CBSE Class 11 Mathematics

Important Questions

Chapter 4

Principle of Mathematical Induction

4 Marks Questions

1. For every integer n, prove that $7^n - 3^n$ *is* divisible by 4.

Ans. P(n) : $7^n - 3^n$ is divisible by 4

For n=1

P(1) : $7^1 - 3^1 = 4$ which is divisible by Thus, P(1) is true

Let P(k) be true

$7^k - 3^k$ *is* divisible by 4

$7^k - 3^k = 4\lambda$, where $\lambda \in N$ (i)

we want to prove that P (k+1) is true whenever P(k) is true

$7^{k+1} - 3^{k+1} = 7^k.7 - 3^k.3$

$= (4\lambda + 3^k).7 - 3^k.3$(from i)

$= 28\lambda + 7.3^k - 3^k.3$

$= 28\lambda + 3^k(7-3)$

$= 4(7\lambda + 3^k)$

Hence

$7^{k+1} - 3^{k+1}$ is divisible by 4

thus P (k+1) is true when P(k) is true.

Therefore by P.M.I. the statement is true for every positive integer n.

2. Prove that $n(n+1)(n+5)$ is multiple of 3.

Ans. $P(n): n(n+1)(n+5)$ is multiple of 3

for n=1

P(1) : 1(1+1) (1+5) = 12 is multiple of 3

let P(k) be true

P(k) : K (k+1) (k+5) is muetiple of 3

$\Rightarrow$ k(k+1)(k+5)=3λ where $\lambda \in N$ (i)

we want to prove that result is true for n=k+1

P(k+1) : (k+1)(k+2)(k+6)

$\Rightarrow$ (K+1)(k+2)(k+6)= $\left[(k+1)(k+2)\right](k+6)$

= k(k+1)(k+2)+6(k+1)(k+2)

=k(k+1)(k+5-3)+6(k+1)(k+2)

=k(k+1)(k+5)-3k(k+1)+6(k+1)(K+2)

=k(k+1)(k+5)+(k+1)$\left[6(k+2)-3k\right]$

=k(k+1)(k+5)+(k+1)(3k+12)

=k(k+1)(k+5)+3(k+1)(k+4)

=3λ+3 (k+1)(k+4) (from i)

=3$\left[\lambda+(K+1)(K+4)\right]$ which is multiple of three

Hence P(k+1) is multiple of 3 .

3. Prove that $10^{2n-1}+1$ is divisible by 11

Ans. $P(n): 10^{2n-1} + 1$ is divisible by 11

for n=1

P(1) $= 10^{2\times 1-1} + 1 = 11$ is divisible by 11 Hence result is true for n=1

let P(k) be true

P(k) : $10^{2k-1} + 1$ is divisible by 11 1

$\Rightarrow$ $10^{2k-1} + 1 = 11\lambda$ where $\lambda \in$ N(i)

we want to prove that result is true for n= k+

$=10^{2(k+1)-1} + 1 = 10^{2k+2-1} + 1$

$= 10^{2k+1} + 1$

$= 10^{2k}.10^{1} + 1$

$= (110\lambda - 10).10 + 1($*from* i)

$= 1100\lambda - 100 + 1$

$= 1100\lambda - 99$

$= 11(100\lambda - 9)$ *is* divisible by 11

Hence by P.M.I. P (k+1) is true whenever P(k) is true.

4. Prove $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{n}\right) = (n+1)$

Ans. $let P(n): \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{n}\right) = (n+1)$

for n=1

P(1): $\left(1+\frac{1}{1}\right) = (1+1) = 2$

which is true

let P(k) be true

P(k) : $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{k}\right)=(k+1)$

we want to prove that P(k+1) is true

P(k+1): $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)..\left(1+\frac{1}{k+1}\right)=(k+2)$

$L.H.S.=\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)....\left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right)$

$=(k+1)\left(1+\frac{1}{k+1}\right)$ $\quad [from(1)]$

$=(k+1)\left(\frac{k+1+1}{K+1}\right)$

=(K+2)

thus P(k+1) is true whenever

P (K) is true.

5. Prove 1.2+2.3+3.4+--+n $(n+1)=\frac{n(n+1)(n+2)}{3}$

Ans. $p(n):1.2+2.3+--n(n+1)=\frac{n(n+1)(n+2)}{3}$

for $n=1$

$p(1):1(1+1)=\frac{1(1+1)(1+2)}{3}$

$p(1)=2=2$

hence $p(1)$ be true

$$p(k):1.2+2.3+---+k(k+1)=\frac{k(k+1)(k+2)}{3}.........(i)$$

we want to prove that

$p(k+1)$:

$$1.2+2.3+---+(k+1)(k+2)=\frac{(k+1)(k+2)(k+3)}{3}$$

$L.H.S.$

$$=1.2+2.3+---+k(k+1)+(k+1)(k+2)$$

$$=\frac{k(k+1)(k+2)}{3}+\frac{(k+1)(k+2)}{1} \qquad [from(i)]$$

$$\frac{k(k+1)(k+2)+3(k+1)(k+2)}{3}$$

$$\frac{(k+1)(k+2)[k+3]}{3}$$

hence $\mathrm{p}(k+1)$ is true whenenes $p(k)$ is true

6. Prove (2n+7)<(n+3)^2

Ans. $p(n):(2n+7)<(n+3)^2$

for $n=1$

$9<(4)^2$

$9<16$

which is true

let p(k) be true

$(2k+7)<(k+3)^2$

now

$2(k+1)+7=(2k+7)+2$

$<(k+3)^2+2=k^2+6k+11$

$<k^2+8k+16=(k+4)^2$

$=(k+3+1)^2$

$\therefore p(k+1): 2(k+1)+7<(k+1+3)^2$

$\Rightarrow p(k+1)$ is true, when ever $p(k)$ is true

hence by PMI $p(k)$ is true for all $n \in N$

7. Prove $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+..+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$

Ans. $p(n): \frac{1}{1.4}+\frac{1}{4.7}+----+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$

for $n=1$

$p(1): \frac{1}{(3-2)(3+1)}=\frac{1}{(3+1)}=\frac{1}{4}$

which is true

let $p(k)$ be true

$p(k): \frac{1}{1.4}+\frac{1}{4.7}+----+\frac{1}{(3k-2)(3k+1)}=\frac{k}{(3k+1)}$ ....(i)

we want to prove that p$(k+1)$ is true

$$p(k+1): \frac{1}{1.4}+\frac{1}{4.7}+---+\frac{1}{(3k+1)(3k+4)}=\frac{k+1}{(3k+4)}$$

$L.H.S.$

$$=\frac{1}{1.4}+\frac{1}{4.7}+---+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)} \qquad [from.......(i)]$$

$$=\frac{k(3k+4)+1}{(3k+1)(3k+4)}$$

$$=\frac{3k^2+4k+1}{(3k+1)(3k+4)}=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$p(k+1)$ is true whenever $p(k)$ is true.

8. Prove 1.2+2.2^2+3.2^3+…+n.2^n = (n-1)2^{n+1}+2

Ans. $p(n): 1.2+2.2^2+3.2^3+---+n.2^n=(n-1)2^{n+1}+2$

$p(n): 1.2+2.2^2+3.2^3+---+n.2^n=(n-1)2^{n+1}+2$

for $n=1$

$p(1): 1.2^1=(1-1)2^2+2$

$2=2$ which is true

let $p(k)$ be true

$p(k): 1.2+2.2^2+---+k.2^k=(k-1)2.^{k+1}+2..........(i)$

we want to prove that $p(k+1)$ is true

$p(k+1): 1.2+2.2^2+---+(k+1)2^{k+1}=k.2^{k+2}+2$

$L.H.S.$

$1.2+2.2^2+---+k.2^k+(k+1)2^{k+1}$ $\quad [\text{from} ....(i)]$

$=(k-1)2^{k+1}+2+(k+1)2^{k+1}$**c**

$=2^{k+1}(k-1+k+1)+2$

$=2^{k+2}k+2$

This $p(k+1)$ is true whenever $p(k)$ is true

9. Prove that 2.7^n + 3.5^n – 5 is divisible by 24 $\forall\ n \in N$

Ans. P(n) : $2.7^n + 3.5^n - 5$ is divisible by 24

for n = 1

P (1) : $2.7^1 + 3.5^1 - 5 = 24$ is divisible by 24

Hence result is true for n = 1

Let P (K) be true

P (K) : $2.7^K + 3.5^K - 5$

$\Rightarrow 2.7^K+3.5^K-5=24\lambda$ when $\lambda \in N$ (i)

we want to prove that P (K+!) is True whenever P (K) is true

$2.7^{K+1} + 3.5^{K+1} - 5 = 2.7^K .7^1 + 3.5^K . 5^1 -5$

$= 7[2.7^K + 3.5^K - 5 - 3.5^K + 5] + 3.5^K.5^1-5$

$= 7[24\lambda - 3.5^K + 5] + 15.5^K-5$ (from i)

$= 7\times 24\lambda\ -21.5^K+35+15.5^K-5$

$= 7\times 24\lambda-6.5^K+30$

$= 7 \times 24\lambda - 6(5^K - 5)$

$= 7 \times 24\lambda - 6.4\,p \left[\because 5^K - 5 \text{ is multiple of } 4\right]$

$= 24(7\lambda - p),$ 24 is divisible by 24

Hence by P M I p (n) is true for all n $\in$ N.

10. Prove that $41^n - 14^n$ is a multiple of 27

Ans. P (n) : $41^n - 14^n$ is a multiple of 27

for n = 1

P (1) : $41^1 - 14 = 27$, which is a multiple of 27

Let P (K) be True

P (K) : $41^K - 14^K$

$\Rightarrow 41^K - 14^K = 27\lambda$, where $\lambda \in N$ (i)

we want to prove that result is true for n = K + 1

$41^{K+1} - 14^{k+1} = 41^K.41 - 14^K.14$

$= (27\lambda + 14^K).41 - 14^K.14(\textit{from } i)$

$= 27\lambda.41 + 14^K.41 - 14^K.14$

$= 27\lambda.41 + 14^K(41 - 14)$

$= 27\lambda.41 + 14^K(27)$ is a multiple of 27

$= 27(41\lambda + 14^K)$

Hence by PMI p (n) is true for ace n $\in$ N.

11. Using induction, prove that $10^n + 3.4^{n+2} + 5$ is divisible by 9 $\forall\ n \in \mathbb{N}$.

Ans. P (n) : $10^n + 3.4^{n+2} + 5$ is divisible by 9

For n = 1

p (1) : $10^1 + 3.4^{1+2} + 5 = 207$, divisible by 9

Hence result is true for n = 1

Let p (K) be true

p (K) : $10^K + 3.4^{K+2} + 5$ is divisible by 9

$\Rightarrow 10^K + 3.4^{K+2} + 5 = 9\lambda$ where $\lambda \in \mathbb{N}$ (i)

we want to prove that result is true for n = K + 1

$10^{(K+1)} + 3.4^{K+1+2} + 5 = 10^{K+1} + 3.4^{K+3} + 5$

$= 10^K.10 + 3.4^K.4^3 + 5$

$= (9\lambda - 3.4^{K+2} - 5).10 + 3.4^K.4^3 + 5$ (from i)

$= 90\lambda - 30.4^{K+2} - 50 + 3.4^{K+3} + 5$

$= 90\lambda - 30.4^{K+2} - 45 + 3.4.\ 4^{K+2}$

$= 90\lambda - 18.4^{K+2} - 45$

$= 9(10\lambda - 2.4^{K+2} - 5)$

which is divisible by 9.

12. Prove that $1^2 + 2^2 + 3^2 + \text{---} + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Ans. Let P(n) : $1^2 + 2^2 + 3^2 + \text{--} + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

for n = 1

$$P(1): 1^2 = \frac{1(2)(3)}{6} = 1$$

which is true

Let P (K) be true

$$P(K): 1^2 + 2^2 + \text{--} + K^2 = \frac{K(K+1)(2K+1)}{6} \quad (1)$$

we want to prove that P(K+1) is true

$$P(K+1): 1^2 + 2^2 + \text{--} + (K+1)^2 = \frac{(K+1)(K+2)(2K+3)}{6}$$

$$L.H.S = \underbrace{1^2 + 2^2 + - - + K^2}_{} + (K+1)^2$$

$$= \frac{K(K+1)(2K+1)}{6} + \frac{(K+1)^2}{1} \quad [\text{from } (1)$$

$$= \frac{K(K+1)(2K+1) + 6(K+1)^2}{6}$$

$$= \frac{(K+1)[K(2K+1) + 6(K+1)]}{6}$$

$$= \frac{(K+1)(2K^2 + K + 6K + 6)}{6}$$

$$= \frac{(K+1)(2K^2 + 7K + 6)}{6}$$

$$= \frac{(K+1)(K+2)(2K+3)}{6}$$

Thus P (K+1) is true, whenever P (K) is true.

Hence, from PMI, the statement P (n) is true for all natural no. n.

13. Prove that $1+3+3^2 + -- + 3^{n-1} = \frac{3^n - 1}{2}$

Ans. Let

P (n) : $1 + 3 + 3^2 + -- + 3^{n-1} = \frac{3^n - 1}{2}$ for n = 1

P (1) $3^{1-1} = \frac{3^1 - 1}{2} = 1$

which is true

Let P (K) be true

P (K) : $1 + 3 + 3^2 + ---- + 3^{K-1} = \frac{3^K - 1}{2}$ (1)

we want to prove that P (K+1) is true

$$P(K+1): 1+3+3^2+---+3^K = \frac{3^{K+1}-1}{2}$$

$$\text{L.H.S} = 1+3+3^2+----+3^{K-1}+3^K$$

$$= \frac{3^K - 1}{2} + 3^K \qquad \text{[From (1)}$$

$$= \frac{3^K - 1 + 2.3^K}{2}$$

$$= \frac{3^K(1+2) - 1}{2}$$

$$= \frac{3^K.3 - 1}{2}$$

$$= \frac{3^{K+1} - 1}{2}$$

Hence p (K+1) is true whenever p (K) is True

14. By induction, prove that $1^2 + 2^2 + 3^2 + \text{--} + n^2 > \frac{n^3}{3} \forall n \in N$

Ans. Let P (n) : $1^2 + 2^2 + 3^2 + \text{---} + n^2 > \frac{n^3}{3}$

for n = 1

12 > $\frac{1}{3}$ which is true

Let P (K) be true

P (K) : $1^2 + 2^2 + 3^2 + \text{--} + K^2 > \frac{K^3}{3} \quad (1)$

we want to prove that P (K + 1) is true

P (K+1) : $1^2 + 2^2 + \text{---} + (K+1)^2$

$=1^2+2^2+\text{--}+K^2+(K+1)^2$

$>\frac{K^3}{3}+(K+1)^2$

$=\frac{1}{3}\left[K^3+3(K+1)^2\right]$

$=\frac{1}{3}\left[K^3+3K^2+3+6K\right]$

$=\frac{1}{3}\left[(K+1)^3+(3K+2)\right]$

$>\frac{1}{3}(K+1)^3$

$\Rightarrow P(K+1)$ is true

Hence by PMI P (n) is true $\forall\ n \in N$

15. Prove by PMI $(ab)^n = a^n b^n$

Ans. Let P (n) : $(ab)^n = a^n b^n$

for n = 1

ab = ab which is true

Let P (K) be true

$(ab)^K = a^K b^K$ (1)

we want to prove that P (K+1) is true

$(ab)^{K+1} = a^{K+1}. b^{K+1}$

L.H.S = $(ab)^{K+1}$

$= (ab)^K . (ab)^1$

$= a^K b^K . (ab)^1$ [from (1)

$= a^{K+1} . b^{K+1}$

$\Rightarrow P(K+1)$ is true.

16. Prove by PMI $a + ar + ar^2 + \text{----} + ar^{n-1} = \dfrac{a(r^n - 1)}{r-1}$

Ans. Let P (n) : $a + ar + ar^2 + \text{--} + ar^{n-1} = \dfrac{a(r^n - 1}{r-1}$

for n = 1

P (1) = a = a which is true

Let P (K) be true

P (K) : $a + ar + ar^2 + -- + ar^{K-1} = \frac{a(r^K - 1)}{r-1}$ (1)

we want to prove that

P (K+1) : $a + ar + ar^2 + -- + ar^K = \frac{a(r^{K+1} - 1)}{r-1}$

L.H.S $= \underbrace{a + ar + ar^2 + -- + ar^{K-1}} + ar^K$

$$= \frac{a(r^K - 1)}{r-1} + ar^K \quad [\text{ from (1)}$$

$$= \frac{a(r^K - 1) + ar^{k+1} - ar^K}{r-1}$$

$$= \frac{ar^K - a + ar^{k+1} - ar^K}{r-1} = \frac{a(r^{K+1} - 1)}{r-1}$$

Thus P (K+1) is true whenever P(K) is true

Hence by PMI P(n) is true for all n $\in$ N

17. Prove that $x^{2n} - y^{2n}$ is divisible by x + y.

Ans. P (n) : $x^{2n} - y^{2n}$ is divisible by x + y

for n = 1

p (1) : $x^2 - y^2 = (x - y)(x + y)$, which is divisible by x + y

Hence result is true for n = 1

Let P (K) be true

p (K) : $x^{2K} – y^{2K}$ is divisible by x + y

$\Rightarrow x^{2K} - y^{2K} = (x+y)\lambda$, where $\lambda \in N$ (i)

we want to prove the result is true for n = K + 1

$x^{2(K+1)} - y^{2(K+1)} = x^{2K+2} - y^{2K+2}$

$= x^{2K}.x^2 - y^{2K}.y^2$

$= \left((x+y)\lambda + y^{2K}\right).x^2 - y^{2K}.y^2 (\textit{from } i)$

$= (x+y)\lambda x^2 + x^2 y^{2K} - y^{2K}.y^2$

$= (x+y)\lambda x^2 + y^{2K}(x^2 - y^2)$

$= (x+y)\lambda x^2 + y^{2K}(x+y)(x-y)$

$= (x+y)\left[x^2\lambda + y^{2K}(x-y)\right]$ is divisible by $(x+y)$

$\Rightarrow$ p (K+1) is true whenever p (K) is true

Hence by P.M.I. p (n) is true $\forall n \in N$

18. Prove that n (n + 1) (2n + 1) is divisible by 6.

Ans.P (n) : n (n+1) (2n+1) is divisible by 6 for n = 1

P (1) : (1) (2) (3) = 6 is divisible by 6

Hence result is true for n = 1

Let P (K) be true

P (K) : K (K+1) (2K+1) is divisible by 6

$\Rightarrow K(K+1)(2K+1) = 6\lambda$ where $\lambda \in N$ (i)

we want to prove that result is true for n = K+1

(K+1) (K+2) (2k+3) = (K+1) (K+2) $\left[(2K+1)+2\right]$

= (K+1) (K+2)(2K+1) + 2 (K+1) (K+2)

$= (K+2)\left[(K+1)(2K+1)\right]$ + 2 (K+1) (K+2)

= K (K+1) (2K+1) + 2 (K+1) (2K+1) + 2 (K+1) (K+2)

$= 6\lambda + 2(K+1)(2K+1) + 2(K+1)(K+2)$ (by i)

$= 6\lambda$ + 2(K+1) $\left[2K+1+K+2\right]$

$= 6\lambda + 2(K+1)(3K+3)$

$= 6\lambda + 6(K+1)(K+1)$

$= 6\left[\lambda + (K+1)(K+1)\right]$

is divisible by 6.

19. Show that 2^{3n} – 1 is divisible by 7 .

Ans. P (n) : 2^{3n} – 1 is divisible by 7

for n = 1

P (1) : 2^3 – 1 = 7 which is divisible by 7

Let P (K) be true

P (K) : 2^{3K} – 1 is divisible by 7

$\Rightarrow 2^{3K} - 1 = 7\lambda$ where $\lambda \in N$ (i)

we want to prove that P(K+1) is true whenever P(K) is true

$2^{3(K+1)} - 1 = 2^{3K+3} - 1$

$= 2^{3K} . 2^3 - 1$

$= (7\lambda+1). 8 - 1$(from i)

$= 56\lambda + 8 - 1$

$= 56\lambda + 7$

$= 7(8\lambda+1)$ which is divisible by 7 s

Thus P (K+1) is true

Hence by P.M.I P (n) is true $\forall$ n $\in \mathbb{N}$

20. Prove by P M I.

1. 2. 3 + 2. 3. 4 + --- + n (n + 1) (n + 2)

$= \dfrac{n(n+1)(n+2)(n+3)}{4}$

Ans. Let P (n) : 1. 2. 3 + 2. 3. 4 + --- + n (n+1) (n+2)

$= \dfrac{n(n+1)(n+2)(n+3)}{4}$

For n = 1

P (1) = 1 (2) (3) = $\dfrac{(1)(2)(3)(4)}{4}$

P (1) = 6 = 6 which is true

Let P (K) be true

P (K) : 1. 2. 3 + 2. 3. 4 + -- + K (K+1) (K+2)

$= \dfrac{K(K+1)(K+2)(K+3)}{4}$ (1)

we want to prove that

P (K+1) n: 1. 2. 3 + 2. 3. 4 + -- + (K+1) (K+2) (K+3) = $\frac{(K+1)(K+2)(K+3)(K+4)}{4}$

L.H.S = 1. 2. 3 + 2. 3. 4 + -- + K (K+1) (K+2) + (K+1) (K+2) (K+3)

$$= \frac{K(K+1)(K+2)(K+3)}{4} + \frac{(K+1)(K+2)(K+3)}{1} \quad [\text{from } (1)$$

$$= \frac{(K+1)(K+2)(K+3)[K+4]}{4}$$

Thus P (K+1) is true whenever P(K) is true.

21. Prove that $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + . + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Ans.P(n) : $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + . + \frac{1}{n(n+1)} = \frac{n}{n+1}$

For n = 1

P (1) = $\frac{1}{2} = \frac{1}{2}$ which is true

Let P (K) be true

$$P(K): \frac{1}{1.2} + \frac{1}{2.3} + \text{ --- } + \frac{1}{K(K+1)} = \frac{K}{K+1} \quad (1)$$

we want to prove that P (K+1) is true

$$\text{P (K+1) : } \frac{1}{1.2} + \frac{1}{2.3} + \text{ --- } + \frac{1}{(K+1)(K+2)} = \frac{K+1}{K+2}$$

$$\text{L.H.S} = \frac{1}{1.2} + \frac{1}{2.3} + \text{ --- } + \frac{1}{K(K+1)} + \frac{1}{(K+1)(K+2)}$$

$$= \frac{K}{K+1} + \frac{1}{(K+1)(K+2)} \qquad [\text{from } (1)$$

$$= \frac{K(K+2)+1}{(K+1)(K+2)}$$

$$= \frac{K^2+2K+1}{(K+1)(K+2)} = \frac{(K+1)^2}{(K+1)(K+2)}$$

$$= \frac{K+1}{K+2}.$$

Thus P (K+1) is for whenever P (K) is true.

22. Show that the sum of the first n odd natural no is n^2.

Ans. Let P (n) : 1 + 3 + 5 + --- + (2n-1) = n^2

For n = 1

P (1) = 1 = 1 which is true

Let P (K) be true

P (K) : 1 + 3 + 5 + --- + (2K-1) = K^2 (1)

we want to prove that P (K+1) is true

P (K+1) : 1 + 3 + 5 + --- + (2K+1) = $(K+1)^2$

L.H.S = $\underbrace{1+3+5+--+(2K-1)}+(2K+1)$

$$= K^2 + 2K + 1 \qquad [\text{From } (1)$$

$$= (K+1)^2$$

Thus P (K+1) is true whenever P(K) is true.

Hence by PMI, P(n) is true for all n $\in$ N.

23. Prove by P M I

$$1^3+2^3+3^3+---+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

Ans.P (n) : $1^3+2^3+3^3+---+n^3=\left(\frac{n(n+1)}{2}\right)^2$

For n = 1

P (1) : $1^3 = 1^3$ which is true

Let P (K) be true

$$P(K):1^3+2^3+--+K^3=\left(\frac{K(K+1)}{2}\right)^2 \quad (1)$$

we want to prove that P (K+1) is true

$$P(K+1):\ 1^3+2^3+--+(K+1)^3=\left(\frac{(K+1)(K+2)}{2}\right)^2$$

$$L.H.S=\underbrace{1^3+2^3+--+K^3}+(K+1)^3$$

$$=\left(\frac{K(K+1)}{2}\right)^2+(K+1)^3 \qquad [\text{from } (1)$$

$$=\frac{K^2(K+1)^2}{4}+\frac{(K+1)^3}{1}$$

$$=\frac{K^2(K+1)^2+4(K+1)^3}{4}$$

$$=\frac{(K+1)^2\left[K^2+4(K+1)\right]}{4}$$

$$=\frac{(K+1)^2(K^2+4K+4)}{4}$$

$$=\frac{(K+1)^2(K+2)^2}{4}$$

$$=\left[\frac{(K+1)(K+2)}{2}\right]$$

Thus P (K+1) is true whenever P (K) is true.

24. Prove. $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)----+\left(1+\frac{(2n+1)}{n^2}\right)=(n+1)^2$

Ans.P (n) : $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)----\left(1+\frac{(2n+1)}{n^2}\right)=(n+1)^2$

For n = 1

P (1) : 4 = 4 which is true

Let P (K) be true

P (K) : $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)---\left(1+\frac{(2K+1)}{K^2}\right)=(K+1)^2 \quad (1)$

We want to power that P (K+1) is true

$$P(K+1):\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)---\left(1+\frac{(2K+3)}{(K+1)^2}\right)=(K+2)^2$$

$$L.H.S=\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)---\left(1+\frac{(2K+1)}{K^2}\right)\left(1+\frac{(2K+3)}{(K+1)^2}\right)$$

$=(K+1)^2\left(1+\frac{(2K+3)}{(K+1)^2}\right)$ $\quad [\because$ from (1)

$=(K+1)^2\left[\frac{(K+1)^2+2K+3}{(K+1)^2}\right]^2$

$=\frac{(K+1)^2(K^2+4K+4)}{(K+1)^2}$

$=\frac{(K+1)^2(K+2)^2}{(K+1)^2}$

$=(K+2)$

Thus P (K+1) is true whenever P (K) is true.

25. Prove that 3^{2n+2} – 8n – 9 is divisible by 8

Ans.P(n) : 3^{2n+2} – 8n – 9 is divisible by 8

For n = 1

P (1) : 3^{2+2} – 8 – 9 = 64

which is divisible by 8

Hence result is true for n = 1

Let P (K) be true

P (K) : 3^{2K+2} – 8K – 9 is divisible by 8

$\Rightarrow 3^{2K+2}-8K-9=8\lambda$, where $\lambda \in N$ (i)

we want to prove that result is true for n = K+1

$3^{2(K+1)+2}-8(K+1)-9=3^{2K+2+2}-8K-8-9$

$= 3^{2K+4} - 8K - 17$

$= 3^{2K}.3^4 - 8K - 17$

$= 3^{2k+2}.3^2 - 8K - 17$

$= (8\lambda + 8K + 9).9 - 8K - 17$

$= 72\lambda + 72K + 81 - 8K - 17$

$= 64\lambda + 64K + 64$

$= 8(8\lambda + 8K + 8)$

(from i)

which is divisible by 8

Hence P(K+1) is true whwnever P (K) is true.

Hence by P.M.I P (n) is true $\forall$ n $\in$ N

26. Prove by PMI.

x^n-y^n is divisible by (x-y) whenever x-y $\neq$ 0

Ans.P (n) : x^n-y^n is divisible by (x-y)

For n = 1

P (1) : x – y is divisible by (x – y)

Let P (K) be true

P (K) : x^K – y^K is divisible by (x – y)

$\Rightarrow x^K - y^K = \lambda(x - y) \, (i)$

we want to prove that P (K+1) is true whenever P (K) is true

$x^{K+1} - y^{K+1} = x^K.x - y^K.y$

$= \left(\lambda(x-y)+y^K\right).x-y^K.y$ (from i)

$= \lambda(x-y).x+y^K.x-y^K y$

$= \lambda(x-y).x+y^K(x-y)$

$= (x-y)\left[\lambda x+y^K\right]$

which is divisible by x-y

Hence P (K+1) is true

27. Prove (x^{2n}-1) is divisible by (x-1).

Ans.P (n) : (x^{2n}-1) is divisible by (x-1).

For n = 1

P (1) : (x^2 – 1) = (x – 1) (x + 1)

which is divisible by (x – 1)

Let P (K) be true

$P(K):\left(x^{2K}-1\right)$ is divisible by x-1 (i)

$\Rightarrow x^{2K}-1=\lambda(x-1)$

we want to prove that P (K+1) is true

$P(K+1) : x^{2(K+1)}-1$

L.H.S

$=x^{2K+2}-1$

$=x^{2K}.x^2-1$

$=(\lambda(x-1)+1).x^2-1$(from i)

$=\lambda(x-1).x^2+x^2-1$

$=\lambda(x-1).x^2+(x-1)(x+1)$

$=(x-1)\left[\lambda x^2+(x+1)\right]$

which is divisible by (x-1)

Hence p(K+1) is true whenever p(k) is true

28. Prove

$$1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+--+\frac{1}{(1+2+--+n)}=\frac{2n}{(n+1)}$$

Ans.P (n) : $1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+--+\frac{1}{(1+2+--+n)}=\frac{2n}{(n+1)}$

$$1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+---+\frac{1}{\frac{n(n+1)}{2}}=\frac{2n}{n+1}$$

$$1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+---+\frac{2}{n(n+1)}=\frac{2n}{n+1}$$

for n = 1

P (1) : $\frac{2}{2}=\frac{2}{2}=1$

which is true

Let p(k) be true

$$p(k): 1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+---+\frac{2}{k(k+1)}=\frac{2k}{k+1} \quad (1)$$

we want to prove that p (k + 1) is true

$$p(k+1) : 1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+---+\frac{2}{(k+1)(k+2)}=\frac{2(k+1)}{k+2}$$

$$L.H.S = 1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+---+\frac{2}{k(k+1)}+\frac{2}{(k+1)(k+2)}$$

$$=\frac{2k}{k+1}+\frac{2}{(k+1)(k+2)} \qquad \text{[from (1)}$$

$$=\frac{2k(k+2)+2}{(k+1)(k+2)}$$

$$=\frac{2k^2+4k+2}{(k+1)(k+2)}$$

$$=\frac{2(k^2+2k+1)}{(k+1)(k+2)}$$

$$=\frac{2(k+1)^2}{(k+1)(k+2)}$$

$$=\frac{2(k+1)}{(k+2)}$$

thus p (k+1) is true whenever p(k) is true

Hence by PMI p(n) is true $\forall$ n $\in$ N.

29. Prove 1.3 + 3.5 + 5.7 + -- + (2n – 1) (2n + 1)

$$=\frac{n\left(4n^2+6n-1\right)}{3}$$

Ans. Let p (n) : 1.3 + 3.5 + -- + (2n-1) (2n+1)

$$=\frac{n\left(4n^2+6n-1\right)}{3}$$

For n = 1

P (1) = (1) (3) = $\frac{1(4+6-1)}{3}$

P(1) = 3 = 3 Hence p (1) is true

Let (k) be true

P(k) : 1.3 + 3.5 + -- + (2k - 1) (2K + 1) = $\frac{k(4k^2+6k-1)}{3}$ (1)

we want to prove that p (k+1) is true

p (k+1) : 1.3 + 3.5 + -- + (2k+1) (2k+3) = $\frac{(k+1)\left[4(k+1)^2+6(k+1)-1\right]}{3}$

L. H. S

$$1.3+3.5+--+(2k-1)(2k+1)+(2k+1)(2k+3)$$

$$=\frac{k(4k^2+6k-1)}{3}+\frac{(2k+1)(2k+3)}{1} \qquad [\text{from } (1)$$

$$=\frac{k\left(4k^2+6k-1\right)+3(2k+1)(2k+3)}{3}$$

$$=\frac{4k^3+18k^2+23k+9}{3}\left[\text{put k = -1 (k+1) is one fator}\right]$$

$$=\frac{(k+1)\left(4k^2+14k+9\right)}{3}$$

Thus p (k+1) is true whenever p (k) is true.

30. Prove by PMI

$3.2^2 + 3^2.2^3 + 3^3.2^4 + -- + 3^n.\ 2^{n+1} = \frac{12}{5}\left(6^n - 1\right) n \in N$.

Ans. Let p (n) : $3.2^2 + 3^2.2^3 + 3^3.2^4 + --- + 3^n.\ 2^{n+1} = \frac{12}{5}\left(6^n - 1\right)$

For n = 1

$p(1): 3^1.2^2 = \frac{12}{5}\left(6^1 - 1\right)$

$p(1) = 12 = 12$

p(1) is true

Let p(k) be true

p(k): $3.2^2 + 3^2.2^3 + -- + 3^k.\ 2^{k+1} = \frac{12}{5}\left(6^k - 1\right)$ (1)

we want to prove that p (k+1) is true

p(k+1): $3.2^2 + 3^2.2^3 + --- + 3^{k+1}.\ 2^{k+2} = \frac{12}{5}\left(6^{k+1} - 1\right)$

$L.H.S = 3.2^2 + 3^2.2^3 + - - + 3^k.2^{k+1} + 3^{k+1}.2^{k+2}$

$= \frac{12}{5}\left(6^k - 1\right) + 3^{k+1}.2^{k+2}$ [from (1)

$= \frac{2}{5}.6.6^k - \frac{12}{5} + 3^{k+1}.2^{k+1}.2^1$

$= \frac{2}{5}6^{k+1} - \frac{12}{5} + 6^{k+1}.2$

$= 6^{k+1}\left(\frac{2}{5} + 2\right) - \frac{12}{5}$

$=6^{k+1}\left(\frac{12}{5}\right)-\frac{12}{5}=\frac{12}{5}\left[6^{k+1}-1\right]$

Thus p(k+1) is truewhenever p(k) is true.

31. Prove 1.3 + **2.3^2** + **3.3^3** + --- + **n.3^n** = $\frac{(2n-1)3^{n+1}+3}{4}$

Ans.P (n) : 1.3 + 2.3^2 + 3.3^3 + --- + n.3^n = $\frac{(2n-1)3^{n+1}+3}{4}$

For n = 1

P (1) : 1. 31 = $\frac{(2-1).3^2+3}{4}$

$p(1): 3=\frac{12\ 3}{4}$

hence p(1) is true

Let p(k) be true

$$p(k): 1.3+2.3^2+3.3^3+\text{--}+k.3^k = \frac{(2k-1)3^{k+1}+3}{4} \quad (1)$$

we want to prove that p (k+1) is true

$$p(k+1): 1.3+2.3^2+3.3^3+\text{--}+(k+1).3^{k+1}=\frac{(2k+1)3^{k+2}+3}{4}$$

$$L.H.S = 1.3+2.3^2+3.3^3+\text{--}+k.3^k+(k+1).3^{k+1}$$

$$=\frac{(2k-1).3^{k+1}+3}{4}+\frac{(k+1)3^{k+1}}{1} \quad \text{[from (1)}$$

$$=\frac{(2k-1).3^{k+1}+3+4(k+1)3^{k+1}}{4}$$

$$=\frac{(2k-1+4k+4).3^{k+1}+3}{4}$$

$$=\frac{(6k+3).3^{k+1}+3}{4}$$

$$=\frac{3(2k+1).3^{k+1}+3}{4}$$

$$=\frac{(2k+1)3^{k+2}+3}{4}$$

Thus p (k+1) is true whenever p(k) is true.

32. Prove $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+---+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$

Ans. P(n) : $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+---+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$

For n = 1

$$p(1):\frac{1}{(2+1)(2+3)}=\frac{1}{3(2+3)}$$

$p(1)=\frac{1}{.15}=\frac{1}{15}$ Hence p (1) is true

Let p (k) be true

p(k) : $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+---+\frac{1}{(2k+1)(2k+3)}=\frac{k}{3(2k+3)}$ (1)

we want to prove that p (k+1) is true

p(k+1): $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+---+\frac{1}{(2k+3)(2k+5)}=\frac{(k+1)}{3(2k+5)}$

$$L.H.S = \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+--+\frac{1}{(2k+1)(2k+3)}+\frac{1}{(2k+3)(2k+5)}$$

$$= \frac{k}{3(2k+3)}+\left(\frac{1}{(2k+3)}\right)\left(\frac{1}{(2k+5)}\right) \quad \text{[from (1)}$$

$$= \frac{k(2k+5)+3}{3(2k+3)(2k+5)}$$

$$= \frac{k+1}{3(2k+5)}$$

Thus p (k+1) is true whenever p (k) is true

Hence p (n) is true for all n $\in$ N.

33. The sum of the cubes of three consecutive natural no. is divisible by 9.

Ans.P(n) $\left[k^3+(k+1)^3+(k+2)^3\right]$ is divisible by 9

For n = 1

P (1) : 1 + 8 + 9 = 18

which is divisible by 9

Let p (k) be true

$p(k):\left[k^3+(k+1)^3+(k+2)^3\right]$ is divisible by 9

$\Rightarrow k^3+(k+1)^3+(k+2)^3 = 9\lambda (i)$

we want to prove that p (k+1) is true

p(k+1) : $(k+1)^3 + (k+2)^3+(k+3)^3$

L.H.S = $(k+1)^3 + (k+2)^3 + (k+3)^3$

$$= (k+1)^3 + (k+2)^3 + k^3 + 9k^2 + 27k + 27$$

$$= \underbrace{k^3 + (k+1)^3 + (k+2)^3} + 9(k^2 + 3k + 3)$$

$$= 9\lambda + 9(k^2 + 3k + 3) \text{ (from i)}$$

$$= 9\left[\lambda + (k^2 + 3k + 3)\right] \text{ which is } \div \text{ by 9.}$$

34. Prove that 12^n + 25^{n-1} is divisible by 13

Ans.P(n) : $12^n + 25^{n-1}$ is divisible by 13

For n = 1

P(1) : $12 + (25)^0 = 13$

which is divisible by 13

Let p (k) be true

P(k) : $12^k + 25^{k-1}$ is divisible by 13

$$\Rightarrow 12^k + 25^{k-1} = 13\lambda (i)$$

we want to prove that result is true for n = k+1

$$12^{(k+1)} + 25^{k+1-1} = 12^k.12^1 + 25^k$$

$$= \left(13\lambda - 25^{k-1}\right).12 + 25^k \ (from\ i)$$

$$= 13 \times 12\lambda - 25^{k-1}.12 + 25^k$$

$$= 13 \times 12\lambda + 25^{k-1}\left(-12 + 25\right)$$

$$= 13\left(12\lambda + 25^{k-1}\right)$$

which is divisible by 13.

35. Prove $11^{n+2} + 12^{2n+1}$ is divisible by 133.

Ans. P(n) : $11^{n+2} + 12^{2n+1}$ is divisible by 133.

For n = 1

P(1) : $11^3 + 12^3 = 3059$

which is divisible by 133

Let p (k) be true

$p(k): 11^{k+2} + 12^{2k+1}$ is divisible by 133

$\Rightarrow 11^{k+2} + 12^{2k+1} = 133\lambda$ (i)

we want to prove that
result is true for n = k+1

$L.H.S = 11^{k+1+2} + 12^{2(k+1)+1}$

$= 11^{k+3} + 12^{2k+2+1}$

$= 11^{k+3} + 12^{2k+3}$

$= 11^k.11^3 + 12^{2k}.12^3$

$= 11^{k+2}.11 + 12^{2k}.12^3$

$= \left(133\lambda - 12^{2k+1}\right).11 + 12^{2k}.12^3$ (from i)

$= 133 \times 11\lambda - 12^{2k+1}.11 + 12^{2k}.12^3$

$= 133 \times 11 \times \lambda + 12^k(-12 \times 11 + 12^3)$

$= 133\left[(11\lambda + 12^k(1596)\right]$

which is $\div$ 133.

36. Prove $1^3 + 2^3 + 3^3 + --- + n^3 = \frac{n^2(n+1)^2}{4}$

Ans. P(n) : $1^3 + 2^3 + 3^3 + --- + n^3 = \frac{n^2(n+1)^2}{4}$

For n = 1

$$p(1): 1^3 = \frac{1^2(2)^2}{4} = 1$$

which is true

Let p(k) be true

$$p(k): 1^3 + 2^3 + --- + k^3 = \frac{k^2(k+1)^2}{4} \quad (1)$$

we want to prove that p (k+1) is true

$$p(k+1): 1^3 + 2^3 + --- + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$

L.H.S = $1^3+2^3+---+k^3+(k+1)^3$ [from (1)

$$= \frac{k^2(k+1)^2}{4} + \frac{(k+1)^3}{1}$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{(k+1)^2[k^2 + 4(k+1)]}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

Thus p(k+1) is true whenever p(k) is true.

37. Prove (a)+ (a + d) + (a + 2d) + -- + [a + (n – 1)d] = $\frac{n}{2}\left[2a+(n-1)d\right]$

Ans. P(n) : (a)+ (a + d) + (a + 2d) + -- + [a + (n – 1)d] = $\frac{n}{2}\left[2a+(n-1)d\right]$

For n = 1

p(1) : a + (1-1) d = $\frac{1}{2}$ 2a + (1-1) d = a

which is true

Let p (k) be true

$$p(k):(a)+(a+d)+(a+2d)+--+\left(a+\left(k-1\right)d\right)=\frac{k}{2}\left[2a+(k-1)d\right] \quad (1)$$

we want to prove that p (k+1) is true

$$p(k+1) : (a) + (a+d) + -- + (a+kd) = \frac{k+1}{2}\left[2a+kd\right]$$

L.H.S = a + (a+d)+--+ a+kd

= a + (a + d) + -- + a + (k-1)d + a + kd

$$=\frac{k}{2}\left[2a+(k-1)d\right]+a+kd \qquad [\text{from } (1)$$

$$= ka + \frac{k}{2}(k-1)d+a+kd$$

$$=\frac{2ak+k^2d-kd+2a+2kd}{2}$$

$$=\frac{2a(k+1)+kd(k+1)}{2}=\frac{(k+1)(2a+kd)}{2}$$

proved.

38. Prove that $2^n > n$ $\forall$ positive integers n.

Ans. Let p (n) : $2^n > n$

For n = 1

P (1) : $2^1 > 1$

Which is true

Let p (k) be true

P (k) : $2^k > k$ (1)

we want to prove that p (k+1) is true

$2^k > k$ by (1)

$\Rightarrow 2^k.2 > 2k$

$2^{k+1} > 2k$

$2^{k+1} > 2k = k+k > k+1$

Hence provtd.

39. Prove $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+---+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Ans. P(n) : $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+---+\frac{1}{n(n+1)}=\frac{n}{n+1}$

For n = 1

$p(1)=\frac{1}{2}=\frac{1}{2}$ which is true

Let p (k) be true

$$p(k): \frac{1}{1.2}+\frac{1}{2.3}+---+\frac{1}{k(k+1)}=\frac{k}{k+1} \quad (1)$$

we want to prove that p (k+1) is true

$$p(k+1): \frac{1}{1.2}+\frac{1}{2.3}+---+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

$$L.H.S=\frac{1}{1.2}+\frac{1}{2.3}+---+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} \quad [\text{from } (1)$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)}$$

$$=\frac{k+1}{k+2}$$

proved.

40. Prove $\frac{1}{3.6}+\frac{1}{6.9}+\frac{1}{9.12}+---+\frac{1}{3n(3n+3)}=\frac{n}{9(n+1)}$.

Ans.P(n) : $\frac{1}{3.6}+\frac{1}{6.9}+\frac{1}{9.12}+---+\frac{1}{3n(3n+3)}=\frac{n}{9(n+1)}$.

For n = 1

$p(1): \frac{1}{3(6)}=\frac{1}{9(2)}=\frac{1}{18}$ which is true

Let p(k) be true

$$p(k): \frac{1}{3.6}+\frac{1}{6.9}+----+\frac{1}{3k(3k+3)}=\frac{k}{9(k+1)} \quad (1)$$

we want to prove that p (k+1) is true

$$p(k+1): \frac{1}{3.6}+\frac{1}{6.9}+----+\frac{1}{3(k+1)(3k+6)}=\frac{k+1}{9(k+2)}$$

$$L.H.S=\frac{1}{3.6}+\frac{1}{6.9}+----+\frac{1}{3k(3k+3)}+\frac{1}{3(k+1)(3k+6)}$$

$$=\frac{k}{9(k+1)}+\frac{1}{3(k+1)3(k+2)} \quad [\text{from } (1)$$

$$=\frac{k(k+2)+1}{9(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{9(k+1)(k+2)}$$

$$=\frac{(k+1)^2}{9(k+1)(k+2)}$$

$$=\frac{k+1}{9(k+2)}$$

proved.

Principles Of Mathematical Induction

1. Give an example of a statement $P(n)$ which is true for all $n \geq 4$ but $P(1)$, $P(2)$ and $P(3)$ are not true. Justify your answer.
2. Give an example of a statement $P(n)$ which is true for all n. Justify your answer.

Prove each of the statements in Exercises 3 - 16 by the Principle of Mathematical Induction :

3. $4^n - 1$ is divisible by 3, for each natural number n.
4. $2^{3n} - 1$ is divisible by 7, for all natural numbers n.
5. $n^3 - 7n + 3$ is divisible by 3, for all natural numbers n.
6. $3^{2n} - 1$ is divisible by 8, for all natural numbers n.
7. For any natural number n, $7^n - 2^n$ is divisible by 5.
8. For any natural number n, $x^n - y^n$ is divisible by $x - y$, where x and y are any integers with $x \neq y$.
9. $n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.
10. $n(n^2 + 5)$ is divisible by 6, for each natural number n.
11. $n^2 < 2^n$ for all natural numbers $n \geq 5$.
12. $2n < (n + 2)!$ for all natural number n.
13. $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}$, for all natural numbers $n \geq 2$.
14. $2 + 4 + 6 + \dots + 2n = n^2 + n$ for all natural numbers n.
15. $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$ for all natural numbers n.
16. $1 + 5 + 9 + \dots + (4n - 3) = n(2n - 1)$ for all natural numbers n.

Use the Principle of Mathematical Induction in the following Exercises.

17. A sequence $a_1, a_2, a_3 \ldots$ is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$ for all natural numbers $k \geq 2$. Show that $a_n = 3.7^{n-1}$ for all natural numbers.

18. A sequence $b_0, b_1, b_2 \ldots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$ for all natural numbers k. Show that $b_n = 5 + 4n$ for all natural number n using mathematical induction.

19. A sequence $d_1, d_2, d_3 \ldots$ is defined by letting $d_1 = 2$ and $d_k = \frac{d_{k-1}}{k}$ for all natural numbers, $k \geq 2$. Show that $d_n = \frac{2}{n!}$ for all $n \in \mathbf{N}$.

20. Prove that for all $n \in \mathbf{N}$

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha+(n-1)\beta)$$

$$= \frac{\cos\left(\alpha+\left(\frac{n-1}{2}\right)\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

21. Prove that, $\cos\theta\cos2\theta\cos2^2\theta \ldots \cos2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n\sin\theta}$, for all $n \in \mathbf{N}$.

22. Prove that, $\sin\theta + \sin2\theta + \sin3\theta + \ldots + \sin n\theta = \frac{\sin\frac{n\theta}{2}\sin\frac{(n+1)}{2}\theta}{\sin\frac{\theta}{2}}$, for all $n \in \mathbf{N}$

23. Show that $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is a natural number for all $n \in \mathbf{N}$.

24. Prove that $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{13}{24}$, for all natural numbers $n > 1$.

25. Prove that number of subsets of a set containing n distinct elements is 2^n, for all $n \in \mathbf{N}$.

Choose the correct answers in Exercises 26 to 30 (M.C.Q.).

26. If $10^n + 3.4^{n+2} + k$ is divisible by 9 for all $n \in \mathbf{N}$, then the least positive integral value of k is

(A) 5 (B) 3 (C) 7 (D) 1

27. For all $n \in \mathbf{N}$, $3.5^{2n+1} + 2^{3n+1}$ is divisible by

(A) 19 (B) 17 (C) 23 (D) 25

28. If $x^n - 1$ is divisible by $x - k$, then the least positive integral value of k is

(A) 1 (B) 2 (C) 3 (D) 4