector $\vec{A} = A\hat{n}$, $\hat{n}$ is the direction of $\vec{A}$ so

$$\vec{R}_x = \hat{i}R_x \text{ and } \vec{R}_y = \hat{j}R_y.$$

Thus, $\vec{R} = \hat{i}R_x + \hat{j}R_y$ ...(i)
But from figure, $R_x = R\cos\theta$ ...(ii)
and $R_y = R\sin\theta$ ...(iii)
Since R and θ are usually known, equations (ii) and (iii) give the magnitude of the components of $\vec{R}$ along x- and y-axes respectively.

Here it is worthy to note once a vector is resolved into its components, the components themselves can be used to specify the vector as:

(1) The magnitude of the vector $\vec{R}$ is obtained by squaring and adding equations (ii) and (iii), *i.e.*

$$R = \sqrt{R_x^2 + R_y^2}$$

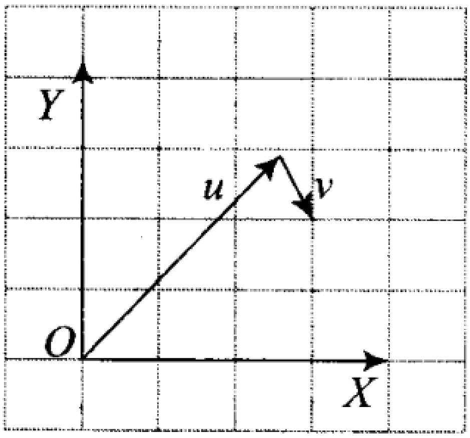
(2) The direction of the vector $\vec{R}$ is obtained by dividing equation (iii) by (ii), *i.e.*

$$\tan\theta = (R_y/R_x) \quad \text{or} \quad \theta = \tan^{-1}(R_y/R_x)$$

In such type of problems we have to resolve the rectangular components according to the diagram.

Clearly from the diagram $\vec{u} = a\hat{i} + b\hat{j}$.

As u is in the first quadrant, hence both of its components a and b will be positive and as v is in fourth quadrant. For $\vec{v} = p\hat{i} + q\hat{j}$, as it is in positive x-direction and located downward hence x-component p will be positive and y-component q will be negative.



Q4. The component of a vector r along X-axis will have maximum value if

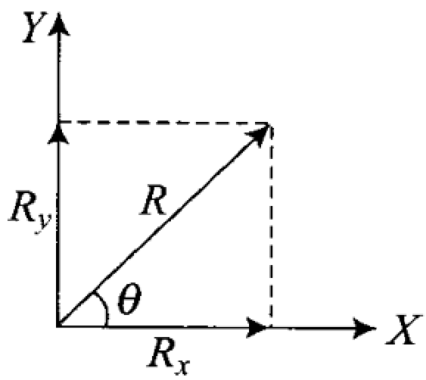
(a) r is along positive Y-axis

(b) r is along positive X-axis

(c) r makes an angle of 45° with the X-axis

(d) r is along negative Y-axis

Sol. **(b)** Consider a vector $\vec{R}$ in X-Y plane as shown in figure. If we draw orthogonal vectors $\vec{R}_x$ and $\vec{R}_y$ along x and y axes respectively, by law of vector addition, $\vec{R} = \vec{R}_x + \vec{R}_y$



The magnitude of component of r along X-axis

$$r_x = |r| \cos\theta$$
$$(r_x)_{\text{maximum}} = |r| (\cos\theta)_{\text{maximum}}$$
$$r_x = |r| \cos\theta$$
$$= |r| \cos 0° = |r| \qquad (\because \cos\theta \text{ is maximum if } \theta = 0°)$$

As $\theta = 0°$,

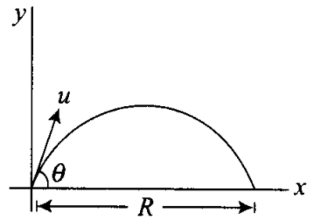
r is along positive x-axis.

Q5. The horizontal range of a projectile fired at an angle of 15° is 50 m. If it is fired with the same speed at an angle of 45°, its range will be

(a) 60 m (b) 71m (c) 100 m (d) 141m

Sol. **(c)** The horizontal range of the projectile fired at an angle is given by

$$\text{Range } R = \frac{u^2 \sin 2\theta}{g}$$



Where, θ is the angle of projection,
u is the initial velocity with which the projectile is fired,
and g is the gravitational acceleration.
According to the problem the angle of projection,
$\theta = 15°$ and horizontal range, $R = 50$ m
By substituting the values in the above relation, we get

$$\Rightarrow \quad R = 50 \text{ m} = \frac{u^2 \sin(2 \times 15°)}{g}$$

$$\Rightarrow \quad 50 \times g = u^2 \sin 30° = u^2 \times \frac{1}{2}$$

$$\Rightarrow \quad 50 \times g \times 2 = u^2$$

$$\Rightarrow \quad u^2 = 50 \times 9.8 \times 2 = 100 \times 9.8 = 980$$

$$\Rightarrow \quad u = \sqrt{980} = \sqrt{49 \times 20} = 7 \times 2 \times \sqrt{5} \text{ m/s} = 14 \times 223 \text{ m/s} = 31.304 \text{ m/s}$$

$$\text{For } \theta = 45°; \quad R = \frac{u^2 \sin 2 \times 45°}{g} = \frac{u^2}{g} \qquad (\because \sin 90° = 1)$$

$$\Rightarrow \quad R = \frac{(14\sqrt{5})^2}{g} = \frac{14 \times 14 \times 5}{9.8} = 100 \text{ m}$$

Q6. Consider the quantities pressure, power, energy, impulse, gravitational potential, electrical charge, temperature, area. Out of these, the only vector quantities are

(a) Impulse, pressure and area

(b) Impulse and area

(c) Area and gravitational potential

(d) Impulse and pressure

Sol: (b) We know that impulse J = F. Δt = Δp, where F is force, At is time duration and Ap is change in momentum. As Δp is a vector quantity, hence impulse is also a vector quantity. Sometimes area can also be treated as vector direction of area vector is perpendicular to its plane.

Q7. In a two dimensional motion, instantaneous speed ==vo== is a positive constant. Then, which of the following are necessarily true?

(a) The average velocity is not zero at any time

(b) Average acceleration must always vanish

(c) Displacements in equal time intervals are equal

(d) Equal path lengths are traversed in equal intervals

Sol :(d) Speed (Instantaneous Speed): The magnitude of the velocity at any instant of time is known as Instantaneous Speed or simply speed at that instant of time. It is denoted by v.
Quantitatively: Speed = distance/ time
Mathematically, it is the time rate at which distance is being travelled by the particle.
• Speed is a scalar quantity. It can never be negative (as shown by speedometer of our vehicle).
• Instantaneous speed is the speed of a particle at a particular instant of time.
Hence, Total distance travelled = Path length = (speed) x time taken Important point: We should be very carefttl with the fact that speed is related with total distance covered not with displacement.

Q8. In a two-dimensional motion, instantaneous speed v_0 is a positive constant. Then, which of the following are necessarily true?
‘(a) The acceleration of the particle is zero.
(b) The acceleration of the particle is bounded.
(c) The acceleration of the particle is necessarily in the plane of motion.
(d) The particle must be undergoing a uniform circular motion.
Sol. (c) This motion is two dimensional and given that instantaneous speed v_0 is positive constant. Acceleration is defined as the rate of change of velocity (instantaneous speed), hence it will also be in the plane of motion.

9. Three vectors $\vec{A}$, $\vec{B}$ and C add up to zero. Find which is false.
(a) $(\vec{A}\times\vec{B})\times\vec{C}$ is not zero unless $\vec{B}$, $\vec{C}$ are parallel
(b) $(\vec{A}\times\vec{B})\cdot\vec{C}$ is not zero unless $\vec{B}$, $\vec{C}$ are parallel
(c) If $\vec{A}$, $\vec{B}$, $\vec{C}$ define a plane, $(\vec{A}\times\vec{B})\times\vec{C}$ is in that plane
(d) $(\vec{A}\times\vec{B})\cdot\vec{C}=|\vec{A}||\vec{B}||\vec{C}|\rightarrow\vec{C}^2=\vec{A}^2+\vec{B}^2$

Sol : (c) These types of problems can be solved by hit and trial method by picking up options one by one

Sum of vectors is given as $\vec{A}+\vec{B}+\vec{C}=0$
Hence, we can say that $\vec{A}$, $\vec{B}$ and $\vec{C}$ are in one plane and are represented by the three sides of a triangle taken in one order. Let us discuss all the options one by one.

(a) We can write

$$\vec{B}\times(\vec{A}+\vec{B}+\vec{C})=\vec{B}\times 0=0$$

$$\Rightarrow \quad \vec{B}\times\vec{A}+\vec{B}\times\vec{B}+\vec{B}\times\vec{C}=0$$

$$\Rightarrow \quad \vec{B}\times\vec{A}+0\times\vec{B}\times\vec{C}=0$$

$$\Rightarrow \quad \vec{B}\times\vec{A}=-\vec{B}\times\vec{C}$$

$$\Rightarrow \quad \vec{A}\times\vec{B}=\vec{B}\times\vec{C}$$

$$\therefore \quad (\vec{A}\times\vec{B})\times\vec{C}=(\vec{B}\times\vec{C})\times\vec{C}$$

It cannot be zero.

If $\vec{B} \parallel \vec{C}$, then $\vec{B} \times \vec{C} = 0$, then $(\vec{B} \times \vec{C}) \times \vec{C} = 0$. Even if $\vec{A} \parallel \vec{C}$ or $\vec{A} \parallel \vec{B}$, then also the cross product in any manner will be zero. So, option (a) is not false.

(b) $(\vec{A} \times \vec{B}).\ \vec{C} = (\vec{B} \times \vec{A}).\ \vec{C} = 0$ whatever be the positions of $\vec{A}$, $\vec{B}$ and $\vec{C}$. If $\vec{B} \parallel \vec{C}$, then $\vec{B} \times \vec{C} = 0$, then $(\vec{B} \times \vec{C}) \times \vec{C} = 0$. So, option (b) is not false.

(c) $(\vec{A} \times \vec{B}) = \vec{X} = \vec{A}\vec{B} \sin \theta$ and the direction of X is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$. $(\vec{A} \times \vec{B}) \times \vec{C} = X \times \vec{C}$. Its direction is in the plane of $\vec{A}$, $\vec{B}$ and $\vec{C}$. So, option (c) is false.

(d) If $\vec{C}^2 = \vec{A}^2 + \vec{B}^2$, then angle between $\vec{A}$ and $\vec{B}$ is 90°.

$\therefore \quad (\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{A}\vec{B} \sin 90°\ X).\ \vec{C} = \vec{A}\vec{B}\ (X \cdot \vec{C}) = ABC \cos 90° = 0$

Hence, option (d) is not false.

10. It is found that $|\vec{A} + \vec{B}| = |A|$. This necessarily implies

(a) $\vec{B} = 0$ (b) $\vec{A}, \vec{B}$ are antiparallel

(c) $\vec{A}, \vec{B}$ are perpendicular (d) $\vec{A} \cdot \vec{B} < 0$

Sol. (b) According to the problem,

$$|\vec{A} + \vec{B}| = |\vec{A}|$$

By squaring both sides, we get

$$|\vec{A} + \vec{B}|^2 = |\vec{A}|^2$$

$$\Rightarrow \quad |\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}||\vec{B}|\cos\theta = |\vec{A}|^2$$

where θ is the angle between $\vec{A}$ and $\vec{B}$.

$$|\vec{B}|\,(|\vec{B}| + 2|\vec{A}|\cos\theta) = 0$$

$$\Rightarrow \quad |\vec{B}| + 2|\vec{A}|\cos\theta = 0$$

$$\Rightarrow \quad \cos\theta = -\frac{|\vec{B}|}{2|\vec{A}|}$$

If A and B are antiparallel, then $\theta = 180°$

Hence from Eq. (i),

$$-1 = -\frac{|\vec{B}|}{2|\vec{A}|} \Rightarrow |\vec{B}| = 2|\vec{A}|$$

Hence, correct answer will be $\vec{A}$ and $\vec{B}$ are antiparallel provided $|\vec{B}| = 2|\vec{A}|$.

It seems like option (a) is also correct but it is not for $|\vec{A} + \vec{B}| = |\vec{A}|$, either $\vec{B} = 0$ or $\vec{B} = -2\vec{A}$, so this option will be false.

More Than One Correct Answer type

Q11. Two particles are projected in air with speed v_0 at angles θ_1 and θ_2 (both acute) to the horizontal, respectively. If the height reached by the first particle is greater than that of the second, then tick the right choices.

(a) Angle of projection: $\theta_1 > \theta_2$
(b) Time of flight: $T_1 > T_2$
(c) Horizontal range: $R_x > R_2$
(d) Total energy: $U_1 > U_2$
Sol: (a, b, c)

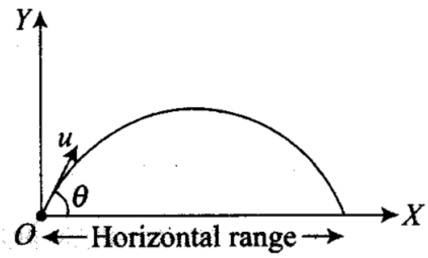
Key concept: When particle is thrown at an angle:

- Time of flight. The total time taken by the projectile to go up and come down to the same level from which it was projected is called time of flight.

For vertical upward motion 0 = u sin θ – gt
=> t = (n sin θ /g)
Now as time taken to go up is equal to the time taken to come down, so
Time of flight T = 2t = 2usin θ/g
Time of flight can also be expressed as: T = 2.u_y /g
(where n is the vertical component of initial velocity).

(ii) Horizontal range : It is the horizontal distance travelled by a body ' during the time of flight. So by using second equation of motion in x-direction.

$$R = u\cos\theta \times T$$
$$= u\cos\theta \times (2u\sin\theta/g)$$
$$= \frac{u^2\sin 2\theta}{g}$$
$$R = \frac{u^2\sin 2\theta}{g}$$



Range of projectile can also be expressed as:

$$R = u\cos\theta \times T = u\cos\theta\frac{2u\sin\theta}{g}$$
$$= \frac{2u\cos\theta\, u\sin\theta}{g} = \frac{2u_xu_y}{g}$$

$\therefore\ R = \frac{2u_xu_y}{g}$ (where u_x and u_y are the horizontal and vertical components of initial velocity)

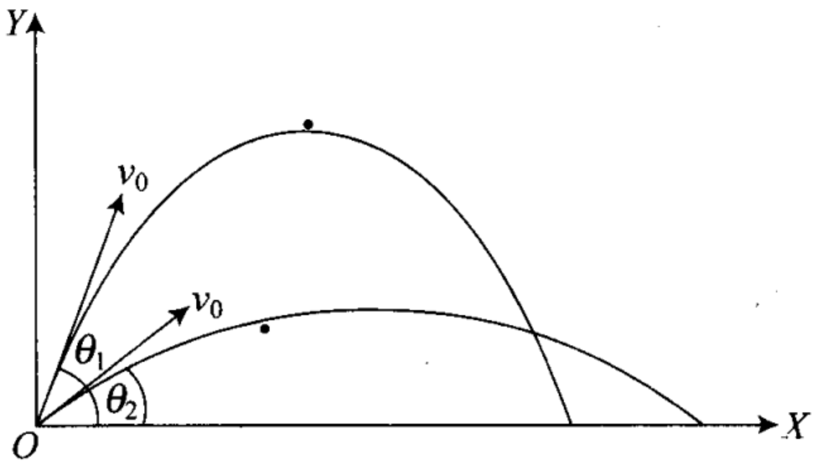
(iii) *Maximum height* : It is the maximum height from the point of projection, a projectile can reach.

So, by using $v^2 = u^2 + 2as$

$$0 = (u\sin\theta)^2 - 2gH$$
$$H = \frac{u^2\sin^2\theta}{2g}$$

Maximum height can also be expressed as:

$H = \frac{u_y^2}{2g}$ (where u_y is the vertical component of initial velocity).



The maximum height reached $H = \dfrac{u^2 \sin^2 \theta}{2g}$, where θ is the angle of projection and u is the initial speed at which projectile is projected.
The maximum height reached for first particle is

$$H_1 = \frac{v_0^2 \sin^2 \theta_1}{2g}$$

The maximum height reached for second particle is

$$H_2 = \frac{v_0^2 \sin^2 \theta_2}{2g}$$

(a) According to the problem,

$\Rightarrow \quad H_1 > H_2$

$\Rightarrow \quad \dfrac{v_0^2 \sin^2 \theta_1}{2g} > \dfrac{v_0^2 \sin^2 \theta_2}{2g}$

$\Rightarrow \quad \sin^2 \theta_1 > \sin^2 \theta_2$

$\Rightarrow \quad \sin^2 \theta_1 - \sin^2 \theta_2 > 0$

$\Rightarrow \quad (\sin \theta_1 - \sin \theta_2)(\sin \theta_1 + \sin \theta_2) > 0$

Thus, either $\sin \theta_1 + \sin \theta_2 > 0$

$\Rightarrow \quad \sin \theta_1 - \sin \theta_2 > 0$

$\Rightarrow \quad \sin \theta_1 > \sin \theta_2$ or $\theta_1 > \theta_2$

Hence option (a) is correct.

(b) Time of fight, $T = \dfrac{2u \sin \theta}{g} = \dfrac{2v_0 \sin \theta}{g}$

Thus, $\quad T_1 = \dfrac{2v_0 \sin \theta_1}{g}$ and $T_2 = \dfrac{2v_0 \sin \theta_2}{g}$

(Here, T_1 = Time of flight of first particle and T_2 = Time of flight of second particle).

As, $\sin \theta_1 > \sin \theta_2$

Thus, $\quad T_1 > T_2$.

Hence option (b) is correct.

(c) We know that Range, $R = \frac{u^2 \sin 2\theta}{g} = \frac{v_0^2 \sin 2\theta}{g}$

$$R_1 = \text{Range of first particle} = \frac{u_0^2 \sin 2\theta_1}{g}$$

$$R_2 = \text{Range of second particle} = \frac{u_0^2 \sin 2\theta_2}{g}$$

Given, $\sin \theta_1 > \sin \theta_2$

$\Rightarrow \quad \sin 2\theta_1 > \sin 2\theta_2$

$\Rightarrow \quad \frac{R_1}{R_2} = \frac{\sin 2\theta_1}{\sin 2\theta_2} > 1$

$\Rightarrow \quad R_1 > R_2$

But if $\theta_1 + \theta_2 = 90°$, then $R_1 = R_2$

Hence option (c) is incorrect.

(d) Total energy for the first particle.

$$U_1 = \text{KE} + \text{PE} = \frac{1}{2} m_1 v_0^2$$

(This value will be constant throughout the journey).

$$U_2 = \text{KE} + \text{PE} = \frac{1}{2} m_2 v_0^2$$

Total energy for the second particle

if $\quad m_1 = m_2$, then $U_1 = U_2$

$m_1 > m_2$, then $U_1 > U_2$

$m_1 < m_2$, then $U_1 < U_2$

Important points about time of flight: For complementary angles of projection θ and $90° - \theta$.

(a) Ratio of time of flight $= \frac{T_1}{T_2} = \frac{2u \sin \theta/g}{2u \sin(90° - \theta)/g} = \tan \theta$

$\Rightarrow \quad \frac{T_1}{T_2} = \tan \theta$

(b) Multiplication of time of flight $= T_1 T_2 = \frac{2u \sin \theta}{g} \frac{2u \cos \theta}{g}$

$\Rightarrow \quad T_1 T_2 = \frac{2R}{g}$

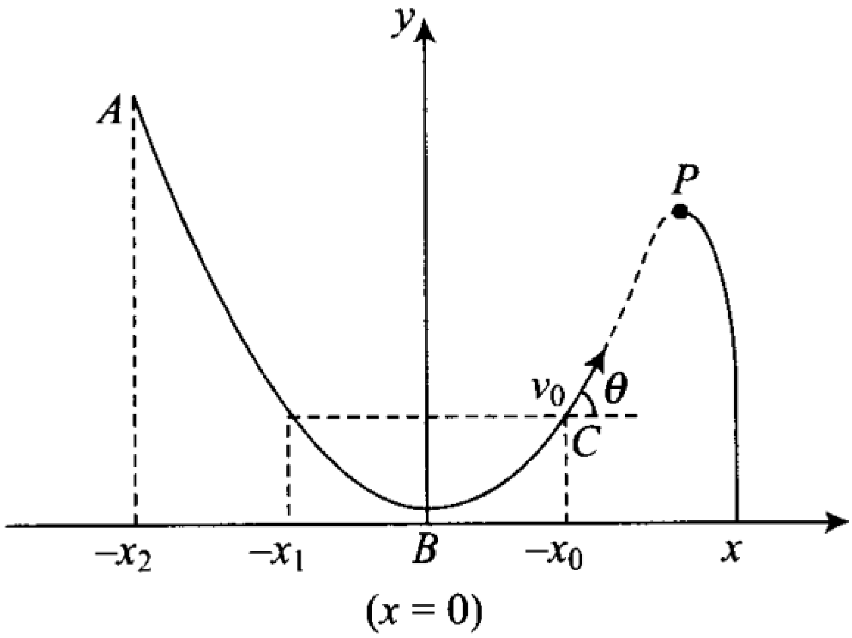
Q12. A particle slides down a frictionless parabolic ($y = x^2$) track (A – B – C) starting from rest at point A (figure). Point B is at the vertex of parabola and point C is at a height less than that of point A. After C, the particle moves freely in air as a projectile. If the particle reaches highest point at P, then

(a) KE at P = KE at B

(b) height at P = height at A

(c) total energy at P = total energy at A

(d) time of travel from AtoB



Solution : (c)

Key concept: In such type of problems, we have to observe the nature of track that if there is a friction or not, as friction is not present in this track, total energy of the particle will remain constant throughout the journey.

According to the problem, the path traversed by the particle on a frictionless track is parabolic, is given by the equation $y = x^2$, thus total energy (KE + PE) will be same throughout the journey.

Hence, total energy at A = total energy at P

At B the particle is having only KE but at P some KE is converted to PE.

So, $(KE)_B > (KE)_P$

Total energy at A = PE = Total energy at B = KE = Total energy at P = PE + KE

The potential energy at A is converted to KE and PE at P, hence (PE)P < (PE)A

Hence, (Height) P < (Height) A

As, Height of P < Height of A

Hence, path length AB > path length BP

Hence, time of travel from A to B ≠ Time of travel from B to P.

13. Following are four different relations about displacement, velocity and acceleration for the motion of a particle in general. Choose the incorrect one(s).

(a) $v_{av} = \frac{1}{2}[v(t_1) + v(t_2)]$ (b) $v_{av} = \frac{r(t_2) - r(t_1)}{t_2 - t_1}$

(c) $r = \frac{1}{2}(v(t_2) - v(t_1)(t_2 - t_1))$ (d) $a_{av} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$

Sol. (a, c) When an object covers a displacement Δr in time Δt, its average velocity is given by $\vec{v}_{avg} = \frac{\overrightarrow{\Delta r}}{\Delta t} = \frac{r_2 - r_1}{t_2 - t_1}$ where r_1 and r_2 are position vectors corresponding to time t_1 and t_2.

If the velocity of an object changes from v_1 to v_2 in time Δt, average acceleration is given by

$$a_{av} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1}$$

But, when acceleration is non-uniform,

$$v_{av} \neq \frac{v_1 + v_2}{2}$$

Option (c) is similar to the relation $\vec{r} = \frac{1}{2}at^2$ which is not correct if initial velocity is given.

So (b) and (d) are the correct relations for the uniform acceleration.

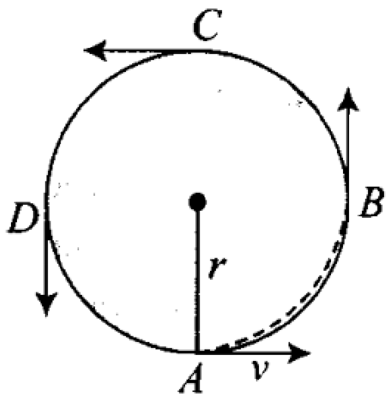
Q14. For a particle performing uniform circular motion, choose the correct statement(s) from the following.

(a) Magnitude of particle velocity (speed) remains constant.

(b) Particle velocity remains directed perpendicular to radius vector.

(c) Direction of acceleration keeps changing as particle moves.

(d) Angular momentum is constant in magnitude but direction keeps changing.



Sol: (a, b, c) While a particle is in uniform circular motion. Then the following statements are true.

(i) speed will be always constant throughout.

(ii) velocity will be always tangential in the direction of motion at a particular point.

(iii) the centripetal acceleration a = v^2/r and its direction will always towards centre of the circular trajectory.

(iv) angular momentum (mvr) is constant in magnitude and direction. And its direction is perpendicular to the plane containing r and v.

Important point: In uniform circular motion, magnitude of linear velocity and centripetal acceleration is constant but direction changes continuously.

15. For two vectors $\vec{A}$ and $\vec{B}$, $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$ is always true when

(a) $|\vec{A}| = |\vec{B}| \neq 0$

(b) $\vec{A} \perp \vec{B}$

(c) $|\vec{A}| = |\vec{B}| \neq 0$ and $\vec{A}$ and $\vec{B}$ are parallel or anti-parallel

(d) when either $|\vec{A}|$ or $|\vec{B}|$ is zero

Sol. **(b, d)** According to the problem, $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$

$$\Rightarrow \quad \sqrt{|\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}||\vec{B}|\cos\theta} = \sqrt{|\vec{A}|^2 + |\vec{B}|^2 - 2|A||B|\cos\theta}$$

$$\Rightarrow \quad |\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}||\vec{B}|\cos\theta = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|\cos\theta$$

$$\Rightarrow \quad 4|\vec{A}||\vec{B}|\cos\theta = 0$$

$$\Rightarrow \quad |\vec{A}||\vec{B}| \cos \theta = 0$$

$$\Rightarrow \quad |\vec{A}| = 0 \text{ or } |\vec{B}| = 0 \text{ or } \cos \theta = 0$$

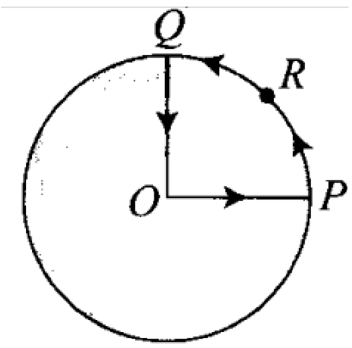
i.e. $\theta = 90°$

When $\theta = 90°$, we can say that $\vec{A} \perp \vec{B}$.

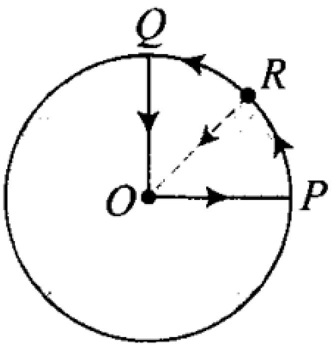
Hence options (b) and (d) are correct.

Very Short Answer Type Questions

Q16 .A cyclist starts from centre O of a circular park of radius 1 km and moves along the path OPRQO as shown in figure**. If he maintains constant speed of 10 ms”^{1}, what is his acceleration at point R in magnitude and direction?**



Ans: According to the problem the path of the cyclist is O-P- R-Q-O.

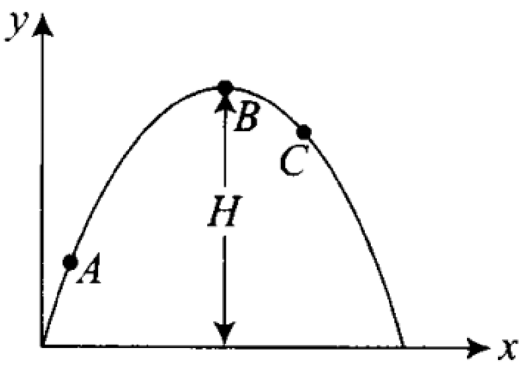


The cyclist is in uniform circular motion and it is given that linear velocity = 10 m/s, R = 1 km = 1000 m. As we know whenever an object is performing circular motion, acceleration is called centripetal acceleration and is always directed towards the centre.So cyclist experiences a centripetal force (acceleration) at point R towards centre.

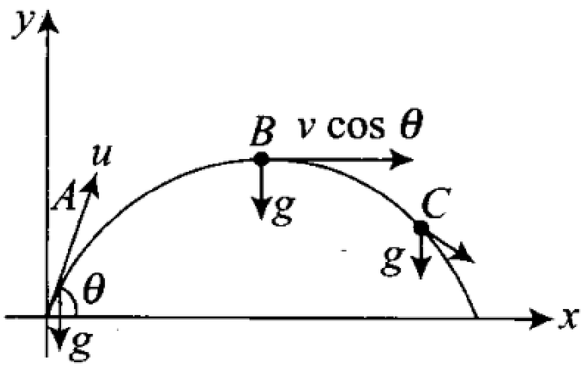
The centripetal acceleration at R is given by the relation, $a_c = \dfrac{v^2}{r}$

$$\Rightarrow \quad a_c = \frac{(10)^2}{1000} = \frac{100}{10^3} = 0.1 \text{ m/s}^2 \text{ along } RO.$$

Q17. A particle is projected in air at some angle to the horizontal, moves along parabola as shown in figure **where x** and y **indicate horizontal and vertical directions, respectively. Show in the diagram, direction of velocity and acceleration at points A, B and C.**

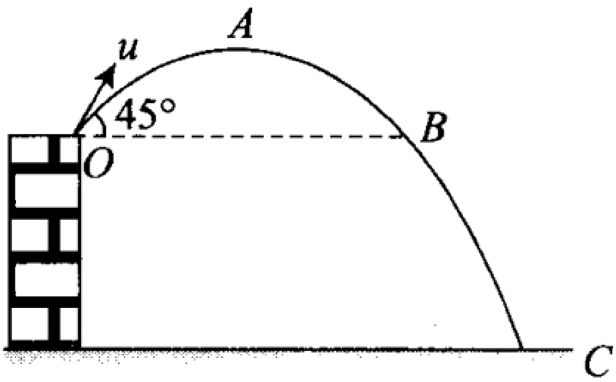


Sol: In projectile motion horizontal component of velocity will always be constant and acceleration is always vertically downward and is equal to g. Direction of velocity will always be tangential to the curve in the direction of motion.
As shown in the diagram in which a particle is projected at an angle 0.



$$v_x = \text{Horizontal component of velocity}$$
$$= v\cos\theta = \text{constant.}$$
$$v_y = \text{Vertical component of velocity}$$
$$= v\sin\theta$$

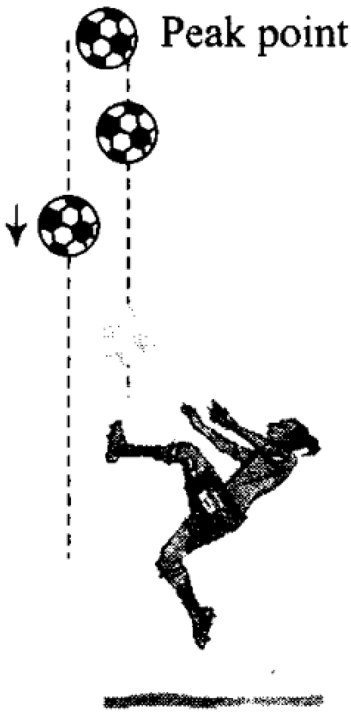
Q18. A ball is thrown from a roof top at an angle
of 45° above the horizontal. It hits the ground a few seconds later. At what point during its motion, does the ball have
(a) greatest speed
(b) smallest speed
(c) greatest acceleration Explain.



Ans: In this problem total mechanical energy of the ball is conserved. As the ball is projected from point O, and covering the path OABC.
At point A it has both kinetic and potential energy.
But at point C it have only kinetic energy, (keeping the ground as reference where PE is zero.)
(a) At point B, it will gain the same speed u and after that speed increases and will be

maximum just before reaching C.
(b) During upward journey from OtoA speed decreases and smallest speed attained by it is at the highest point, i.e., at point A.
(c) Acceleration is always constant throughout the journey and is vertically downward equal to g.

Q19. A football is kicked into the air vertically upwards. What is **its (a) acceleration and (b) velocity at the highest point?**



Sol: (a) The situation is shown in the diagram below in which a football is kicked into the air vertically upwards. Acceleration of the football will always be vertical downward and is called acceleration due to gravity (g).
(b) When the football reaches the highest point it is momentarily at rest and at that moment its velocity will be zero as it is continuously retarded by acceleration due to gravity (g).

20. $\vec{A}$, $\vec{B}$ and $\vec{C}$ are three non-collinear, non-co-planar vectors. What can you say about direction of $\vec{A} \times (\vec{B} \times \vec{C})$?

Key concept: Collinear vectors: When the vectors under consideration can share the same support or have a common support then the considered vectors are collinear.
Coplanar vectors: Three (or more) vectors are called coplanar vector if they lie in the same plane. Two (free) vectors are always coplanar.

$$\vec{B} \times \vec{C} = BC \sin\theta\, \hat{n}$$

where $\hat{n}$ = unit vector perpendicular to the plane containing $\vec{B}$ and $\vec{C}$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{A} \times \hat{n}(BC \sin\theta)$$
$$= (BC \sin\theta) A \sin\alpha\, \hat{p}$$

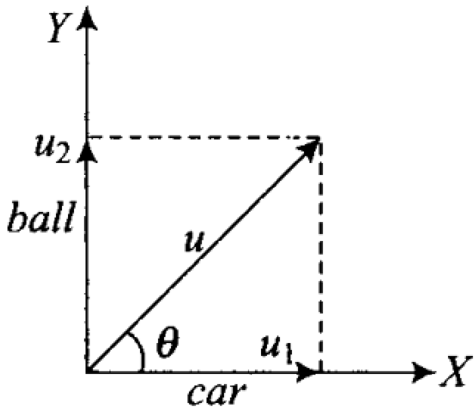
where $\hat{p}$ = unit vector perpendicular to both $\hat{n}$ and $\vec{A}$.
Here perpendicular to $\hat{n}$ and $\vec{A}$ means in the plane of $\vec{B}$ and $\vec{C}$.
Hence $\vec{A} \times (\vec{B} \times \vec{C})$ will lie in the plane of $\vec{B}$ and $\vec{C}$, and is perpendicular to $\vec{A}$.

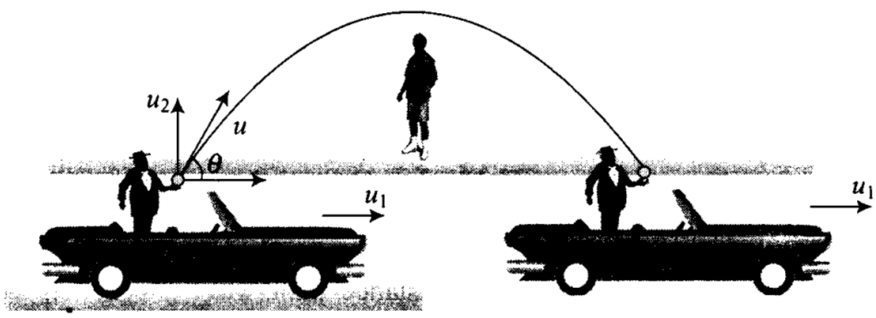
Q21. A boy travelling in an open car moving on a levelled road with constant speed tosses a ball vertically up in the air and catches it back. Sketch the motion of the ball as observed by a boy standing on the footpath. Give explanation **to support your diagram.**
Sol: With respect to the observer standing on the footpath ball is thrown with velocity u at an

angle θ with the horizontal, hence it seems as a projectile. So path of the ball will be parabolic. The horizontal speed of the ball is same as that of the car, therefore, ball as well car travels equal horizontal ult distance. Due to its vertical speed, the ball follows a parabolic path.

u_1 = car speed, u_2 = initial vertical speed of ball

$$u = \sqrt{u_1^2 + u_2^2}, \theta = \tan^{-1}\left(\frac{u_2}{u_1}\right)$$

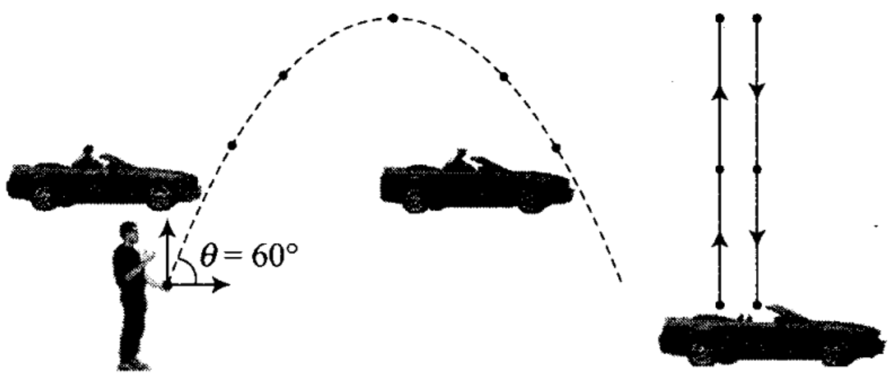




Important point: We must be very clear that we are working with respect to ground. When we observe with respect to the car, motion will be along vertical direction only.

Q22. A boy throws a ball in air **at 60° to the horizontal along a road with a speed of 10 m/s (36 km/h). Another boy sitting in a passing by car observes the ball. Sketch the motion of the ball as observed by the boy in the car, if** car **has a speed of 18 km/h. Give** explanation **to support your diagram.**

Sol: The situation is shown in the below diagram.



According to the problem the boy standing on ground throws the ball at an angle of 60° with horizontal at a speed of 10 m/s.

$\therefore$ Horizontal component of velocity, $u_x = 10 \cos \theta$

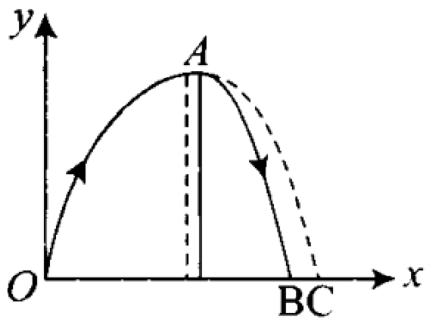
$$u_x = (10 \text{ m/s}) \cos 60° = 10 \times \frac{1}{2} = 5 \text{ m/s}$$

Vertical component of velocity, $u_y = 10 \sin \theta$

$$u_y = (10 \text{ m/s}) \sin 60° = 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3} \text{ m/s}$$

Speed of the car = 18 km/h = 5 m/s
As horizontal speed of ball and car is same, hence relative velocity of ball w.r.t car in the horizontal direction will be zero.
Only vertical motion of the ball will be observed by the boy in the car, as shown in above diagram.

Q23. In dealing with motion **of** projectile **in air, we ignore** effect **of air resistance on** motion**. This gives trajectory as a parabola as you have studied. What would the trajectory look like if air resistance is included? Sketch such a trajectory and explain why you have drawn it that way.**
Sol:



When we are dealing with projectile motion generally we neglect the air resistance. But if air resistance is included the horizontal component of velocity will not be constant and obviously trajectory will change.
Due to air resistance, particle energy as well as horizontal component of velocity keep on decreasing making the fall steeper than rise as shown in the figure.
When we are neglecting air resistance path was symmetric parabola (OAC). When air resistance is considered path is asymmetric parabola (OAB).

Short Answer Type Questions
Q24. A fighter plane is flying horizontally at an altitude of 1.5 km with speed 720 km/h. At what angle of sight (w.r.t. horizontal) when the target is seen, should the pilot drop the bomb in order to attack the target ?

Sol.

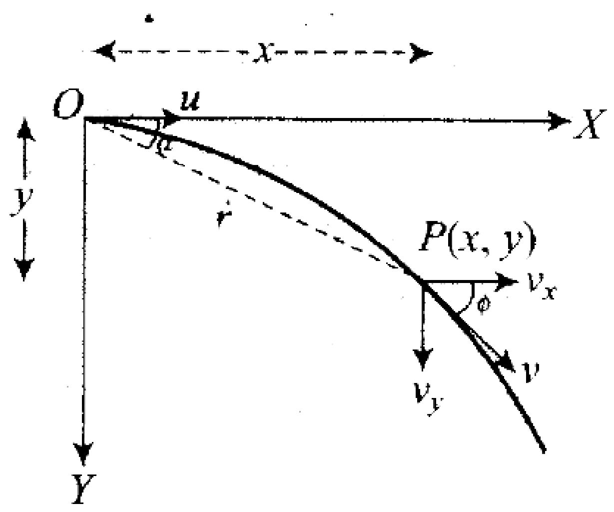
Key concept:

1. *Displacement of Projectile* ($\vec{r}$): After time t, horizontal displacement $x = ut$ and vertical displacement $y = \frac{1}{2}gt^2$.

   So, the position vector $\vec{r} = ut\,\hat{i} + \frac{1}{2}gt^2\,\hat{j}$

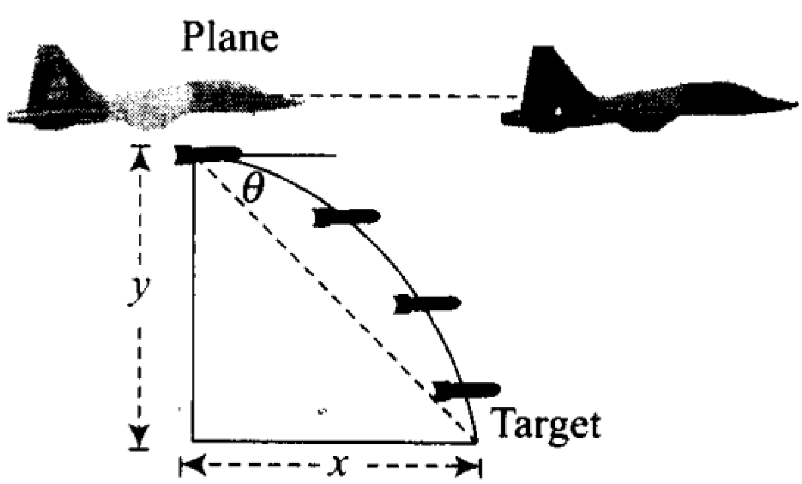
   Therefore $r = ut\sqrt{1+\left(\frac{gt}{2u}\right)^2}$ and $\alpha = \tan^{-1}\left(\frac{gt}{2u}\right)$

   $\alpha = \tan^{-1}\left(\sqrt{\frac{gy}{2}}\Big/u\right)$ $\left(\text{as } t = \sqrt{\frac{2y}{g}}\right)$



2. When an object is dropped/released by any moving vehicle. Then initial velocity of the object is same as the moving vehicle.

When the bomb is dropped from Plane the plane which is moving horizontally. So, the bomb will have same initial velocity as that of plane along horizontal direction.
The situation is shown in the diagram below. Let a fighter plane, when it be plane at position P, drops a bomb to hit a target T.
Let the target is seen at an angle θ with horizontal.



And the speed of the plane $= 720 \text{ km/h} = 720 \times \frac{5}{18} \text{ m/s} = 200 \text{ m/s}$

Altitude of the plane $y = 1.5 \text{ km} = 1500 \text{ m}$

If bomb hits the target after time t, then horizontal distance travelled by the bomb is

$$x = u \times t = 200t \qquad \ldots\text{(i)}$$

Vertical distance travelled by the bomb

$$y = \frac{1}{2}gt^2 \Rightarrow 1500 = \frac{1}{2} \times 9.8t^2$$

$$\Rightarrow \quad t^2 = \frac{1500}{4.9} \Rightarrow t = \sqrt{\frac{1500}{49}} = 17.49 \text{ s}$$

Using value of t in Eq. (i),

$$x = 200 \times 17.49 \text{ m}$$

Now, $\tan\theta = \frac{y}{x} = \frac{1500}{200 \times 17.49} = 49287 = \tan 23°12'$

$$\Rightarrow \quad \theta = 23°12'$$

Important point: Angle is with respect to target. As seen by observer in the plane, motion of the bomb will be vertically downward below the plane. .

Q25. (a) Earth can be thought of as a sphere of radius 6400 km. Any object (or a person) is performing circular motion around the axis of the earth due to the earth rotation (period 1 day). What is the acceleration of object on the surface of the earth (at equator) towards its centre? What is it at latitude 9? How does these accelerations compare with g = 9.8 m/s^2?
(b) Earth also moves in circular orbit around the sun once every year with an orbital radius of 1.5 x 10^{11} What is the acceleration of the earth (or any object on the surface of the earth) towards the centre of the sun? How does this acceleration compare with g = 9.8 m/s^2?

Sol. (a) According to the problem,

Radius of the earth $(R) = 6400 \text{ km} = 6.4 \times 10^6$ m.

Time period $(T) = 1 \text{ day} = 24 \times 60 \times 60 \text{ s} = 86400 \text{ s}$

Centripetal acceleration,

$$(a_c) = \omega^2 R = \frac{4\pi^2 R}{T} = \frac{4 \times (22/7)^2 \times 6.4 \times 10^6}{(24 \times 60 \times 60)^2}$$

$$= \frac{4 \times 484 \times 64 \times 10^6}{49 \times (24 \times 3600)^2} = 0.034 \text{ m/s}^2$$

At equator, latitude $\theta = 0°$

$$\therefore \quad \frac{a_c}{g} = \frac{0.034}{9.8} = \frac{1}{288}$$

(b) Orbital radius of the earth around the sun $(R) = 1.5 \times 10^{11}$ m

Time period $= 1 \text{ yr} = 365 \text{ days} = 365 \times 24 \times 60 \times 60 \text{ s} = 3.15 \times 10^7$ s

Centripetal acceleration

$$(a_c) = R\omega^2 = \frac{4\pi^2 R}{T^2} = \frac{4 \times (22/7)^2 \times 1.5 \times 10^{11}}{(3.15 \times 10^7)^2} = 5.97 \times 10^{-3} \text{ m/s}^2$$

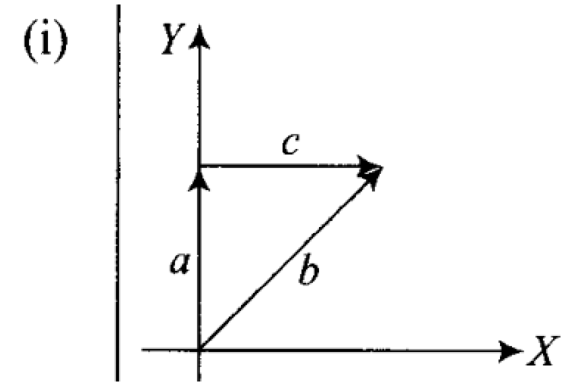
$$\frac{a_c}{g} = \frac{5.97 \times 10^{-3}}{9.8} = \frac{1}{1642}$$

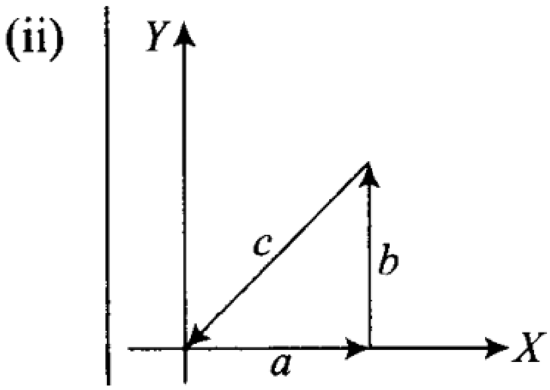
Q26. Given below in Column I are the relations between vectors a, b and c and in Column II are the orientations of a, b and c in the AY-plane. Match the relation in Column I to correct orientations in Column II.

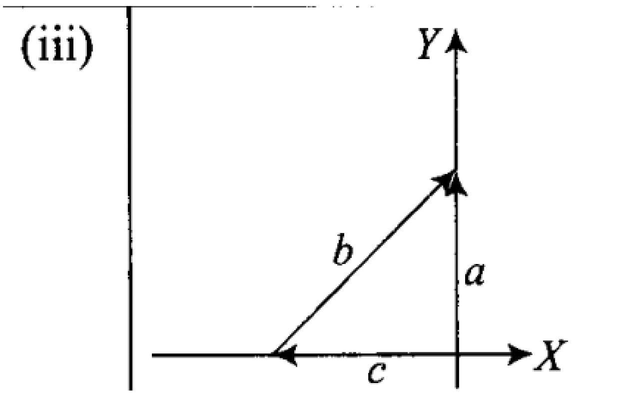
Column I

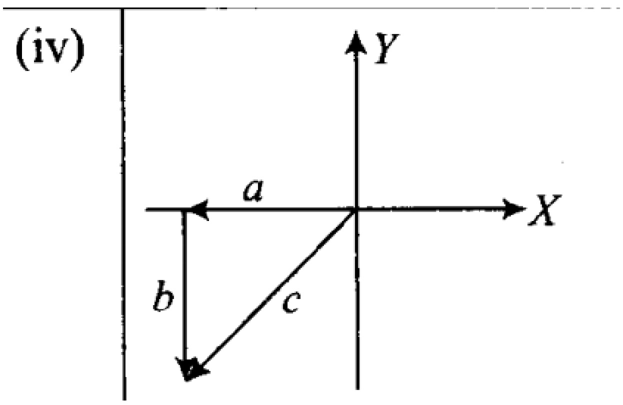
(a) a + b = c
(b) a − c = b
(c) b − a = c
(d) a+b+c = 0

Column II



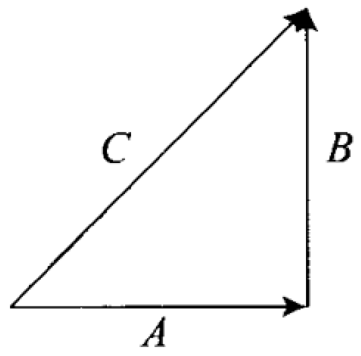






Sol: We apply triangulr law of addition
Triangular law of vector addition: Two vectors are considered as two sides of a triangle taken in the same order. The third side or completing side of the triangle is the resultant taken in the opposite order.
or
We can say that vectors are arranged head to tail, this graphical method is called the head-to-tail method. The two vectors and their resultant form three sides of a triangle, so this method

is also known as triangle method of vector addition.



As shown in the diagram below in which vectors A and B are corrected by head and tail.

Resultant vector C = A + B

(a) from (iv), it is clear that c = a + b

(b) from (iii), c + b = a => a- c = b

(c) from (i), b = a + c => b-a = c

(d) from (ii), -c = a + b => a + b + c = 0

27. If $|\vec{A}| = 2$ and $|\vec{B}| = 4$, then match the relation in Column I with the angle θ between $\vec{A}$ and $\vec{B}$ in Column II.

Column I		Column II	
(a)	$\vec{A} \cdot \vec{B} = 0$	(i)	$\theta = 0°$
(b)	$\vec{A} \cdot \vec{B} = +8$	(ii)	$\theta = 90°$
(c)	$\vec{A} \cdot \vec{B} = 4$	(iii)	$\theta = 180°$
(d)	$\vec{A} \cdot \vec{B} = -8$	(iv)	$\theta = 60°$

Sol. According to the problem $|\vec{A}| = 2$ and $|\vec{B}| = 4$

(a) $\vec{A} \cdot \vec{B} = AB \cos\theta = 0 \Rightarrow 2 \times 4 \cos\theta = 0$

$\Rightarrow \cos\theta = 0 = \cos 90° \Rightarrow \theta = 90°$

$\therefore$ Option (a) matches with option (ii).

(b) $\vec{A} \cdot \vec{B} = AB \cos\theta = 8 \Rightarrow 2 \times 4 \cos\theta = 8$

$\Rightarrow \cos\theta = 1 = \cos 0° \Rightarrow \theta = 0°$

$\therefore$ Option (b) matches with option (i).

(c) $\vec{A} \cdot \vec{B} = AB \cos\theta = 4 \Rightarrow 2 \times 4 \cos\theta = 4$

$\Rightarrow \cos\theta = \frac{1}{2} = \cos 60° \Rightarrow \theta = 60°$

$\therefore$ Option (c) matches with option (iv).

(d) $\vec{A} \cdot \vec{B} = AB \cos\theta = -8 \Rightarrow 2 \times 4 \cos\theta = -8$

$\Rightarrow \cos\theta = -1 = \cos 180° \Rightarrow \theta = 180°$

Option (d) matches with option (iii).

28. If $|\vec{A}| = 2$ and $|\vec{B}| = 4$, then match the relations in Column I with the angle θ between $\vec{A}$ and $\vec{B}$ in Column II.

Column I		Column II	
(a)	$\lvert\vec{A}\times\vec{B}\rvert = 0$	(i)	$\theta = 30°$
(b)	$\lvert\vec{A}\times\vec{B}\rvert = 8$	(ii)	$\theta = 45°$
(c)	$\lvert\vec{A}\times\vec{B}\rvert = 4$	(iii)	$\theta = 90°$
(d)	$\lvert\vec{A}\times\vec{B}\rvert = 4\sqrt{2}$	(iv)	$\theta = 0°$

Sol. According to the problem $|\vec{A}| = 2$ and $|\vec{B}| = 4$.

(a) $|\vec{A}\times\vec{B}| = \vec{A}\vec{B}\sin\theta = 0$

$\Rightarrow \quad 2\times 4\times\sin\theta = 0$

$\Rightarrow \quad \sin\theta = 0 = \sin 0° \quad \Rightarrow \quad \theta = 0°$

$\therefore$ Option (a) matches with option (iv).

(b) $|\vec{A}\times\vec{B}| = \vec{A}\vec{B}\sin\theta = 8$

$\Rightarrow \quad 2\times 4\sin\theta = 8$

$\Rightarrow \quad \sin\theta = 1 = \sin 90° \Rightarrow \theta = 90°$

$\therefore$ Option (b) matches with option (iii).

(c) $|\vec{A}\times\vec{B}| = \vec{A}\vec{B}\sin\theta = 4 \Rightarrow 2\times 4\sin\theta = 4$

$$\sin\theta = \frac{1}{2} = \sin 30° \Rightarrow \theta = 30°$$

$\therefore$ Option (c) matches with option (i).

(d) $|\vec{A}\times\vec{B}| = AB\sin\theta = 4\sqrt{2}$

$\Rightarrow \quad 2\times 4\sin\theta = 4\sqrt{2}$

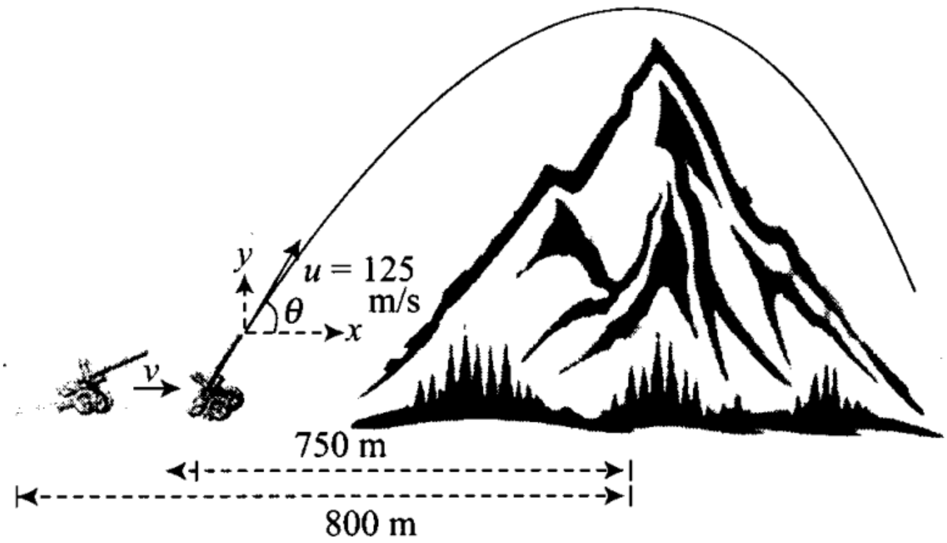
$$\Rightarrow \quad \sin\theta = \frac{1}{\sqrt{2}} = \sin 45° \Rightarrow \theta = 45°$$

$\therefore$ Option (d) matches with option (ii).

Long Answer Type Questions

Q29. A hill is 500 m high. Supplies are to be sent across the hill using a canon that can hurl packets at a speed of 125 m/s over the hill. The canon is located at a distance of 800 m from the foot of hill **and can be moved on the ground at a speed of 2** m/s, **so that its distance from the hill can be adjusted. What is the shortest time in which a packet can reach on the ground across the hill? Take g= 10 m/s^2.**

Sol. According to the problem, speed of packets =125 m/s , height of the hill = 500 m, distance between the cannon and the foot of the hill, d = 800 m



To cross the hill in shortest time, then the vertical component of the velocity should be minimum so that it just crosses the height of hill.

$$u_y = \sqrt{2gh} \geq \sqrt{2 \times 10 \times 500} \geq 100 \text{ m/s}$$

But $u^2 = u_x^2 + u_y^2$

$\therefore$ Horizontal component of initial velocity,

$$u_x = \sqrt{u^2 - u_y^2} = \sqrt{(125)^2 - (100)^2} = 75 \text{ m/s}$$

Time taken by the packet to reach the top of the hill,

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 500}{10}} = 10 \text{ s}$$

Time taken to reach the ground from the top of the hill $t' = t = 10$ s.
Horizontal distance travelled in 10 s,

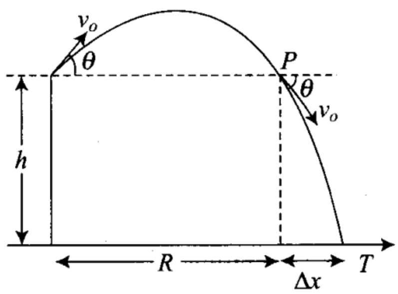
$$x = u_x \times t = 75 \times 10 = 750 \text{ m}$$

Distance through which canon has to be moved = 800 – 750 = 50 m Speed with which canon can move = 2 m/s

$\therefore$ Time taken by canon $\frac{50}{2} \Rightarrow t'' = 25$ s

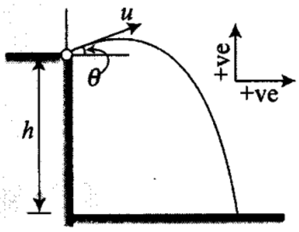
$\therefore$ Total time taken by a packet to reach on the ground $= t'' + t + t'$
$= 25 + 10 + 10 = 45$ s

30. A gun can fire shells with maximum speed v_0 and the maximum horizontal range that can achieved is $R = \frac{v_0^2}{g}$. If a target farther away by distance Δx (beyond R) has to be hit with the same gun, show that it could be achieved by raising the gun to a height at least $h = \Delta x\left[1 + \frac{\Delta x}{R}\right]$.



Sol.

Key concept: Let an object is projected with initial velocity u at an angle θ above horizontal.



Component of the initial velocity in horizontal direction $u_x = u \cos \theta$

Component of the initial velocity in vertical direction $u_y = u \sin \theta$

Acceleration of the particle $a_y = -g$

Equations of horizontal motion: Range, $R = u \cos \theta t$

Equation of vertical motion: $-h = u \sin \theta \, t - \frac{1}{2} g t^2$

From the above equation,

$$-h = u \sin \theta t - \frac{1}{2} g t^2$$

$$\Rightarrow \quad g t^2 - 2u \sin \theta t - 2h = 0$$

$$\Rightarrow \quad t = \frac{u \sin \theta}{g} \pm \sqrt{\frac{u^2 \sin^2 \theta}{g} + \frac{2h}{g}}$$

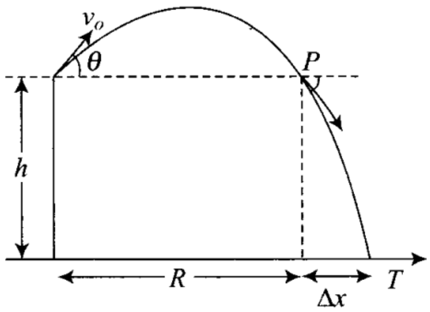
According to the diagram, target T is at horizontal distance x = R +Δx and y = -h from point of projection.If a projectile will have maximum range when it is projected at an angle of 45° to the horizontal and the maximum range will be Maximum range: For range to be maximum it will be assumed to constant and then the rate of change of range with time will be equal to zero as shown.

$$\frac{dR}{d\theta} = 0 \Rightarrow \frac{d}{d\theta}\left[\frac{u^2 \sin 2\theta}{g}\right] = 0$$

$\Rightarrow \quad \cos \theta = 0 \;\; i.e., \; 2\theta = 90° \Rightarrow \; \theta = 45°$

and $R_{max} = (u^2/g)$

where, u is speed of projection of the projectile.



i.e. $\quad R = \dfrac{v_0^2}{g}$ ...(i)

The gun is placed at a height h from the ground so that it can hit the target. Let vertically downward direction is taken as negative.

Horizontal component of initial velocity = $v_0 \cos \theta$

Vertical component of initial velocity = $v_0 \sin \theta$

Displacement along y-axis (taking motion in vertical direction)

$$-h = (v_0 \sin \theta)t + \frac{1}{2}(-g)t^2 \qquad \text{...(ii)}$$

Displacement along x-axis (taking motion in horizontal direction)

$$\Rightarrow \quad t = \frac{(R + \Delta x)}{v_0 \cos \theta} \qquad \text{...(iii)}$$

Substituting value of t in Eq. (ii), we get

$$h = (-v_0 \sin\theta) \times \left(\frac{R + \Delta x}{v_0 \cos\theta}\right) + \frac{1}{2} g \left(\frac{R + \Delta x}{v_0 \cos\theta}\right)^2$$

$$h = -(R + \Delta x) \tan\theta + \frac{1}{2} g \frac{(R + \Delta x)^2}{v_0^2 \cos^2\theta}$$

As angle of projection is $\theta = 45°$, therefore

$$h = -(R + \Delta x) + \tan 45° + \frac{1}{2} g \frac{(R + \Delta x)^2}{v_0^2 \cos^2 45°}$$

$$h = -(R + \Delta x) \times 1 + \frac{1}{2} g \frac{(R + \Delta x)^2}{v_0^2 (1/2)}$$

$$\left(\because \tan 45° = 1 \text{ and } \cos 45° = \frac{1}{\sqrt{2}}\right)$$

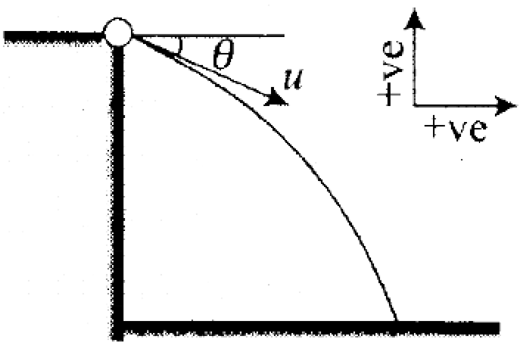
$$h = -(R + \Delta x) + \frac{(R + \Delta x)^2}{R} \qquad \text{[using Eq.(i), } R = v_0^2/g\text{]}$$

$$= -(R + \Delta x) + \frac{1}{R}(R^2 + \Delta x^2 + 2R\Delta x)$$

$$= -R - \Delta x + \left(R + \frac{\Delta x^2}{R} + 2\Delta x\right) = \Delta x + \frac{\Delta x^2}{R}$$

$$h = \Delta x\left(1 + \frac{\Delta x}{R}\right)$$

Important point: We should not confuse with the positive direction of motion. May be vertically upward direction or vertically downward direction is taken as positive according to convenience. And this problem can also be solved by taking motion from point P to T. From point P in the diagram projection at speed v0 at an angle θ below horizontal with height h and horizontal range Δx. Then this will be analyzed from the alongside diagram

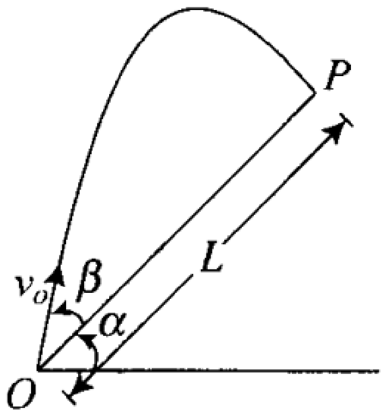


Q31. A particle is projected in air at an angle β to a surface which itself is inclined at an angle α to the horizontal (figure).

(a) Find an expression of range on the plane surface (distance on the plane from the point of projection at which particle will hit the surface).

(b) Time of flight.

(c) **β** at which range will be maximum



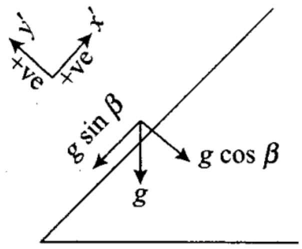
Sol.

Key concept: The Projectile Motion on an Inclined Plane

In analyzing the projectile in inclined plane, we can assign our frame of reference parallel and perpendicular to the inclined plane.

In $x' - y'$ plane, $a'_x = -g\sin\beta$; $a'_y = -g\cos\beta$

In projectile motion along inclined plane, the displacement of the particle normal to the plane direction will be zero.

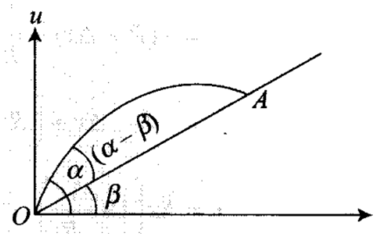


Since $y' = v_0\sin(\alpha-\beta)t - \dfrac{1}{2}g\cos\beta t^2$ at $t = T$, $y' = 0$, where

$$T = \text{Time of flight} \Rightarrow T = \frac{2v_0\sin(\alpha-\beta)}{g\cos\beta}$$

$$\Rightarrow \quad x' = v_0\cos(\alpha-\beta) - \frac{1}{2}g\sin\beta t^2$$

Putting $t = T$ from above equation, the range up the incline is



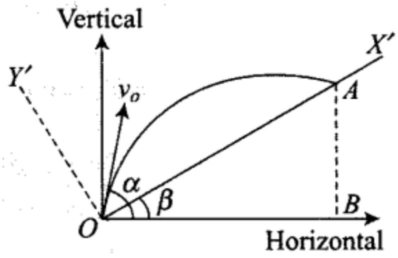
$$R = v_0\cos(\alpha-\beta)\frac{2v_0\sin(\alpha-\beta)}{g\cos\beta} - \frac{1}{2}g\sin\beta\left\{\frac{2V_0\sin(\alpha-\beta)}{g\cos\beta}\right\}^2$$

$$\Rightarrow \quad R = \frac{v_0^2[\sin(2\alpha-\beta)-\sin\beta]}{g\cos^2\beta}$$

For a given value of v_0 and β, range is maximum when $\sin(2\alpha - \beta)$ is naximum.

$$i.e., \quad \sin(2\alpha-\beta)=1 \Rightarrow 2\alpha-\beta=\frac{\pi}{2} \Rightarrow \alpha=\frac{\beta}{2}+\frac{\pi}{4}$$

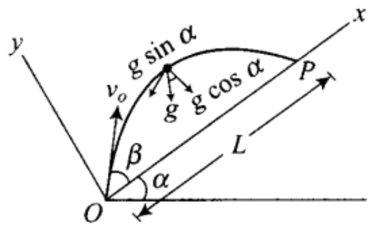
So when $\alpha = \frac{\beta}{2}+\frac{\pi}{4}$, range is maximum on an inclined plane.



$$\therefore \quad R_{\max} = \frac{v_0^2(1-\sin\beta)}{g(1-\sin^2\beta)}$$

$$\Rightarrow \quad R_{\max} = \frac{v_0^2}{g(1+\sin\beta)} \text{ up the plane}$$

Let us discuss motion along incline plane in detail.
Mutually perpendicular x- and y-axes are shown in the diagram.
Assume particle moves from point O to P in time T which is time of flight.
Let as solve part (b) first.



(b) Considering motion along Y-direction perpendicular to the inclined plane.
For the journey O to P.
$y = 0$, $u_y = v_0 \sin\beta$, $a_y = -g\cos\alpha$, $t = T$
Applying equation,

$$y = u_y t + \frac{1}{2}a_y t^2$$

$$\Rightarrow \quad 0 = v_0\sin\beta T + \frac{1}{2}(-g\cos\alpha)T^2$$

$$\Rightarrow \quad T\left[v_0\sin\beta - \frac{g\cos\alpha}{2}T\right] = 0$$

$$\Rightarrow \quad T = \frac{2v_0\sin\beta}{g\cos\alpha}$$

As $T = 0$, corresponding to point O.

Hence, $T = \text{time of flight} = \dfrac{2v_0\sin\beta}{g\cos\alpha}$

(a) Considering motion in **X**-direction parallel to the inclined plane for journey O to P along X.

$$x = L, u_x = v_0 \cos \beta, \quad a_x = -g \sin \alpha$$

$$t = T = \frac{2v_0 \sin \beta}{g \cos \alpha}$$

$$x = u_x t + \frac{1}{2} a_x t^2$$

$$\Rightarrow \quad L = v_0 \cos \beta T + \frac{1}{2}(-g \sin \alpha) T^2$$

$$\Rightarrow \quad L = v_0 \cos \beta T - \frac{1}{2} g \sin \alpha T^2$$

$$= T\left[v_0 \cos \beta - \frac{1}{2} g \sin \alpha \times \frac{2v_0 \sin \beta}{g \cos \alpha}\right]$$

$$= \frac{2v_0 \sin \beta}{g \cos \alpha}\left[v_0 \cos \beta - \frac{v_0 \sin \alpha \sin \beta}{\cos \alpha}\right]$$

$$= \frac{2v_0^2 \sin \beta}{g \cos^2 \alpha}[\cos \beta \cdot \cos \alpha - \sin \alpha \cdot \sin \beta]$$

$$. \Rightarrow \quad L = \frac{2v_0^2 \sin \beta}{g \cos^2 \alpha} \cos(\alpha + \beta)$$

(c) The angle β for which the (L) to be maximum, $\sin \beta \cdot \cos(\alpha + \beta)$ should be maximum.

Let $\quad z = \sin \beta \cdot \cos(\alpha + \beta)$

$$= \sin \beta[\cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta]$$

$$= \frac{1}{2}[\cos \alpha \cdot \sin 2\beta - 2 \sin \alpha \cdot \sin^2 \beta]$$

$$= \frac{1}{2}[\sin 2\beta \cdot \cos \alpha - \sin \alpha(1 - \cos 2\beta)]$$

$$= \frac{1}{2}[\sin 2\beta \cdot \cos \alpha - \sin \alpha + \sin \alpha \cdot \cos 2\beta]$$

$$= \frac{1}{2}[\sin 2\beta \cdot \cos \alpha + \cos 2\beta \cdot \sin \alpha - \sin \alpha]$$

$$= \frac{1}{2}[\sin(2\beta + \alpha) - \sin \alpha]$$

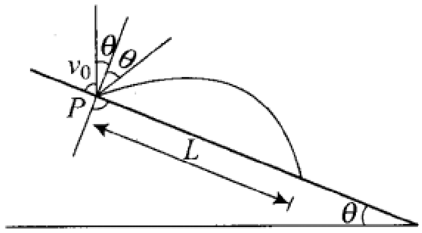
For z to be maximum,

$$\sin(2 = \beta + \alpha) = \text{maximum} = 1$$

$$\Rightarrow \quad 2\beta + \alpha = \frac{\pi}{2} \quad \text{or,} \quad \beta = \frac{\pi}{4} - \frac{\alpha}{2}$$

Q32. A particle falling vertically from a height hits a plane surface inclined to horizontal at an angle **θ** with speed v0 and rebounds elastically. Find the distance along the plane where it will hit second time.

Sol:



Key concept: If in a collision, kinetic energy after collision is equal to kinetic energy before collision, the collision is said to be perfectly elastic. So, when a particle collide or rebound elastically after and before striking speed will remain same.
After that it will be only motion down an inclined plane. Conventionally, angles measured in the anticlockwise direction are taken positive, and those measured in the clockwise direction are taken negative.

$\therefore$ Repeating β by $-\beta$ in the above results, the new results can be obtained as follows:

(a) Time of flight $(T) = \dfrac{2u\sin(\alpha+\beta)}{g\cos\beta}$

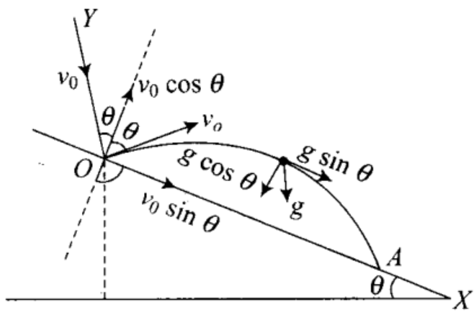
(b) Range down an inclined plane: $R = \dfrac{2u^2\sin(\alpha+\beta)\cos\alpha}{g\cos^2\beta}$

(c) Maximum range down an inclined plane $R_{\max} = \dfrac{u^2}{g(1-\sin\beta)}$

So, we select x and y-axes as shown in the diagram.
Along y-axis motion of the projectile from O to A.

$y = 0, u_y = v_0\cos\theta$

$a_y = -g\cos\theta, t = T$



Applying equation of kinematics (along y- axis),

$$y = u_y t + \frac{1}{2}a_y t^2$$

$$\Rightarrow \quad 0 = v_0\cos\theta T + \frac{1}{2}(-g\cos\theta)T^2$$

$$\Rightarrow \quad T\left[v_0\cos\theta - \frac{g\cos\theta T}{2}\right] = 0$$

$$T = \frac{2v_0\cos\theta}{g\cos\theta}$$

As T = 0, corresponds to point <9.
Hence, $T = 2v_0/g$
Now considering motion along x-axis.
$x = L, u_x = v_0 \sin\theta, a_x = g\sin\theta, t = T = 2v_0/g$

Applying equation of kinematics (displacement along x-axis),

$$x = u_x t + \frac{1}{2} a_x t^2$$

$$\Rightarrow \quad L = v_0 \sin \theta t + \frac{1}{2} g \sin \theta t^2$$

$$= (v_0 \sin \theta)(T) + \frac{1}{2} g \sin \theta T^2$$

$$= (v_0 \sin \theta)\left(\frac{2v_0}{g}\right) + \frac{1}{2} g \sin \theta \times \left(\frac{2v_0}{g}\right)$$

$$= \frac{2v_0^2}{g} \sin \theta + \frac{1}{2} g \sin \theta \times \frac{4v_0^2}{g^2} = \frac{2v_0^2}{g}[\sin \theta + \sin \theta]$$

$$= \frac{4v_0^2}{g} \sin \theta$$

Q33. A girl riding a bicycle with a speed of 5 m/s towards north direction, observes rain falling vertically down. If she increases her speed to 10 m/s, rain appears to meet her at 45° to the vertical. What is the speed of the rain? In what direction does rain fall as observed by a ground based observer?

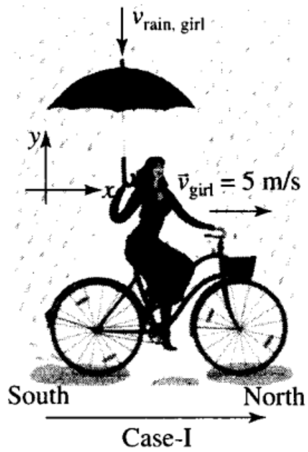
Sol: V_{rg} is the velocity of rain appears to the girl.

We must draw all vectors in the reference frame of ground-based observer.

Assume north to be $\hat{i}$ direction and vertically downward to be $(-\hat{j})$.

Let the rain velocity

$$v_r = a\hat{i} + b\hat{j}$$



Case I: According to the problem, velocity of girl $= v_g = (5 \text{ m/s})\hat{i}$

Let v_{rg} = velocity of rain w.r.t girl

$$= v_r - v_g = (a\hat{i} + b\hat{j}) - 5\hat{i} = (a-5)\hat{i} + b\hat{j}$$

According to question, rain appears to fall vertically downward.

Hence, $a - 5 = 0 \Rightarrow a = 5$

Case II: Now velocity of the girl after increasing her speed,

$$v_g = (10 \text{ m/s})\hat{i}$$

$$\therefore \quad v_{rg} = v_r - v_g$$

$$= (a\hat{i} + b\hat{j}) - 10\hat{i} = (a-10)\hat{i} + b\hat{j}$$

According to question rain appears to fall at 45° to the vertical, hence $\tan 45° = \frac{b}{a-10} = 1$

$$\Rightarrow \quad b = a - 10 = 5 - 10 = -5$$

Hence, velocity of rain $= a\hat{i} + b\hat{j}$

$$\Rightarrow \quad v_r = 5\hat{i} - 5\hat{j}$$

Speed of rain

$$= |v_r| = \sqrt{(5)^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2} \text{ m/s}$$

Q34. A river is flowing due east with a speed 3 m/s. A swimmer can swim in still water at a speed of 4 m/s (figure).

(a) If swimmer, starts swimming due north, what will be his resultant velocity (magnitude and direction)?

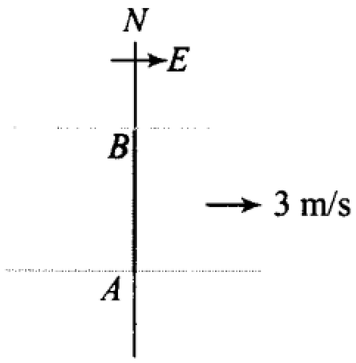
(b) If he wants to start from point A on south bank –

and reach opposite point B on north bank,

(i) which direction should he swim?

(ii) what will be his resultant speed?

(c) From two different cases as mentioned in (a) and (b) above, in which case will he reach opposite bank in shorter time?



Sol:

Key concept: Analysis of Relative Velocity in Different Situations:

The motion of a boat (or a swimmer) in running water, say, a river.

$\vec{V}_{MR}$ = velocity of man relative to river flow

$\vec{V}_{RE}$ = velocity of river flow relative to earth

$\vec{V}_{ME}$ = velocity of man relative to earth

Denoting $|\vec{V}_{MR}| = v, |\vec{V}_{RE}| = u, |\vec{V}_{ME}| = v_R$

Case (i): Man swims towards flow

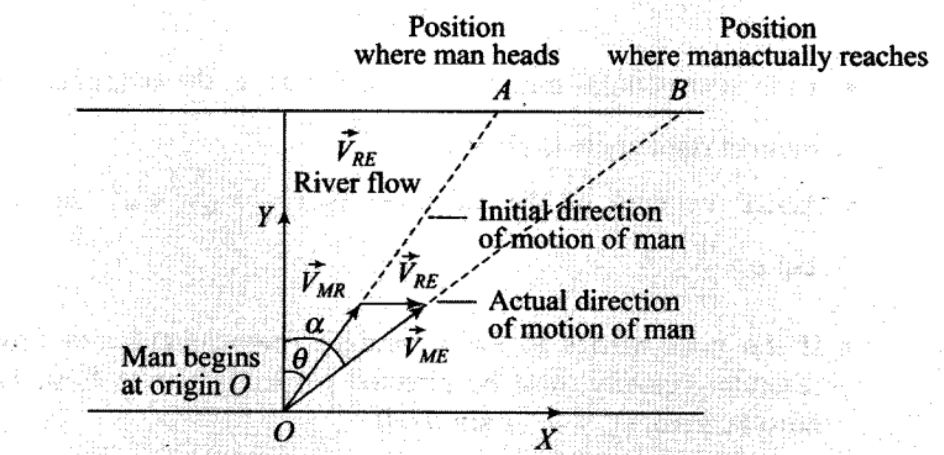


Fig. (a)

In unit vector notation.

$$\vec{V}_{MR} = v\sin\theta\hat{i} + v\cos\theta\hat{j};\ \vec{V}_{ME} = u\hat{i} \quad \text{as } \vec{V}_{ME} = \vec{V}_{MR} + \vec{V}_{RE}$$

$$\vec{V}_R = (v\sin\theta\hat{i} + v\cos\theta\hat{j}) + u\hat{i}$$

$$= (v\sin\theta + u)\hat{i} + v\cos\theta\hat{j}$$

Suppose the man reaches the point $B(x, y)$ and time of motion is t. We have

$$x = v_{Rx}t = (v\sin\theta + u)t \quad \text{...(i)}$$

$$y = v_{Ry}t = v\cos\theta t \quad \text{...(ii)}$$

We have used the fact that the motion along the x- and y-axes takes place with the respective to x- and y-components of resultant velocity.

Case (ii): Man swims against flow

$$\vec{v}_R = (-v\sin\theta + u)\hat{i} + v\cos\theta\hat{j}$$

$$x = v_{Rx}\,t = (u - v\sin\theta)t \quad \text{...(iii)}$$

$$y = v_{Ry}\,t = v\cos\theta\,t \quad \text{...(iv)}$$

- If width of the river is d, then time taken to cross the river is $t = \dfrac{y}{v_{Ry}} = \dfrac{d}{v\cos\theta}$

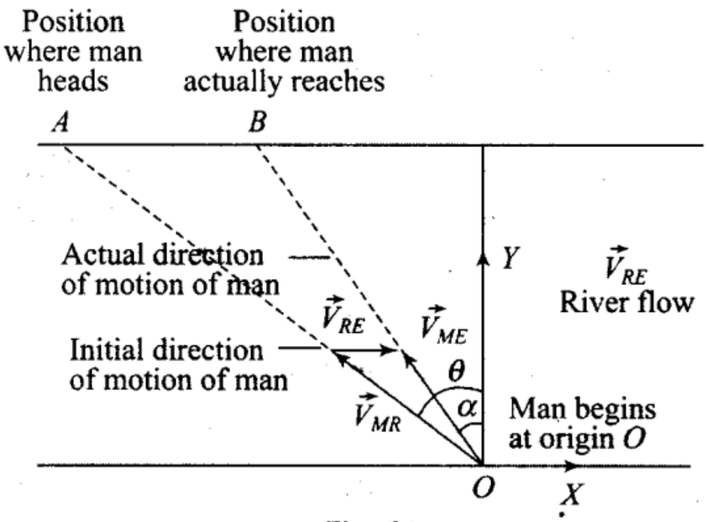
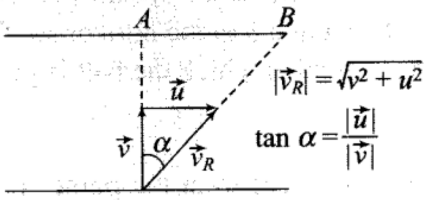
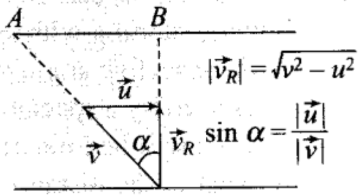


Fig. (b)

- Time of crossing is minimum if $\cos\theta = 1$, i.e. the swimmer begins to swim at right angle to flow $t_{\text{min.}} = \dfrac{d}{v}$.
- Actual velocity of the swimmer is $|\vec{v}_R| = \sqrt{v^2 + u^2}$ at an angle $\tan\alpha = \dfrac{|\vec{u}|}{|\vec{v}|}$.
- If the man wishes to swim straight (perpendicular to flow), his resultant velocity must be directed perpendicular to flow, *i.e.*, v_{Rx}, must be zero. $v_{Rx} = u - v\sin\theta = 0$



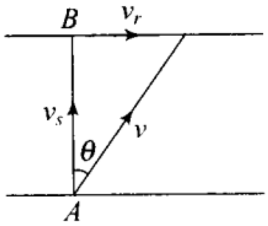
Crossing the river
in minimum time
Fig. (c)



Crossing the river
along shortest path
Fig. (d)

According to the problem,
Speed of the river (v_r) = 3 m/s (east)
Velocity of swimmer (with respect to the river), (v_S) = 4 m/s (north)
(a) As the swimmer starts swimming due north, then y-component of his resultant velocity is 4 m/s (north) and x-component (produced by river flow) is 3 m/s (east). Then his resultant velocity will be

$$v = \sqrt{v_r^2 + v_s^2} = \sqrt{(3)^2 + (4)^2}$$
$$= \sqrt{9+16} = \sqrt{25} = 5 \text{ m/s}$$
$$\tan \theta = \frac{v_r}{v_s} = \frac{3}{4} = 0.75 = \tan 36°54'$$

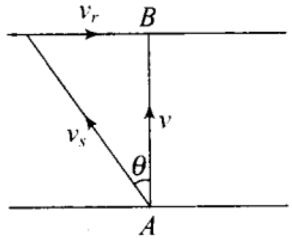


Hence, $\theta = 36°54'$ N or $\theta = 37°$ N

(b) If he wants to reach on the point directly opposite to him (points B), the swimmer should swim at an angle θ of north.
Resultant speed of the swimmer

Resultant speed of the swimmer

$$v = \sqrt{v_s^2 - v_r^2} = \sqrt{(4)^2 - (3)^2}$$
$$= \sqrt{16-9} = \sqrt{7} \text{ m/s}$$
$$\tan \theta = \frac{v_r}{v} = \frac{3}{\sqrt{7}}$$
$$\theta = \tan^{-1}\left(\frac{3}{\sqrt{7}}\right) \text{of north}$$



(c) In case (a),

Time taken by the swimmer to cross the river, $t_1 = \frac{d}{v_s} = \frac{d}{4} s$

In case (b),
Time taken by the swimmer to cross the river,

$$t_1 = \frac{d}{v} = \frac{d}{\sqrt{7}}$$

As $\frac{d}{4} < \frac{d}{\sqrt{7}}$, therefore $t_1 < t_2$.

Hence, the swimmer will cross the river in shorter time in case (a).

Q32. A cricket fielder can throw the Cricket ball with a speed v_0. If he throws the ball while running with speed u at an angle θ to the horizontal, find

(a) the effective angle to the horizontal at which the ball is projected in air as seen by a spectator.

(b) what will be time of flight?

(c) what is the distance (horizontal range) from the point of projection at which the ball will land?

(d) find **θ** at which he should throw the ball that would maximize the horizontal range as found in (c).

(e) how does **θ** for maximum range change if u > v_0, u = v_0, u < v_0?

(f) how does **θ** in (e) compare with that for u = 0 (i.e., 45°)?

Sol:. The observer on ground (spectator) observes that the x-component of ball is more because of the speed of fielder. As shown in the adjacent diagram,

So, initial velocity in x-direction

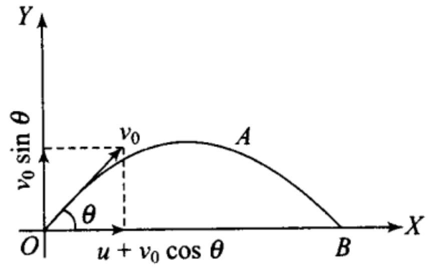
So, initial velocity in x-direction,

$$u_x = u + v_0 \cos\theta$$

Initial velocity in y-direction,

$$u_y = v_0 \sin\theta$$

where angle of projection is θ.



(a) Now, the angle of projection with horizontal seen by spectator will be

$$\tan\theta = \frac{u_y}{u_x} = \frac{u_0 \sin\theta}{u + u_0 \cos\theta}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{v_0 \sin\theta}{u + v_0 \cos\theta}\right)$$

(b) As net displacement along y-axis is zero over time period T (time of flight).

$$y = 0,\ u_y = v_0 \sin\theta,\ a_y = -g,\ t = T$$

We know that $y = u_y t + \frac{1}{2} a_y t^2$

$$\Rightarrow \quad 0 = v_0 \sin\theta T + \frac{1}{2}(-g)T^2$$

$$\Rightarrow \quad T\left[v_0 \sin\theta - \frac{g}{2}T\right] = 0 \Rightarrow T = 0, \frac{2v_0 \sin\theta}{g}$$

$T = 0$, corresponds to point O.

Hence, $T = \dfrac{2u_0 \sin\theta}{g}$

(c) Horizontal range,

$$R = (u + v_0 \cos\theta)T = (u + v_0 \cos\theta)\frac{2v_0 \sin\theta}{g}$$

$$= \frac{v_0}{g}[2u \sin\theta + v_0 \sin 2\theta]$$

(d) For horizontal range to be maximum, $\frac{dR}{d\theta} = 0$

$$\Rightarrow \quad \frac{v_0}{g}[2u \cos\theta + v_0 \cos 2\theta \times 2] = 0$$

$$\Rightarrow \quad 2u\cos\theta + 2v_0[2\cos^2\theta - 1] = 0$$

$$\Rightarrow \quad 4v_0 \cos^2\theta + 2u\cos\theta - 2v_0 = 0$$

$$\Rightarrow \quad 2v_0\cos^2\theta + u\cos\theta - v_0 = 0$$

$$\Rightarrow \quad \cos\theta = \frac{-u \pm \sqrt{u^2 + 8v_0^{\,2}}}{4v_0}$$

$$\Rightarrow \quad \theta_{max} = \cos^{-1}\left[\frac{-u \pm \sqrt{u^2+8v_0^{\,2}}}{4v_0}\right] = \cos^{-1}\left[\frac{-u + \sqrt{u^2+8v_0^{\,2}}}{4v_0}\right]$$

(e) If $u = v_0$,

$$\cos\theta = \frac{-v_0 \pm \sqrt{v_0^2 + 8v_0^2}}{4v_0} = \frac{-1+3}{4} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = 60°$$

If $u << v_0$, then $8v_0^2 + u^2 \approx 8v_0^2$

$$\theta_{max} = \cos^{-1}\left[\frac{-u \pm 2\sqrt{2}v_0}{4v_0}\right] = \cos^{-1}\left[\frac{1}{\sqrt{2}} - \frac{u}{4v_0}\right]$$

If $u << v_0$ then $\theta_{max} \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

If $u > u_0$ and $u >> v_0$

$$\theta_{max} = \cos^{-1}\left[\frac{-u \pm u}{4v_0}\right] = 0$$

$$\Rightarrow \quad \theta_{max} = \frac{\pi}{2}$$

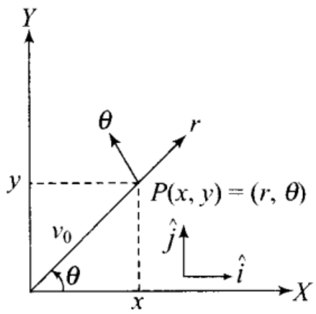
(f) If $u = 0, \theta_{max} = \cos^{-1}\left[\frac{0 \pm \sqrt{8v_0^2}}{4v_0}\right]$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$$

Hence, when $u = 0$, $\theta \geq 45°$

36. Motion in two dimensions, in a plane can be studied by expressing position, velocity and acceleration as vectors in Cartesian coordinates $\vec{A} = \vec{A}_x\hat{i} + \vec{A}_y\hat{j}$, where $\hat{i}$ and $\hat{j}$ are unit vectors along x- and y-directions, respectively and

A_x and A_y are corresponding components of A. Motion can also be studied by expressing vectors in circular polar coordinates as $\vec{A} = \vec{A}_r\hat{r} + A_\theta\hat{\theta}$, where $\hat{r} = \frac{\vec{r}}{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$ and $\hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}$ are unit vectors along direction in which r and θ are increasing.



(a) Express $\hat{i}$ and $\hat{j}$ in terms of $\hat{r}$ and $\hat{\theta}$.

(b) Show that both $\hat{r}$ and $\hat{\theta}$ are unit vectors and are perpendicular to each other.

(c) Show that $\frac{d}{dt}(\hat{r}) = \omega\hat{\theta}$, where

$$\omega = \frac{d\theta}{dt} \text{ and } \frac{d}{dt}(\hat{\theta}) = -\theta\hat{r}.$$

(d) For a particle moving along a spiral given by $r = a\theta\hat{r}$, where $a = 1$ (unit), find dimensions of a.

(e) Find velocity and acceleration in polar vector representation for particle moving along spiral described in (d) above.

Sol. (a) According to the problem, unit vector

$$\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j} \quad \ldots\text{(i)}$$

$$\hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j} \quad \ldots\text{(ii)}$$

Multiplying Eq. (i) by $\sin\theta$ and Eq. (ii) with $\cos\theta$ and adding

$$\Rightarrow \hat{r}\sin\theta + \hat{\theta}\cos\theta = \sin\theta\cdot\cos\theta\hat{i} + \sin^2\theta\hat{j} + \cos^2\theta\hat{j} - \sin\theta\cdot\cos\theta\hat{i}$$

$$= \hat{j}(\cos^2\theta + \sin^2\theta) = \hat{j}$$

$$\hat{r}\sin\theta + \hat{\theta}\cos\theta = \hat{j}$$

By Eq. (i) $x\cos\theta -$ Eq.(ii) $x\sin\theta$

$$n(\hat{r}\cos\theta - \hat{\theta}\sin\theta) = \hat{i}$$

(b) $\hat{r}\cdot\hat{\theta} = (\cos\theta\hat{i} + \sin\theta\hat{j})\cdot(-\sin\theta\hat{i} + \cos\theta\hat{j})$

$$= -\cos\theta\cdot\sin\theta + \sin\theta\cdot\cos\theta = 0$$

$\Rightarrow \quad \theta = 90°$, angle between $\hat{r}$ and $\hat{\theta}$.

(c) Given, $\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}(\cos\theta\hat{i} + \sin\theta\hat{j})$$

$$= -\sin\theta\cdot\frac{d\theta}{dt}\hat{i} + \cos\theta\cdot\frac{d\theta}{dt}\hat{j}$$

$$= \omega[-\sin\theta\hat{i} + \cos\theta\hat{j}] \quad \left[\because \theta = \frac{d\theta}{dt}\right]$$

(d) Given, $r = a\theta\hat{r}$, here writing dimension.

$$[r] = [a][\theta][\hat{r}] \Rightarrow L = \frac{[a]}{1}$$

$$\Rightarrow \quad [a] = L = [M^0L^1T^0]$$

(e) Given, $a = 1$ unit, $r = \theta\hat{r} = \theta[\cos\theta\hat{i} + \sin\theta\hat{j}]$

By differentiating this equation, we get

$$\text{Velocity} \quad v = \frac{dr}{dt} = \frac{d\theta}{dt}\hat{r} + \theta\frac{d}{dt}\hat{r}$$

$$= \frac{d\theta}{dt}\hat{r} + \theta\frac{d}{dt}[(\cos\theta\hat{i} + \sin\theta\hat{j})]$$

$$= \frac{d\theta}{dt}\hat{r} + \theta\left[(-\sin\theta\hat{i} + \cos\theta\hat{j})\frac{d\theta}{dt}\right]$$

$$= \frac{d\theta}{dt}\hat{r} + \theta\hat{\theta}\omega = \omega\hat{r} + \omega\theta\hat{\theta}$$

By differentiating this equation, we get
Acceleration,

$$a = \frac{d}{dt}[\omega\hat{r} + \omega\theta\hat{\theta}]$$

$$= \frac{d}{dt}\left[\frac{d\theta}{dt}\hat{r} + \frac{d\theta}{dt}(\theta\hat{\theta})\right]$$

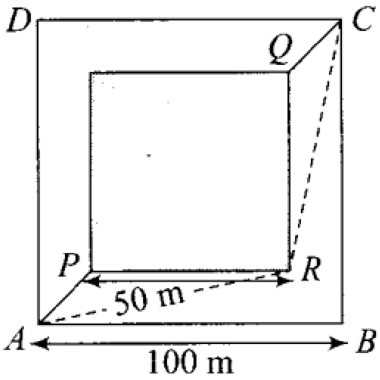
$$= \frac{d^2\theta}{dt^2}\hat{r} + \frac{d\theta}{dt}\cdot\frac{d\hat{r}}{dt} + \frac{d^2\theta}{dt^2}\theta\hat{\theta} + \frac{d\theta}{dt}\frac{d}{dt}(\theta\hat{\theta})\frac{d^2\theta}{dt^2}\hat{r}$$

$$+ \omega[-\sin\theta\hat{i} + \sin\theta\hat{j}] + \frac{d^2\theta}{dt^2}\theta\hat{\theta} + \frac{\omega d}{dt}(\theta\hat{\theta})$$

$$= \frac{d^2\theta}{dt^2}\hat{r} + \omega^2\hat{\theta} + \frac{d^2\theta}{dt^2} \times \theta\hat{\theta} + \omega^2\hat{\theta} + \omega^2\theta(-\hat{r})$$

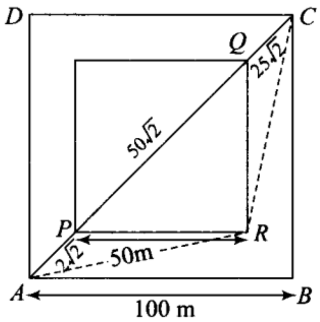
$$\left(\frac{d^2\theta}{dtv^2} - \omega^2\right)\hat{r} + \left(2\omega^2 + \frac{d^2\theta}{dt^2}\theta\right)\theta$$

Q37. A man wants to reach from A to the opposite comer of the square C. The sides of the square are 100 m. A central square of 50 m x 50 m is filled with sand. Outside this square, he can walk at a speed 1 m/s. In the central square, he can walk only at a speed of v m/s (v < 1). What is smallest value of v for which he can reach faster via a straight path through the sand than any path in the square outside the sand?



Sol. As shown in the adjacent diagram.

The man walks through the sand on the path $APQC$ via straight line, so, time taken by him to go from A to C is



$$T_{\text{sand}} = \frac{AP + QC}{1} + \frac{PQ}{v}$$

$$= \frac{25\sqrt{2} + 25\sqrt{2}}{1} + \frac{50\sqrt{2}}{v}$$

$$= 50\sqrt{2} + \frac{50\sqrt{2}}{v} = 50\sqrt{2}\left(\frac{1}{v} + 1\right)$$

Clearly from figure the shortest path outside the sand will be ARC. Time taken to go from A to C via this path is

$$T_{\text{outside}} = \frac{AR + RC}{1} s$$

Clearly, $AR = \sqrt{75^2 + 25^2} = \sqrt{75 \times 75 + 25 \times 25}$

$$= 5 \times 5\sqrt{9 + 1} = 25\sqrt{10}\,\text{m}$$

$$RC = AR = \sqrt{75^2 + 25^2} = 25\sqrt{10}\,\text{m}$$

$\Rightarrow \quad T_{\text{outside}}\ 2AR = 2 \times 25\sqrt{10}\,\text{s} = 50\sqrt{10}\,\text{s}$

For $T_{\text{sand}} < T_{\text{outside}}$

$$\Rightarrow \quad 50\sqrt{2}\left(\frac{1}{v} + 1\right) < 2 \times 25\sqrt{10}$$

$$\Rightarrow \quad \frac{2\sqrt{2}}{2}\left(\frac{1}{v} + 1\right)\sqrt{10}$$

$$\Rightarrow \quad \frac{1}{v} + 1 < \frac{2\sqrt{10}}{2\sqrt{2}} = \frac{\sqrt{5}}{2} \times 2 = \sqrt{5}$$

$$\Rightarrow \quad \frac{1}{v} < \frac{\sqrt{5}}{2} \times 2 - 1 \Rightarrow \frac{1}{v} < \sqrt{5} - 1$$

$$\Rightarrow \quad v > \frac{1}{\sqrt{5} - 1} \approx 0.81\,\text{m/s} \Rightarrow v > 0.81\,\text{m/s}$$