# Relations and Functions

1. **Ordered Pair**

If a pair of elements written in a small brackets and grouped together in a particular order, then such a pair is called ordered pair. The ordered pair of two elements $a$ and $b$ is denoted by $(a, b)$, where $a$ is first element and $b$ is second element.

**Two ordered pairs $(a, b)$ and $(c, d)$ are equal,** if their corresponding elements are equal i.e. $a = c$ and $b = d$.

2. **Cartesian Products of Sets**

For any two non-empty sets $A$ and $B$, the set of all ordered pairs $(a, b)$ of elements $a \in A$ and $b \in B$ is called the cartesian product of sets $A$ and $B$ and is denoted by $A \times B$.

Thus, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

If $A = \phi$ or $B = \phi$, then $A \times B = \phi$.

**Note** $A \times B \neq B \times A$

3. **Number of Elements in Cartesian Product of Two Sets**

(i) If there are $p$ elements in set $A$ and $q$ elements in set $B$, then there will be $pq$ elements in $A \times B$ i.e. if $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$.

(ii) If $A$ and $B$ are non-empty sets and either $A$ or $B$ is an infinite set, then $A \times B$ will also be an infinite set.

(iii) If $A$ or $B$ is the null set or an empty set, then $A \times B$ will also be an empty set i.e. $A \times B = \phi$

4. **Ordered Triplet**

An ordered triplet is a list of 3 elements written in a particular order and enclosed in small bracket.

e.g. (1, 2, 3) is an ordered triplet. Ordered triplet is also called **3-tuple.**

5. **Cartesian Product of Three Sets**

If $A$, $B$ and $C$ are three sets, then

$$A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$$

6. **Relations**

A relation $R$ from a non-empty set $A$ to a non-empty set $B$ is a subset of the cartesian product $A \times B$, i.e. $R \subseteq A \times B$. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$.

The second element is called image of first element. The set of all first elements in a relation $R$, is called the **domain** of the relation $R$ and the set of all second elements is called the **range** of $R$. The set $B$ is called the **codomain** of relation $R$.

Thus, if $R = \{(a, b) : a \in A, b \in B\}$, then domain $(R) = \{a : (a, b) \in R\}$ and range $(R) = \{b : (a, b) \in R\}$

e.g. Let $R = \{(1, 2), (1, 5), (3, 4), (6, 7)\}$ be a relation from set $A = \{1, 3, 6, 7\}$ to set $B = \{2, 4, 5, 6, 7\}$.

Then, domain $= \{1, 3, 6\}$, range $= \{2, 4, 5, 7\}$ and codomain $=$ set $B = \{2, 4, 5, 6, 7\}$

Clearly, Range $\subseteq$ Codomain

**Note** *If $n(A) = m$, $n(B) = n$, then $n(A \times B) = mn$ and the total number of possible relations from set $A$ to set $B = 2^{mn}$*

7. **Representation of a Relation**

A relation can be represented algebraically by roster form or by set-builder form and visually, it can be represented by an arrow diagram.

(i) **Roster form** In this form, we represent the relation by the set of all ordered pairs belongs to $R$.

(ii) **Set-builder form** In this form, we represent the relation $R$ from set $A$ to set $B$ as $R = \{(a, b) : a \in A, b \in B$ and the rule which relate the elements of $A$ and $B\}$

8. **Inverse Relation**

Let $A$ and $B$ be two sets and $R$ be a relation from set $A$ to a set $B$. Then, the inverse of relation $R$, denoted by $R^{-1}$, is a relation from $B$ to $A$ and it is defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

$\therefore$ Domain of $R$ = Range of $R^{-1}$,
and Range of $R$ = Domain of $R^{-1}$

9. **Functions**

A relation $f$ from a non-empty set $A$ to non-empty set $B$ is said to be function, if every element of set $A$ has one and only one image in set $B$.

In other words, a function $f$ is a special type of relation from a non-empty set $A$ to a non-empty set $B$ such that the domain of $f$ is $A$ and no two distinct ordered pairs in $f$ have the same first element. If $f$ is a function from a set $A$ to a set $B$, then we write $f : A \rightarrow B$ and it is read as $f$ is function from $A$ to $B$ or $f$ map $A$ to $B$ and $(a, b) \in f$, then $f(a) = b$, where $b$ is called image of $a$ under $f$ and $a$ is called the pre-image of $b$ under $f$.

If $f : A \rightarrow B$, then, the set $A$ is called the **domain** of function $f$ and the set $B$ is called the **codomain** of $f$. The subset of $B$ containing the images of elements of $A$ is called the **range** of the function.

**Note** *Every relation is a function but converse is not true.*

## 10. Real Functions

A function $f : A \to B$ is called a **real valued function**, if $B$ is a subset of $R$ (set of all real numbers). If $A$ and $B$ both are subsets of $R$, then $f$ is called a real function.

## 11. Types of Functions

(i) **Identity function** The function $f : R \to R$ defined by $f(x) = x$ for each $x \in R$ is called identity function.
Domain of $f = R$; Range of $f = R$

(ii) **Constant function** The function $f : R \to R$ defined by $f(x) = c, \forall x \in R$, where c is a constant $\in R$, is called a constant function. Domain of $f = R$; Range of $f = \{c\}$

(iii) **Polynomial function** A real function $f : R \to R$ defined by $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where $n$ is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n \in R$ for each $x \in R$, is called a polynomial function. If $a_n \neq 0$, then $n$ is called the degree of the polynomial. The domain of a polynonial function is $R$ and range depends on the polynomial representing the function.

(iv) **Rational function** A function of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of $x$ defined in a domain and $g(x) \neq 0$, is called a rational function.

(v) **Modulus or Absolute value function** The real function $f : R \to R$ defined by

$$f(x) = |x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

is called the modulus function.
Domain of $f = R$, Range of $f = R^+ \cup \{0\}$ i.e. $[0, \infty)$

(vi) **Signum function** The real function $f : R \to R$ defined

$$\text{by } f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$$

is called the signum function.
Domain of $f = R$; Range of $f = \{-1, 0, 1\}$

(vii) **Greatest integer or Step function** The real function $f : R \to R$ defined by $f(x) = [x]$, is called the greatest integer function, where $[x]$ = integral part of $x$ or greatest integer less than or equal to $x$.
Domain of $f = R$; Range of $f$ = Integer

(viii) **Least** (smallest) **integer or Ceiling function** The function $f : R \to R$ defined by $f(x) = \lceil x \rceil$ for all $x \in R$ is called the least integer function or the ceiling function. Thus, for any real number $x$, $\lceil x \rceil$ or $(x)$ = smallest integer greater than or equal to $x$.
Domain of $f = R$; range of $f = I$

(ix) **Square root function** The square root of a function is defined as $f(x) = \sqrt{x}$, whose domain is $R^+$ i.e. $[0, \infty)$ and range is $R^+$ i.e. $[0, \infty)$.

(x) **Reciprocal function** The function $f : R - \{0\} \to R$ defined by $f(x) = \frac{1}{x}$ is called the reciprocal function.
Its domain as well as range is $R - \{0\}$.

(xi) **Exponential function** Let $a\ (\neq 1)$ be a positive real number. Then, the function $f : R \to R$, defined by $f(x) = a^x$, is called the exponential function. Its domain is $R$ and range is $(0, \infty)$.

(xii) **Logarithmic function** Let $a\ (\neq 1)$ be a positive real number. Then, the function $f : (0, \infty) \to R$, defined by $f(x) = \log_a x$, is called the logarithmic function. Its domain is $(0, \infty)$ and range is $R$.

## 12. Algebra of Real Functions

Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions with domain $D_1$ and $D_2$ respectively. Then,

(i) $(f + g)(x) = f(x) + g(x), \forall x \in D_1 \cap D_2$

(ii) $(f - g)(x) = f(x) - g(x), \forall x \in D_1 \cap D_2$

(iii) $(fg)(x) = f(x) \cdot g(x), \forall x \in D_1 \cap D_2$

(iv) $\frac{f}{g}(x) = \frac{f(x)}{g(x)}, \forall x \in D_1 \cap D_2 - \{x : g(x) = 0\}$

(v) $(cf)(x) = c \cdot f(x), \forall x \in D_1$, where c is a scalar (real number).