STRAIGHT LINE

BASIC THEOREMS & RESULTS OF TRIANGLES

(a) Two polygons are similar if **(i)** their corresponding angles are equal, **(ii)** the length of their corresponding sides are proportional. (Both condition are independent & necessary)

In case of a triangle, any one of the condition is sufficient, other satisfies automatically.

(b) **Thales Theorem (Basic Proportionality Theorem) :** In a triangle, a line drawn parallel to one side, to intersect the other sides in distinct points, divides the two sides in the same ratio.

Converse : If a line divides any two sides of a triangle in the same ratio then the line must be parallel to the third side.

(c) **Similarity Theorem**

(i) **AAA Similarity** : If in two triangles, corresponding angles are equal i.e. two triangles are equiangular, then the triangles are similar.

(ii) **SSS Similarity** : If the corresponding sides of two triangles are proportional, then they are similar.

(iii) **SAS Similarity :** If in two triangles, one pair of corresponding sides are proportional and the included angles are equal then the two triangles are similar.

(iv) If two triangles are similar then

(i) They are equiangular.

(ii) The ratio of the corresponding **(I)** Sides (all), **(II)** Perimeters, **(III)** Medians,

(IV) Angle bisector segments, **(V)** Altitudes are same (converse also true)

(iii) The ratio of the areas is equal to the ratio of the squares of corresponding

(I) Sides (all), **(II)** Perimeters, **(III)** Medians, **(IV)** Angle bisector segments,

(V) Altitudes (converse also true)

(d) **Congruency Theorem**

Congruent Triangles : Two triangles are congruent, iff one of them can be made to superpose on the other, so as to cover it exactly.

Sufficient-Conditions (criteria) for Congruence of Triangles :

(i) **Side-Angle-Side (SAS) :** Two triangles are congruent, if two sides and the included angle of one triangle are equal to the corresponding sides and the included angle of other triangle.

(ii) **Angle-Side-Angle (ASA) :** Two triangles are congruent if two angles and the included side of one triangle are equal to the corresponding two angles and the included side of the other triangle.

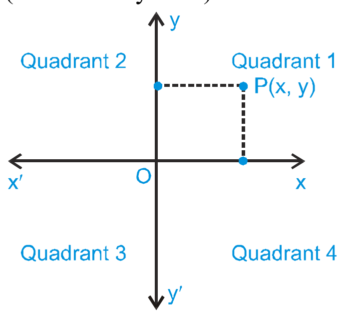
(iii) **Angle-Angle-Side(AAS) :** If any two angles and a non-included side of one triangle are equal to the corresponding to angles & the non-included side of the other triangle then the two triangles are congruent.

(iv) **Side-Side-Side (SSS) :** Two triangles are congruent if the three sides of one triangle are equal to the corresponding three sides of the other triangle.

(v) **Right Angle-Hypotenuse-Side(RHS) :** Two right-triangles are congruent, if the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and one side of other triangle.

Rectangular Cartesian Co-ordinate Systems

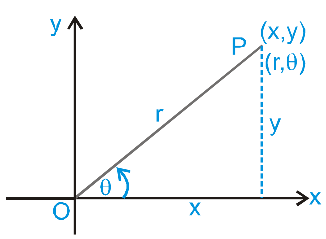
We shall right now focus on two-dimensional co-ordinate geometry in which two perpendicular lines called co-ordinate axes (x-axis and y-axis) are used to locate a point in the plane.



O is called origin. Any point P in this plane can be represented by a unique ordered pair (x, y), which are called co-ordinates of that point. x is called x co-ordinate or abscissa and y is called y co-ordinate or ordinate. The two perpendicular lines xox′ and yoy′ divide the plane in four regions which are called quadrants, numbered as shown in the figure.

Polar Co-ordinate System

A coordinate system in which the position of a point is determined by the length of a line segment from a fixed origin together with the angle that the line segment makes with a fixed line. The origin is called the pole and the line segment is the radius vector (r).

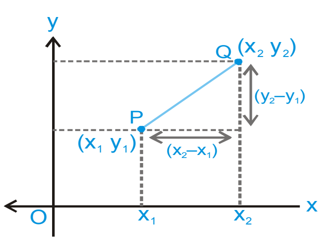


The angle θ between the polar axis and the radius vector is called the vectorial angle. By convention, positive values of θ are measured in an anticlockwise sense, negative values in clockwise sense. The coordinates of the point are then specified as (r, θ).

If (x,y) are cartesian co-ordinates of a point P, then : $x = r\cos\theta$, $y = r\sin\theta$

and $\quad r = \sqrt{x^2 + y^2}$, $\quad \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

Distance Formula



The distance between the points $A(x_1,y_1)$ and $B(x_2,y_2)$ is $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

ETOOS KEY POINTS

(i) Three given points A,B and C are collinear, when sum of any two distances out of AB,BC, CA is equal to the remaining third otherwise the points will be the vertices of a triangle.

(ii) Let A,B,C & D be the four given points in a plane. Then the quadrilateral will be :

(a) Square if AB = BC = CD = DA & AC = BD ; $AC \perp BD$

(b) Rhombus if AB = BC = CD = DA and AC ≠ BD ; $AC \perp BD$

(c) Parallelogram if AB = DC, BC = AD; AC ≠ BD ; $AC \not\perp BD$

(d) Rectangle if AB = CD, BC = DA, AC = BD ; $AC \not\perp BD$

Ex. Find the value of x, if the distance between the points (x, –1) and (3, 2) is 5.

Sol. Let P(x ,–1) and Q(3, 2) be the given points. Then PQ = 5 (given)

$$\sqrt{(x-3)^2+(-1-2)^2} = 5$$

$\Rightarrow$ $(x-3)^2 + 9 = 25$ $\Rightarrow$ $x = 7$ or $x = -1$

Ex. Find the number of points on x-axis which are at a distance c(c < 3) from the point (2, 3).

Sol. Let a point on x-axis is $(x_1, 0)$ then its distance from the point (2, 3)

$$= \sqrt{(x_1-2)^2+9} = c \quad \text{or} \quad (x_1-2)^2 = c^2 - 9$$

$\therefore$ $x_1 - 2 = \pm\sqrt{c^2-9}$ since $c < 3$ $\Rightarrow$ $c^2 - 9 < 0$

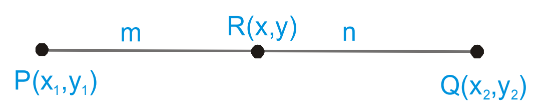
$\therefore$ x_1 will be imaginary.

SECTION FORMULA

The co-ordinates of a point dividing a line joining the points $P(x_1,y_1)$ and $Q(x_2,y_2)$ in the ratio m:n is given by :

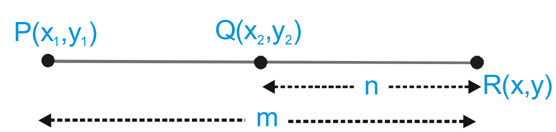
(a) **For Internal Division :** P - R - Q $\Rightarrow$ R divides line segment PQ, internally.

$$(x, y) \equiv \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$



(b) **For External Division :** R - P - Q or P - Q - R $\Rightarrow$ R divides line segment PQ, externally.

$$(x, y) \equiv \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$$



$\frac{(PR)}{(QR)} < 1$ $\Rightarrow$ R lies on the left of P & $\frac{(PR)}{(QR)} > 1$ $\Rightarrow$ R lies on the right of Q

ETOOS KEY POINTS

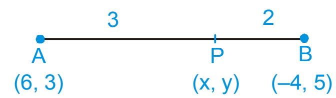
Harmonic Conjugate : If P divides AB internally in the ratio m : n & Q divides AB externally in the ratio m : n then P & Q are said to be harmonic conjugate of each other w.r.t. AB.

Mathematically ; $\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}$ i.e. AP, AB & AQ are in **H.P.**

Ex. Find the co-ordinates of the point which divides the line segment joining the points (6, 3) and (– 4, 5) in the ratio 3 : 2 **(i)** internally and **(ii)** externally.

Sol. Let P (x, y) be the required point.

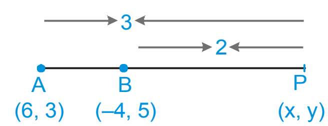
(i) For internal division :



$$x = \frac{3\times -4 + 2\times 6}{3+2} \text{ and } y = \frac{3\times 5 + 2\times 3}{3+2} \text{ or } x = 0 \text{ and } y = \frac{21}{5}$$

So the co-ordinates of P are $\left(0, \frac{21}{5}\right)$

(ii) For external division



$$x = \frac{3\times -42\times 6}{3-2} \quad \text{and} \quad y = \frac{3\times 5 - 2\times 3}{3-2}$$

or $x = -24$ and $y = 9$

So the co-ordinates of P are (–24, 9)

Ex. Determine the ratio in which $y - x + 2 = 0$ divides the line joining (3, –1) and (8, 9).

Sol. Suppose the line $y - x + 2 = 0$ divides the line segment joining A(3, –1) and B(8, 9) in the ratio $\lambda : 1$ at a point P, then the co-ordinates of the point P are $\left(\frac{8\lambda+3}{\lambda+1}, \frac{9\lambda-1}{\lambda+1}\right)$

But P lies on $y - x + 2 = 0$ therefore $\left(\frac{9\lambda-1}{\lambda+1}\right) - \left(\frac{8\lambda+3}{\lambda+1}\right) + 2 = 0$

$\Rightarrow$ $9\lambda - 1 - 8\lambda - 3 + 2\lambda + 2 = 0$

$\Rightarrow$ $3\lambda - 2 = 0$ or $\lambda = \frac{2}{3}$

So, the required ratio is $\frac{2}{3} : 1$, i.e., 2 : 3 (internally) since here λ is positive.

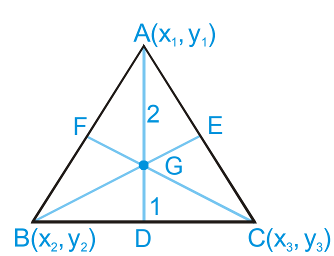
Co-ordinates of Some Particular Points

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of any triangle ABC, then

(a) Centroid

The centroid is the point of intersection of the medians (line joining the mid point of sides and opposite vertices). Centroid divides each median in the ratio of 2 : 1.

Co-ordinates of centroid $G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

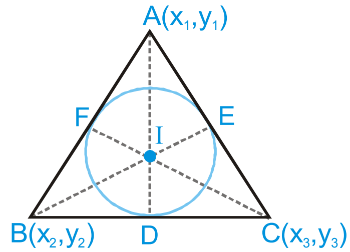


(b) Incenter

The incenter is the point of intersection of internal bisectors of the angles of a triangle. Also it is a centre of the circle touching all the sides of a triangle.

Co-ordinates of incenter I $\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$

where a, b, c are the sides of triangle ABC.



ETOOS KEY POINTS

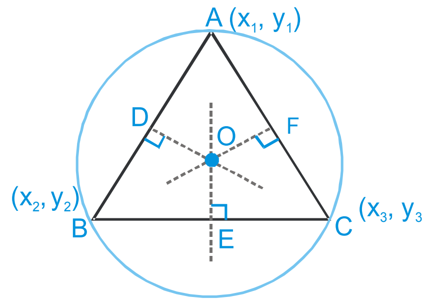
(i) Angle bisector divides the opposite sides in the ratio of remaining sides. e.g. $\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$

(ii) Incenter divides the angle bisectors in the ratio $(b+c):a, (c+a):b, (a+b):c$.

(c) Circumcenter

It is the point of intersection of perpendicular bisectors of the sides of a triangle. If O is the circumcenter of any triangle ABC, then

$OA^2 = OB^2 = OC^2$. Also it is a centre of a circle touching all the vertices of a triangle.

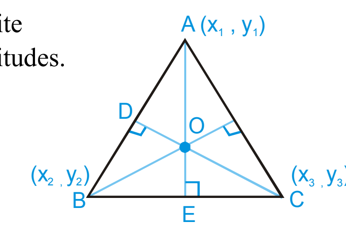


Co-ordinates of circumcenter $\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$

❖ If the triangle is right angled, then its circumcenter is the mid point of hypotenuse.

(d) Orthocenter

It is the point of intersection of perpendiculars drawn from vertices on the opposite sides of a triangle and it can be obtained by solving the equation of any two altitudes.



ETOOS KEY POINTS

(i) If a triangle is right angled, then orthocenter is the point where right angle is formed.

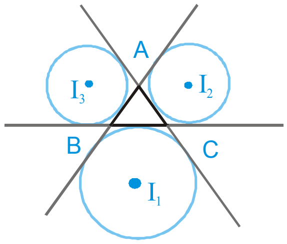
(ii) If the triangle is equilateral, then centroid, incentre, orthocenter, circumcenter, coincide.

(iii) Orthocentre, centroid and circumcentre are always collinear and centroid divides the line joining orthocentre and circumcentre in the ratio 2 : 1

(iv) In an isosceles triangle centroid, orthocentre, incentre & circumcentre lie on the same line.

(e) Ex-centers

The centre of a circle which touches side BC and the extended portions of sides AB and AC is called the ex-centre of ΔABC with respect to the vertex A. It is denoted by I_1 and its coordinates are



$$I_1\left(\frac{-ax_1+bx_2+cx_3}{-a+b+c},\frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$$

Similarly ex-centers of ΔABC with respect to vertices B and C are denoted by I_2 and I_3 respectively , and

$$I_2\left(\frac{ax_1-bx_2+cx_3}{a-b+c},\frac{ay_1-by_2+cy_3}{a-b+c}\right),\quad I_3\left(\frac{ax_1+bx_2-cx_3}{a+b-c},\frac{ay_1+by_2-cy_3}{a+b-c}\right)$$

ETOOS KEY POINTS

(i) Incentre and Excentre are harmonic conjugate of each other w.r.t. the angle bisector on which they lie.

(ii) Orthocentre, Centroid & Circumcentre are always collinear & centroid divides the line joining orthocentre & circumcentre in the ratio 2 : 1.

(iii) In an isosceles triangle G, O, I & C lie on the same line and in an equilateral triangle, all these four points coincide.

(iv) In a right angled triangle orthocentre is at right angled vertex and circumcentre is mid point of hypotenuse

(v) In case of an obtuse angled triangle circumcentre and orthocentre both are out side the triangle.

Ex. Find the co-ordinates of **(i)** centroid **(ii)** in-centre of the triangle whose vertices are (0, 6), (8, 12) and (8, 0).

Sol. **(i)** We know that the co-ordinates of the centroid of a triangle whose angular points are (x_1, y_1), (x_2, y_2) (x_3, y_3)

are $\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$

So the co-ordinates of the centroid of a triangle whose vertices are (0, 6), (8, 12) and (8, 0) are

$\left(\frac{0+8+8}{3},\frac{6+12+0}{3}\right)$ or $\left(\frac{16}{3},6\right)$.

(ii) Let A (0, 6), B (8, 12) and C(8, 0) be the vertices of triangle ABC.

Then $c = AB = \sqrt{(0-8)^2+(6-12)^2} = 10$, $b = CA = \sqrt{(0-8)^2+(6-0)^2} = 10$

and $a = BC = \sqrt{(8-8)^2+(12-0)^2} = 12$.

The co-ordinates of the in-centre are $\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right)$

or $\left(\frac{12\times0+10\times8+10\times8}{12+10+10},\frac{12\times6+10\times12+10\times0}{12+10+10}\right)$

or $\left(\frac{160}{32},\frac{192}{32}\right)$ or (5, 6)

Ex. If $\left(\frac{5}{3}, 3\right)$ is the centroid of a triangle and its two vertices are (0, 1) and (2, 3), then find its third vertex, circumcentre, circumradius & orthocentre.

Sol. Let the third vertex of triangle be (x, y), then

$\frac{5}{3} = \frac{x+0+2}{3} \Rightarrow x = 3$ and $3 = \frac{y+1+3}{3} \Rightarrow y = 5$. So third vertex is (3, 5).

Now three vertices are A(0, 1), B(2, 3) and C(3, 5)

Let circumcentre be P(h, k),

then $AP = BP = CP = R$ (circumradius)

$\Rightarrow$ $AP^2 = BP^2 = CP^2 = R^2$

$h^2 + (k-1)^2 = (h-2)^2 + (k-3)^2 = (h-3)^2 + (k-5)^2 = R^2$ ...... **(i)**

from the first two equations, we have

$h + k = 3$ ...... **(ii)**

from the first and third equation, we obtain

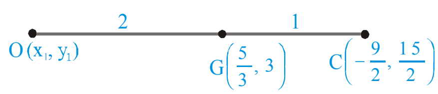
$6h + 6k = 33$ ...... **(iii)**

On solving, **(ii)** & **(iii)**, we get

$h = -\frac{9}{2}$, $k = \frac{15}{2}$

substituting these values in **(i)**, we have

$R = \frac{5}{2}\sqrt{10}$



Let $O(x_1, y_1)$ be the orthocentre, then $\frac{x_1 + 2\left(-\frac{9}{2}\right)}{3} = \frac{5}{3}$

$\Rightarrow$ $x_1 = 14$, $\frac{y_1 + 2\left(\frac{15}{2}\right)}{3} = 3$

$\Rightarrow$ $y_1 = -6$. Hence orthocentre of the triangle is (14, –6).

LOCUS

The locus of a moving point is the path traced out by that point under one or more geometrical conditions.

(a) Equation of Locus

The equation to a locus is the relation which exists between the coordinates of any point on the path, and which holds for no other point except those lying on the path.

(b) Procedure for Finding the Equation of the Locus of a Point :

(i) If we are finding the equation of the locus of a point P, assign coordinates (h, k) to P.

(ii) Express the given condition as equations in terms of the known quantities to facilitate calculations. We sometimes include some unknown quantities known as parameters.

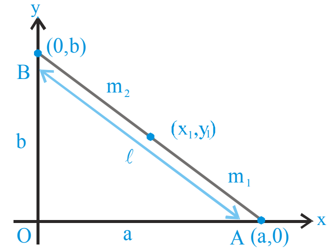
(iii) Eliminate the parameters, so that the eliminant contains only h, k and known quantities.

(iv) Replace h by x, and k by y, in the eliminant. The resulting equation would be the equation of the locus of P.

Ex. The ends of the rod of length ℓ moves on two mutually perpendicular lines, find the locus of the point on the rod which divides it in the ratio $m_1 : m_2$

Sol. Let (x_1, y_1) be the point that divide the rod $AB = \ell$, in the ratio $m_1 : m_2$, and $OA = a$, $OB = b$ (say)

$\therefore\ a^2 + b^2 = \ell^2$ ..... **(i)**



Now $x_1 = \left(\dfrac{m_2 a}{m_1 + m_2}\right) \Rightarrow a = \left(\dfrac{m_1 + m_2}{m_2}\right)x_1$

$y_1 = \left(\dfrac{m_1 b}{m_1 + m_2}\right) \Rightarrow b = \left(\dfrac{m_1 + m_2}{m_1}\right)y_1$

putting these values in **(i)** $\dfrac{(m_1 + m_2)^2}{m_2^2}x_1^2 + \dfrac{(m_1 + m_2)^2}{m_1^2}y_1^2 = \ell^2$

$\therefore$ Locus of (x_1, y_1) is $m_1^2 x^2 + m_2^2 y^2 = \left(\dfrac{m_1 m_2 \ell}{m_1 + m_2}\right)^2$

AREA OF A TRIANGLE

If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of triangle ABC, then its area is equal to

$\Delta\, ABC = \dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$, provided the vertices are considered in the counter clockwise sense. The above formula will give a $(-)$ ve area if the vertices (x_i, y_i), $i = 1, 2, 3$ are placed in the clockwise sense.

ETOOS KEY POINTS

Area of n-sided polygon formed by points (x_1, y_1) ; (x_2, y_2); ....;(x_n, y_n) is given by

$$\frac{1}{2}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \ldots\ldots\ldots\ldots \begin{vmatrix} x_{n-1} & x_n \\ y_{n-1} & y_n \end{vmatrix} + \begin{vmatrix} x_n & x_1 \\ y_n & y_1 \end{vmatrix}\right).$$

Here vertices are taken in order.

(i) If the area of triangle joining three points is zero, then the points are collinear.

(ii) **Area of Equilateral triangle :** If altitude of any equilateral triangle is P, then its area $= \dfrac{P^2}{\sqrt{3}}$. If 'a' be the side of equilateral triangle, then its area $= \left(\dfrac{a^2\sqrt{3}}{4}\right)$.

Ex. If the vertices of a triangle are (1, 2), (4, –6) and (3, 5) then find its area.

Sol. $\Delta = \dfrac{1}{2}\left[1(-6-5) + 4(5-2) + 3(2+6)\right] = \dfrac{1}{2}\left[-11 + 12 + 24\right] = \dfrac{25}{2}\textit{square units}$

Ex. If the co-ordinates of two points A and B are (3, 4) and (5, –2) respectively. Find the co-ordinates of any point P if PA = PB and Area of ΔPAB = 10.

Sol. Let the co-ordinates of P be (x, y). Then

PA = PB $\Rightarrow$ $PA^2 = PB^2$

$\Rightarrow$ $(x-3)^2 + (y-4)^2 = (x-5)^2 + (y+2)^2$ $\Rightarrow$ $x - 3y - 1 = 0$

Now, Area of ΔPAB = 10

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ 5 & -2 & 1 \end{vmatrix} = \pm 10$$

$\Rightarrow$ $6x + 2y - 26 = \pm 20$

$\Rightarrow$ $6x + 2y - 46 = 0$ or $6x + 2y - 6 = 0$

$\Rightarrow$ $3x + y - 23 = 0$ or $3x + y - 3 = 0$

Solving $x - 3y - 1 = 0$ and $3x + y - 23 = 0$ we get $x = 7, y = 2$.

Solving $x - 3y - 1 = 0$ and $3x + y - 3 = 0$, we get $x = 1, y = 0$.

Thus, the co-ordinates of P are (7, 2) or (1, 0)

Ex. Prove that the co-ordinates of the vertices of an equilateral triangle can not all be rational.

Sol. Let $A(x_1\ y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC. If possible let $x_1, y_1, x_2, y_2, x_3, y_3$ be all rational.

Now area of $\Delta ABC = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| = \text{Rational}$ ...... **(i)**

Since ΔABC is equilateral

$\therefore$ Area of $\Delta ABC = \frac{\sqrt{3}}{4}(side)^2 = \frac{\sqrt{3}}{4}(AB)^2 = \frac{\sqrt{3}}{4}\{(x_1 - x_2)^2 + (y_1 - y_2)^2\} = \text{Irrational}$ ...... **(ii)**

From **(i)** and **(ii)**,

Rational = Irrational

which is contradiction.

Hence $x_1, y_1, x_2, y_2, x_3, y_3$ cannot all be rational.

CONDITIONS FOR COLLINEARITY OF THREE GIVEN POINTS

Three given points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are collinear if any one of the following conditions are satisfied.

(i) Area of triangle ABC is zero i.e. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(ii) Slope of AB = slope of BC = slope of AC. i.e. $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{y_3 - y_1}{x_3 - x_1}$

(iii) Find the equation of line passing through 2 given points, if the third point satisfies the given equation of the line, then three points are collinear.

(iv) AC = AB + BC or AB ~ BC

(v) A divides the line segment BC in some ratio.

Ex. Show that the points (1, 1), (2, 3) and (3, 5) are collinear.

Sol. Let (1, 1) (2, 3) and (3, 5) be the co-ordinates of the points A, B and C respectively.

$$\text{Slope of AB} = \frac{3-1}{2-1} = 2 \quad \text{and} \quad \text{Slope of BC} = \frac{5-3}{3-2} = 2$$

$\therefore$ Slope of AB = slope of BC

$\therefore$ AB & BC are parallel

$\therefore$ A, B, C are collinear because B is on both lines AB and BC.

STRAIGHT LINE

A relation between x and y which is satisfied by co-ordinates of every point lying on a line is called equation of the straight line. Here, remember that every one degree equation in variable x and y always represents a straight line i.e. $\mathbf{ax + by + c = 0 \; ; a \;\&\; b \neq 0}$ simultaneously.

(a) Equation of a line parallel to x-axis at a distance 'a' is **y = a** or **y = – a**

(b) Equation of x-axis is **y = 0**

(c) Equation of a line parallel to y-axis at a distance 'b' is **x = b** or **x = – b**

(d) Equation of y-axis is **x = 0**

Slope of Line

If θ is the angle at which a straight line is inclined to the positive direction of x–axis,& $0° \le \theta < 180°, \theta \ne 90°$, then the slope of the line, denoted by m, is defined by $m = \tan\theta$. If θ is 90°, m does not exist, but the line is parallel to the y–axis. If $\theta = 0$, then m = 0 & the line is parallel to the x-axis.

If A (x_1, y_1) & B (x_2, y_2), $x_1 \neq x_2$, are points on a straight line, then the slope m of the line is given by : $m = \left(\frac{y_1 - y_2}{x_1 - x_2}\right)$.

ETOOS KEY POINTS

(i) If $\theta = 90°$, **m does not exist** and line is parallel to **y-axis.**

(ii) If $\theta = 0°$, **m = 0** and the line is parallel to **x-axis.**

(iii) Let m_1 and m_2 be slopes of two given lines (none of them is parallel to y-axis)

- **(a)** If lines are parallel, $\mathbf{m_1 = m_2}$ and vice-versa.
- **(b)** If lines are perpendicular, $\mathbf{m_1 m_2 = -1}$ and vice-versa

Ex. What is the slope of a line whose inclination with the positive direction of x-axis is :

(i) 0° **(ii)** 90° **(iii)** 120° **(iv)** 150°

Sol. **(i)** Here $\theta = 0°$

Slope = $\tan\theta = \tan 0° = 0$.

(ii) Here $\theta = 90°$

$\therefore$ The slope of line is not defined.

(iii) Here $\theta = 120°$

$\therefore$ Slope = $\tan\theta = \tan 120° = \tan(180° - 60°) = -\tan 60° = -\sqrt{3}$.

(iv) Here $\theta = 150°$

$\therefore$ Slope = $\tan\theta = \tan 150° = \tan(180° - 30°) = -\tan 30° = -\dfrac{1}{\sqrt{3}}$

Ex. Find the slope of the line passing through the points :

(i) A = (1, 6) and B = (– 4, 2) **(ii)** A = (5, 9) and B = (2, 9)

Sol. **(i)** Let A = (1, 6) and B = (– 4, 2)

$\therefore$ Slope of AB $= \dfrac{2-6}{-4-1} = \dfrac{-4}{-5} = \dfrac{4}{5}$ $\left(\text{Using slope} = \dfrac{y_2 - y_1}{x_2 - x_1}\right)$

(ii) Let A = (5, 9), B = (2, 9)

$\therefore$ Slope of AB $= \dfrac{9-9}{2-5} = \dfrac{0}{-3} = 0$

EQUATION OF STRAIGHT LINE IN VARIOUS FORM

Slope Intercept Form

Let m be the slope of a line and c its intercept on y-axis. Then the equation of this straight line is written as

$\mathbf{y = mx + c}$

If the line passes through origin, its equation is written as y = mx

Point - Slope Form

$\mathbf{y - y_1 = m(x - x_1)}$ is the equation of a straight line whose slope is m & which passes through the point (x_1, y_1).

Two Point Form

$\mathbf{y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$ is the equation of a straight line which passes through the points (x_1, y_1) & (x_2, y_2).

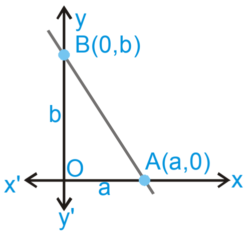
Determinant Form

Equation of line passing through (x_1, y_1) and (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Intercept Form

If a and b are the intercepts made by a line on the axes of x and y,

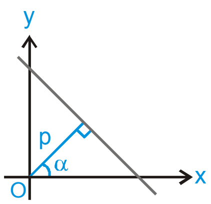
its equation is written as : $\mathbf{\dfrac{x}{a} + \dfrac{y}{b} = 1}$



(i) Length of intercept of line between the coordinate axes $= \sqrt{a^2 + b^2}$

Normal Form (Perpendicular Form)

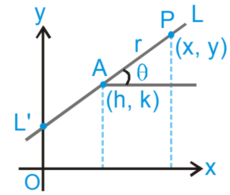
If p is the length of perpendicular on a line from the origin, and α the angle which this perpendicular makes with positive x-axis, then the equation of this line is written as : $x\cos\alpha + y\sin\alpha = p$ (p is always positive) where $0 \le \alpha < 2\pi$.



(ii) Area of triangle AOB $= \dfrac{1}{2} OA.OB = \left|\dfrac{1}{2}ab\right|$

Parametric Form

To find the equation of a straight line which passes through a given point A(h, k) and makes a given angle θ with the positive direction of the x-axis. P(x, y) is any point on the line LAL'.



Let AP = r, then $x - h = r\cos\theta,\ y - k = r\sin\theta$ & $\dfrac{x-h}{\cos\theta} = \dfrac{y-k}{\sin\theta} = r$

is the equation of the straight line LAL'.

Any point P on the line will be of the form $(h + r\cos\theta, k + r\sin\theta)$, where $|r|$ gives the distance of the point P from the fixed point (h, k).

General Form

We know that a first degree equation in x and y, $ax + by + c = 0$ always represents a straight line. This form is known as general form of straight line.

(i) Slope of this line $= \dfrac{-a}{b} = -\dfrac{\text{coeff. of } x}{\text{coeff. of } y}$

(ii) Intercept by this line on x-axis $= -\dfrac{c}{a}$ and intercept by this line on y-axis $= -\dfrac{c}{b}$

(iii) To change the general form of a line to normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{a^2 + b^2}$.

Ex. Find the equation of a line passing through (2, –3) and inclined at an angle of 135º with the positive direction of x-axis.

Sol. Here, m = slope of the line = tan 135º = tan (90º + 45º) = – cot 45º = –1, $x_1 = 2$, $y_1 = -3$

So, the equation of the line is $y - y_1 = m(x - x_1)$

i.e. $y - (-3) = -1(x - 2)$ or $y + 3 = -x + 2$ or $x + y + 1 = 0$

Ex. Find the equation of a line with slope –1 and cutting off an intercept of 4 units on negative direction of y-axis.

Sol. Here $m = -1$ and $c = -4$. So, the equation of the line is $y = mx + c$

i.e. $y = -x - 4$ or $x + y + 4 = 0$

Ex. Find the equation of the line joining the points (– 1, 3) and (4, – 2)

Sol. Here the two points are $(x_1, y_1) = (-1, 3)$ and $(x_2, y_2) = (4, -2)$.

So, the equation of the line in two-point form is

$$y - 3 = \frac{3-(-2)}{-1-4}(x+1) \quad \Rightarrow \quad y - 3 = -x - 1$$

$\Rightarrow$ $x + y - 2 = 0$

Ex. Find the equation of the line which passes through the point (3, 4) and the sum of its intercepts on the axes is 14.

Sol. Let the equation of the line be $\dfrac{x}{a} + \dfrac{y}{b} = 1$ ....**(i)**

This passes through (3, 4), therefore $\dfrac{3}{a} + \dfrac{4}{b} = 1$ ....**(ii)**

It is given that $a + b = 14$

$\Rightarrow$ $b = 14 - a$.

Putting b = 14 – a in **(ii)**, we get $\frac{3}{a} + \frac{4}{14-a} = 1$

$\Rightarrow$ $a^2 - 13a + 42 = 0$

$\Rightarrow$ $(a-7)(a-6) = 0$

$\Rightarrow$ $a = 7, 6$

For a = 7, b = 14 – 7 = 7 and for a = 6, b = 14 – 6 = 8.

Putting the values of a and b in **(i)**, we get the equations of the lines

$$\frac{x}{7} + \frac{y}{7} = 1 \quad \text{and} \quad \frac{x}{6} + \frac{y}{8} = 1$$

or $x + y = 7$ and $4x + 3y = 24$

Ex. Find the equation of the line which is at a distance 3 from the origin and the perpendicular from the origin to the line makes an angle of 30º with the positive direction of the x-axis.

Sol. Here $p = 3, \alpha = 30^\circ$

$\therefore$ Equation of the line in the normal form is

$$x \cos 30^\circ + y \sin 30^\circ = 3 \quad \text{or} \quad x\frac{\sqrt{3}}{2} + \frac{y}{2} = 3 \quad \text{or} \quad \sqrt{3}\,x + y = 6$$

Ex. Find the equation of the line through the point A(2, 3) and making an angle of 45º with the x-axis. Also determine the length of intercept on it between A and the line x + y + 1 = 0

Sol. The equation of a line through A and making an angle of 45º with the x-axis is

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} \quad \text{or} \quad \frac{x-2}{\frac{1}{\sqrt{2}}} = \frac{y-3}{\frac{1}{\sqrt{2}}} \quad \text{or} \quad x - y + 1 = 0$$

Suppose this line meets the line x + y + 1 = 0 at P such that AP = r. Then the co-ordinates of P are given by

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} = r$$

$\Rightarrow$ $x = 2 + r\cos 45^\circ, \quad y = 3 + r \sin 45^\circ$

$\Rightarrow$ $x = 2 + \frac{r}{\sqrt{2}}, \quad y = 3 + \frac{r}{\sqrt{2}}$

Thus, the co-ordinates of P are $\left(2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$

Since P lies on x + y + 1 = 0, so $2 + \frac{r}{\sqrt{2}} + 3 + \frac{r}{\sqrt{2}} + 1 = 0$

$\Rightarrow$ $\sqrt{2}\,r = -6$ $\Rightarrow$ $r = -3\sqrt{2}$ $\Rightarrow$ length AP = $|r| = 3\sqrt{2}$

Ex. Find slope, x-intercept & y-intercept of the line 2x – 3y + 5 = 0.

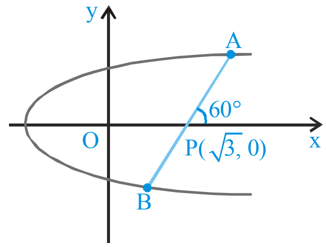
Sol. Here, $a = 2, b = -3, c = 5$

$\therefore$ slope $= -\frac{a}{b} = \frac{2}{3}$

x-intercept $= -\frac{c}{a} = -\frac{5}{2}$ y-intercept $= \frac{5}{3}$

Ex. If the line $y - \sqrt{3}x + 3 = 0$ cuts the parabola $y^2 = x + 2$ at A and B, then find the value of PA.PB {where $P \equiv (\sqrt{3}, 0)$}

Sol. Slope of line $y - \sqrt{3}x + 3 = 0$ is $\sqrt{3}$



If line makes an angle θ with x-axis, then $\tan\theta = \sqrt{3}$

$\therefore \quad \theta = 60°$

$$\frac{x-\sqrt{3}}{\cos 60°} = \frac{y-0}{\sin 60°} = r$$

$\Rightarrow \quad \left(\sqrt{3}+\frac{r}{2}, \frac{r\sqrt{3}}{2}\right)$ be a point on the parabola $y^2 = x + 2$

then $\quad \frac{3}{4}r^2 = \sqrt{3} + \frac{r}{2} + 2 \quad \Rightarrow \quad 3r^2 - 2r - 4(2+\sqrt{3}) = 0$

$\therefore \quad PA.PB = r_1r_2 = \left|\frac{-4(2+\sqrt{3})}{3}\right| = \frac{4(2+\sqrt{3})}{3}$

ANGLE BETWEEN TWO STRAIGHT LINES

If m_1 & m_2 are the slopes of two intersecting straight lines ($m_1 m_2 \neq -1$) & θ is the acute angle between them,

then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$.

ETOOS KEY POINTS

(i) There are two angles formed between two lines but usually the acute angle is taken as the angle between the lines. So we shall find θ from the above formula only by taking positive value of tanθ.

(ii) Let m_1, m_2, m_3 are the slopes of three lines $L_1 = 0$; $L_2 = 0$; $L_3 = 0$ where $m_1 > m_2 > m_3$ then the interior angles of the Δ ABC found by these formulas are given by, $\tan A = \frac{m_1 - m_2}{1 + m_1 m_2}$; $\tan B = \frac{m_2 - m_3}{1 + m_2 m_3}$ & $\tan C = \frac{m_3 - m_1}{1 + m_3 m_1}$

(iii) If equation of lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then these line are -

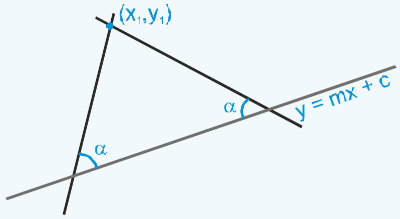
(a) Parallel $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(b) Perpendicular $\Leftrightarrow a_1a_2+b_1b_2=0$

(c) Coincident $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iv) The equation of lines passing through point (x_1, y_1) and making angle α with the line $y = mx + c$ are given by :

$(y - y_1) = \tan(\theta - \alpha)(x - x_1)$ & $(y - y_1) = \tan(\theta + \alpha)(x - x_1)$, where $\tan\theta = m$.



Ex. The acute angle between two lines is $\pi/4$ and slope of one of them is 1/2. Find the slope of the other line.

Sol. If θ be the acute angle between the lines with slopes m_1 and m_2, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

Let $\theta = \frac{\pi}{4}$ and $m_1 = \frac{1}{2}$

$\therefore \quad \tan\frac{\pi}{4} = \left|\frac{\frac{1}{2} - m_2}{1 + \frac{1}{2}m_2}\right| \quad \Rightarrow \quad 1 = \left|\frac{1 - 2m_2}{2 + m_2}\right| \quad \Rightarrow \quad \frac{1 - 2m_2}{2 + m_2} = +1$ or -1

Now $\frac{1 - 2m_2}{2 + m_2} = 1 \quad \Rightarrow \quad m_2 = -\frac{1}{3}$ and $\frac{1 - 2m_2}{2 + m_2} = -1$

$\Rightarrow \quad m_2 = 3.$

$\therefore$ The slope of the other line is either – 1/3 or 3

PARALLEL LINES

(i) When two straight lines are parallel their slopes are equal. Thus any line parallel to $y = mx + c$ is of the type $y = mx + d$, where 'd' is a parameter.

(ii) Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are parallel if $\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$.

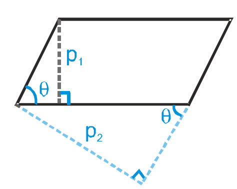
Thus any line parallel to $ax + by + c = 0$ is of the type $ax + by + k = 0$, where k is a parameter.

(iii) The distance between two parallel lines with equations $ax + by + c_1 = 0$ &

$ax + by + c_2 = 0$ is $= \left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$.

Note that coefficients of x & y in both the equations must be same.

(iv) The area of the parallelogram $= \frac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides. Note that area of the parallelogram bounded by the lines $y = m_1 x + c_1$, $y = m_1 x + c_2$ and $y = m_2 x + d_1$, $y = m_2 x + d_2$ is given by $\left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.



Ex. Find the equation of the straight line that has y-intercept 4 and is parallel to the straight line $2x - 3y = 7$.

Sol. Given line is $2x - 3y = 7$

(1) $\Rightarrow \quad 3y = 2x - 7 \quad \Rightarrow \quad y = \frac{2}{3}x - \frac{7}{3}$

$\therefore$ Slope of (1) is 2/3

The required line is parallel to (1), so its slope is also 2/3, y-intercept of required line = 4

$\therefore$ By using $y = mx + c$ form, the equation of the required line is

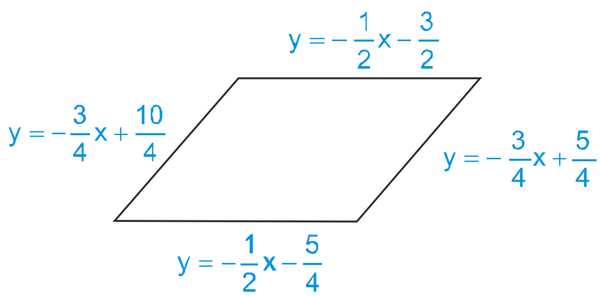
$y = \frac{2}{3}x + 4$ or $2x - 3y + 12 = 0$

Ex. If the straight line $3x + 4y + 5 - k(x + y + 3) = 0$ is parallel to y-axis, then find the value of k.

Sol. A straight line is parallel to y-axis, if its y - coefficient is zero, i.e. $4 - k = 0$ i.e. $k = 4$

Ex. Find the area of the parallelogram whose sides are $x + 2y + 3 = 0$, $3x + 4y - 5 = 0$, $2x + 4y + 5 = 0$ and $3x + 4y - 10 = 0$

Sol.

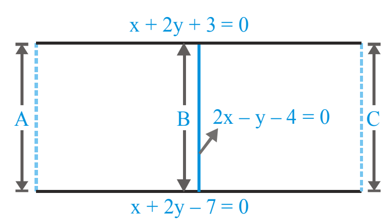


Here, $c_1 = -\frac{3}{2}$, $c_2 = -\frac{5}{4}$, $d_1 = \frac{10}{4}$, $d_2 = \frac{5}{4}$,

$m_1 = -\frac{1}{2}$, $m_2 = -\frac{3}{4}$

$$\therefore \quad \text{Area} = \left| \frac{\left(-\frac{3}{2} + \frac{5}{4}\right)\left(\frac{10}{4} - \frac{5}{4}\right)}{\left(-\frac{1}{2} + \frac{3}{4}\right)} \right| = \frac{5}{4} \text{ sq. units}$$

Ex. Three lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ form 3 sides of two squares. Find the equation of remaining sides of these squares.



Sol. Distance between the two parallel lines is $\frac{|7+3|}{\sqrt{5}} = 2\sqrt{5}$.

The equations of sides A and C are of the form

$2x - y + k = 0$.

Since distance between sides A and B

= distance between sides

B and C $\frac{|k-(-4)|}{\sqrt{5}} = 2\sqrt{5} \quad \Rightarrow \quad \frac{k+4}{\sqrt{5}} = \pm 2\sqrt{5} \quad \Rightarrow \quad k = 6, -14.$

Hence the fourth sides of the two squares are **(i)** $2x - y + 6 = 0$ **(ii)** $2x - y - 14 = 0$.

PERPENDICULAR LINES

(i) When two lines of slopes m_1 & m_2 are at right angles, the product of their slopes is -1, i.e. $m_1 m_2 = -1$. Thus any line perpendicular to $y = mx + c$ is of the form

$$y = -\frac{1}{m}x + d, \text{ where 'd' is any parameter.}$$

(ii) Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are perpendicular if $aa' + bb' = 0$. Thus any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$, where 'k' is any parameter.

Ex. If $x + 4y - 5 = 0$ and $4x + ky + 7 = 0$ are two perpendicular lines then find k.

Sol. $m_1 = -\dfrac{1}{4}$ $\qquad m_2 = -\dfrac{4}{k}$

Two lines are perpendicular if $m_1 m_2 = -1$

$$\Rightarrow \quad \left(-\frac{1}{4}\right) \times \left(-\frac{4}{k}\right) = -1 \quad \Rightarrow \quad k = -1$$

Ex. Find the equation of the straight line that passes through the point (3, 4) and perpendicular to the line $3x + 2y + 5 = 0$

Sol. The equation of a line perpendicular to $3x + 2y + 5 = 0$ is

$$2x - 3y + \lambda = 0 \qquad \text{...... (i)}$$

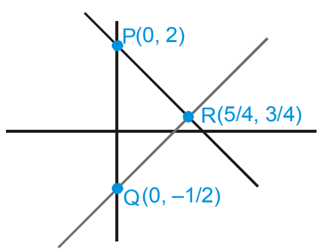
This passes through the point (3, 4)

$\therefore \quad 3 \times 2 - 3 \times 4 + \lambda = 0 \quad \Rightarrow \quad \lambda = 6$

Putting $\lambda = 6$ in **(i)**, we get $2x - 3y + 6 = 0$, which is the required equation.

Ex. A line L passes through the points (1, 1) and (0, 2) and another line M which is perpendicular to L passes through the point (0, –1/2).Then find the area of the triangle formed by these lines with y-axis.

Sol. Equation of the line L is $y - 1 = \dfrac{-1}{1}(x - 1) \Rightarrow y = -x + 2$



Equation of the line M is $y = x - 1/2$.

If these lines meet y-axis at P (0, –1/2) and Q (0, 2) then $PQ = 5/2$.

Also x-coordinate of their point of intersection R = 5/4

$$\therefore \quad \text{area of the } \Delta PQR = \frac{1}{2}\left(\frac{5}{2} \times \frac{5}{4}\right) = 25/16.$$

POSITION OF THE POINT (x_1, y_1) RELATIVE OF THE LINE $ax + by + c = 0$

If $ax_1 + by_1 + c$ is of the same sign as c, then the point (x_1, y_1) lie on the origin side of $ax + by + c = 0$. But if the sign of $ax_1 + by_1 + c$ is opposite to that of c, the point (x_1, y_1) will lie on the non–origin side of $ax + by + c = 0$.

In general two points (x_1, y_1) and (x_2, y_2) will lie on same side or opposite side of $ax + by + c = 0$ according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of same or opposite sign respectively.

Ex. Show that (1, 4) and (0, –3) lie on the opposite sides of the line $x + 3y + 7 = 0$.

Sol. At (1, 4), the value of $x + 3y + 7 = 1 + 3(4) + 7$

$$= 20 > 0.$$

At (0, – 3), the value of $x + 3y + 7 = 0 + 3(-3) + 7$

$$= -2 < 0$$

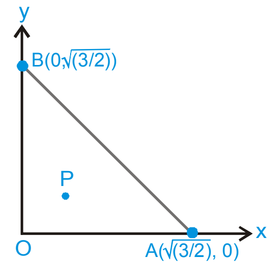
$\therefore$ The points (1, 4) and (0, – 3) are on the opposite sides of the given line.

Ex. Let P $(\sin\theta, \cos\theta)$ $(0 \le \theta \le 2\pi)$ be a point and let OAB be a triangle with vertices (0, 0), $\left(\sqrt{\frac{3}{2}}, 0\right)$ and $\left(0, \sqrt{\frac{3}{2}}\right)$. Find θ if P lies inside the ΔOAB.

Sol. Equations of lines along OA, OB and AB are $y = 0$, $x = 0$ and $x + y = \sqrt{\frac{3}{2}}$ respectively. Now P and B will lie on the same side of $y = 0$ if $\cos\theta > 0$. Similarly P and A will lie on the same side of $x = 0$ if $\sin\theta > 0$ and P and O will lie on the same side of $x + y = \sqrt{\frac{3}{2}}$ if $\sin\theta + \cos\theta < \sqrt{\frac{3}{2}}$.

Hence P will lie inside the ΔABC, if $\sin\theta > 0$, $\cos\theta > 0$ and $\sin\theta + \cos\theta < \sqrt{\frac{3}{2}}$.

Now $\sin\theta + \cos\theta < \sqrt{\frac{3}{2}}$



$\Rightarrow \quad \sin\left(\theta + \frac{\pi}{4}\right) < \frac{\sqrt{3}}{2}$

i.e. $\quad 0 < \theta + \frac{\pi}{4} < \pi/3$

or $\quad \frac{2\pi}{3} < \theta + \frac{\pi}{4} < \pi$

Since $\sin\theta > 0$ and $\cos\theta > 0$, so $0 < \theta < \frac{\pi}{12}$ or $\frac{5\pi}{12} < \theta < \frac{3\pi}{4}$.

THE RATIO IN WHICH A GIVEN LINE DIVIDES THE LINE SEGMENT JOINING TWO POINTS

Let the given line $ax + by + c = 0$ divide the line segment joining $A(x_1, y_1)$ & $B(x_2, y_2)$ in the ratio m : n, then $\frac{m}{n} = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$. If A & B are on the same side of the given line then m/n is negative but if A & B are on opposite sides of the given line, then m/n is positive

Ex. Find the ratio in which the line joining the points A (1, 2) and B(– 3, 4) is divided by the line $x + y – 5 = 0$.

Sol. Let the line $x + y = 5$ divides AB in the ratio k : 1 at P

$\therefore \quad$ co-ordinate of P are $\left(\frac{-3k+1}{k+1}, \frac{4k+2}{k+1}\right)$

Since P lies on $x + y – 5 = 0$

$\therefore \quad \frac{-3k+1}{k+1} + \frac{4k+2}{k+1} - 5 = 0 \quad \Rightarrow \quad k = -\frac{1}{2}$

$\therefore \quad$ Required ratio is 1 : 2 externally.

LENGTH OF PERPENDICULAR FROM A POINT ON A LINE

Length of perpendicular from a point (x_1,y_1) on the line $ax + by + c = 0$ is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$

In particular, the length of the perpendicular from the origin on the line $ax + by + c = 0$ is $P = \dfrac{|c|}{\sqrt{a^2+b^2}}$

Ex. Find the distance between the line $12x - 5y + 9 = 0$ and the point $(2, 1)$

Sol. The required distance $= \left|\dfrac{12\times 2 - 5\times 1 + 9}{\sqrt{12^2+(-5)^2}}\right| = \dfrac{|24-5+9|}{13} = \dfrac{28}{13}$

Ex. If the algebraic sum of perpendiculars from n given points on a variable straight line is zero then prove that the variable straight line passes through a fixed point.

Sol. Let n given points be (x_i, y_i) where $i = 1, 2.... n$ and the variable straight line is $ax + by + c = 0$.

Given that $\displaystyle\sum_{i=1}^{n}\left(\frac{ax_i + by_i + c}{\sqrt{a^2+b^2}}\right) = 0$

$\Rightarrow$ $a\Sigma x_i + b\Sigma y_i + cn = 0$

$\Rightarrow$ $a\dfrac{\Sigma x_i}{n} + b\dfrac{\Sigma y_i}{n} + c = 0$.

Hence the variable straight line always passes through the fixed point $\left(\dfrac{\Sigma x_i}{n}, \dfrac{\Sigma y_i}{n}\right)$.

CONDITION OF CONCURRENCY

Three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ & $a_3x + b_3y + c_3 = 0$ are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

ETOOS KEY POINTS

(i) If three constants A, B & C (not all zero) can be found such that
$A(a_1x + b_1y + c_1) + B(a_2x + b_2y + c_2) + C(a_3x + b_3y + c_3) \equiv 0$, then the three straight lines are concurrent.

(ii) To test the concurrency of three lines, first find out the point of intersection of any two of the three lines. If this point lies on the remaining line (i.e. coordinates of the point satisfy the equation of the line) then the three lines are concurrent otherwise not concurrent.

Ex. Prove that the straight lines $4x + 7y = 9$, $5x - 8y + 15 = 0$ and $9x - y + 6 = 0$ are concurrent.

Sol. Given lines are

$4x + 7y - 9 = 0$ ......**(i)**

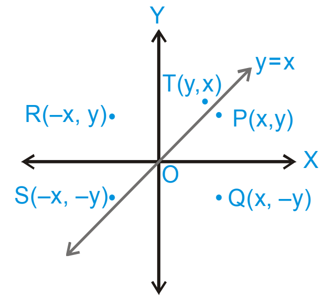
$5x - 8y + 15 = 0$ ......**(ii)**

and $9x - y + 6 = 0$ ......**(iii)**

$$\Delta = \begin{vmatrix} 4 & 7 & -9 \\ 5 & -8 & 15 \\ 9 & -1 & 6 \end{vmatrix} = 4(-48 + 15) - 7(30 - 135) - 9(-5 + 72) = -132 + 735 - 603 = 0$$

Hence lines **(i), (ii)** and **(ii)** are concurrent.

REFLECTION OF A POINT ABOUT A LINE



Let P(x, y) be any point, then its image with respect to

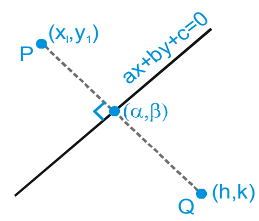
(a) x-axis is Q(x, –y)

(b) y-axis is R(–x, y)

(c) origin is S(–x,–y)

(d) line y = x is T(y, x)

(e) Reflection of a point about any arbitrary line : The image (h,k) of a point $P(x_1, y_1)$ about the line ax + by + c = 0 is given by following formula.



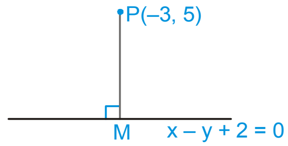
$$\frac{h-x_1}{a}=\frac{k-y_1}{b}=-2\frac{(ax_1+by_1+c)}{a^2+b^2}$$

and the foot of perpendicular (α,β) from a point (x_1, y_1) on the line ax + by + c = 0 is given by following formula.

$$\frac{\alpha-x_1}{a}=\frac{\beta-y_1}{b}=-\frac{ax_1+by_1+c}{a^2+b^2}$$

Ex. Find the foot of perpendicular of the line drawn from P (– 3, 5) on the line x – y + 2 = 0.

Sol. Slope of PM = – 1



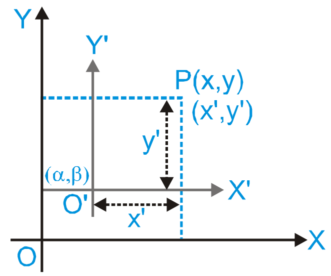
∴ Equation of PM is

x + y – 2 = 0 ....**(i)**

solving equation **(i)** with x – y + 2 = 0, we get co-ordinates of M (0, 2)

TRANSFORMATION OF AXES

(a) Shifting of Origin Without Rotation of Axes



Let P (x, y) with respect to axes OX and OY.

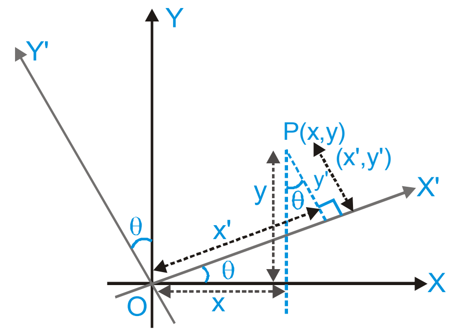
Let O' (α, β) is new origin with respect to axes OX and OY and let P (x', y') with respect to axes O'X' and O'Y', where OX and O'X' are parallel and OY and O'Y' are parallel.

Then **x = x' + α, y = y' + β**

or **x' = x – α, y' = y – β**

Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting x + α in place of x and y + β in place of y.

(b) Rotation of Axes Without Shifting the Origin



Let O be the origin. Let P (x, y) with respect to axes OX and OY and let P (x', y') with respect to axes OX' and OY' where ∠X'OX = ∠YOY' = θ, where θ is measured in anticlockwise direction.

then $x = x' \cos\theta - y' \sin\theta$

$y = x' \sin\theta + y' \cos\theta$

and $x' = x \cos\theta + y \sin\theta$

$y' = -x \sin\theta + y \cos\theta$

The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

New \ Old	$x \downarrow$	$y \downarrow$
$x' \rightarrow$	cos	
$y' \rightarrow$	−sin	

Ex. Through what angle should the axes be rotated so that the equation $9x^2 - 2\sqrt{3}\,xy + 7y^2 = 10$ may be changed to $3x^2 + 5y^2 = 5$?

Sol. Let angle be θ then replacing (x, y) by (x cosθ – y sinθ, x sinθ + y cosθ)

then $9x^2 - 2\sqrt{3}xy + 7y^2 = 10$ becomes

$$9(x\cos\theta - y\sin\theta)^2 - 2\sqrt{3}(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + 7(x\sin\theta + y\cos\theta)^2 = 10$$

$$\Rightarrow \quad x^2(9\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta + 7\sin^2\theta) + 2xy(-9\sin\theta\cos\theta - \sqrt{3}\cos2\theta + 7\sin\theta\cos\theta)$$

$$+ y^2(9\cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta + 7\cos^2\theta) = 10$$

On comparing with $3x^2 + 5y^2 = 5$ (coefficient of xy = 0)

We get $-9\sin\theta\cos\theta - \sqrt{3}\cos2\theta + 7\sin\theta\cos\theta = 0$

or $\sin2\theta = -\sqrt{3}\cos2\theta$

or $\tan2\theta = -\sqrt{3} = \tan(180° - 60°)$

or $2\theta = 120°$

$\therefore \quad \theta = 60°$

EQUATION OF BISECTORS OF ANGLES BETWEEN TWO LINES

If equation of two intersecting lines are $a_1x+b_1y+c_1=0$ and $a_2x + b_2y+c_2=0$, then equation of bisectors of the angles between these lines are written as :

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \qquad \text{......(i)}$$

(a) Equation of Bisector of Angle Containing Origin

If the equation of the lines are written with constant terms $\mathbf{c_1}$ and $\mathbf{c_2}$ positive, then the equation of the bisectors of the angle containing the origin is obtained by taking positive sign in **(i)**

(b) Equation of Bisector of Acute/Obtuse Angles

To find the equation of the bisector of the acute or obtuse angle :

(i) let ϕ be the angle between one of the two bisectors and one of two given lines. Then if $\tan\phi < 1$

i.e. $\phi < 45°$ i.e. $2\phi < 90°$, the angle bisector will be bisector of acute angle.

(ii) See whether the constant terms c_1 and c_2 in the two equation are +ve or not. If not then multiply both sides of given equation by –1 to make the constant terms positive.

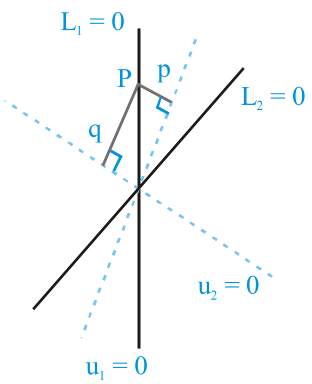
Determine the sign of $a_1a_2 + b_1b_2$

If sign of $a_1a_2 + b_1b_2$	For obtuse angle bisector	For acute angle bisector
+	use + sign in eq. (i)	use – sign in eq. (i)
–	use – sign in eq. (i)	use + sign in eq. (i)

i.e. if $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to + sign gives obtuse angle bisector

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

(iii) Another way of identifying an acute and obtuse angle bisector is as follows : Let $L_1 = 0$ & $L_2 = 0$ are the given lines & $u_1 = 0$ and $u_2 = 0$ are the bisectors between $L_1 = 0$ & $L_2 = 0$. Take a point P on any one of the lines $L_1 = 0$ or $L_2 = 0$ and drop perpendicular on $u_1 = 0$ & $u_2 = 0$ as shown . If,



$|p| < |q| \Rightarrow u_1$ is the acute angle bisector .

$|p| > |q| \Rightarrow u_1$ is the obtuse angle bisector .

$|p| = |q| \Rightarrow$ the lines L_1 & L_2 are perpendicular .

ETOOS KEY POINTS

Equation of straight lines passing through $P(x_1, y_1)$ & equally inclined with the lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisectors between these two lines & passing through the point P.

Ex. Find the equations of the bisectors of the angle between the straight lines $3x - 4y + 7 = 0$ and $12x - 5y - 8 = 0$.

Sol. The equations of the bisectors of the angles between $3x - 4y + 7 = 0$ and $12x - 5y - 8 = 0$ are

$$\frac{3x-4y+7}{\sqrt{3^2+(-4)^2}} = \pm \frac{12x-5y-8}{\sqrt{12^2+(-5)^2}}$$

or $$\frac{3x-4y+7}{5} = \pm \frac{12x-5y-8}{13}$$

or $39x - 52y + 91 = \pm (60x - 25y - 40)$

Taking the positive sign, we get $21x + 27y - 131 = 0$ as one bisector

Taking the negative sign, we get $99x - 77y + 51 = 0$ as the other bisector.

Ex. For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the

(i) bisector of the obtuse angle between them.

(ii) bisector of the acute angle between them.

(iii) bisector of the angle which contains origin.

(iv) bisector of the angle which contains (1, 2).

Sol. Equations of bisectors of the angles between the given lines are

$$\frac{4x+3y-6}{\sqrt{4^2+3^2}} = \pm \frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 9x - 7y - 41 = 0 \quad \text{and} \quad 7x + 9y - 3 = 0$$

If θ is the acute angle between the line $4x + 3y - 6 = 0$ and the bisector

$9x - 7y - 41 = 0$, then $$\tan\theta = \left|\frac{-\frac{4}{3}-\frac{9}{7}}{1+\left(\frac{-4}{3}\right)\frac{9}{7}}\right| = \frac{11}{3} > 1$$

Hence

(i) bisector of the obtuse angle is $9x - 7y - 41 = 0$

(ii) bisector of the acute angle is $7x + 9y - 3 = 0$

(iii) bisector of the angle which contains origin

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 7x+9y-3=0$$

(iv) $L_1(1, 2) = 4 \times 1 + 3 \times 2 - 6 = 4 > 0$

$L_2(1, 2) = 5 \times 1 + 12 \times 2 + 9 = 38 > 0$

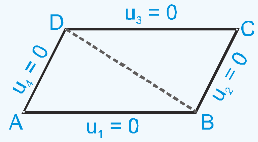
+ve sign will give the required bisector, $\dfrac{4x+3y-6}{5} = +\dfrac{5x+12y+9}{13}$

$\Rightarrow$ $9x - 7y - 41 = 0$.

FAMILY OF STRAIGHT LINES

The equation of a family of straight lines passing through the point of intersection of the lines, $L_1 \equiv a_1x + b_1y + c_1 = 0$ & $L_2 \equiv a_2x + b_2y + c_2 = 0$ is given by $L_1 + k L_2 = 0$ i.e. $(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$, where k is an arbitrary real number.

(i) If $u_1 = ax + by + c$, $u_2 = a'x + b'y + d$, $u_3 = ax + by + c'$, $u_4 = a'x + b'y + d'$ then $u_1 = 0; u_2 = 0; u_3 = 0\ ; u_4 = 0$ form a parallelogram.
The diagonal BD can be given by $u_2u_3 - u_1u_4 = 0$.



(ii) The diagonal AC is also given by $u_1 + \lambda u_4 = 0$ and $u_2 + \mu u_3 = 0$, if the two equations are identical for some real λ and μ.
[For getting the values of λ & μ compare the coefficients of x, y & the constant terms].

Ex. Find the equation of the straight line which passes through the point (2, –3) and the point of intersection of the lines $x + y + 4 = 0$ and $3x - y - 8 = 0$.

Sol. Any line through the intersection of the lines $x + y + 4 = 0$ and $3x - y - 8 = 0$ has the equation

$$(x + y + 4) + \lambda (3x - y - 8) = 0 \qquad ......\text{(i)}$$

This will pass through (2, –3) if

$$(2 - 3 + 4) + \lambda (6 + 3 - 8) = 0 \quad \text{or} \quad 3 + \lambda = 0 \Rightarrow \lambda = -3.$$

Putting the value of λ in (i), the required line is $(x + y + 4) + (-3)(3x - y - 8) = 0$

or $\quad -8x + 4y + 28 = 0 \quad$ or $\quad 2x - y - 7 = 0$

Ex. Prove that each member of the family of straight lines

$(3\sin\theta + 4\cos\theta)x + (2\sin\theta - 7\cos\theta)y + (\sin\theta + 2\cos\theta) = 0$ (θ is a parameter) passes through a fixed point.

Sol. The given family of straight lines can be rewritten as

$$(3x + 2y + 1)\sin\theta + (4x - 7y + 2)\cos\theta = 0$$

or, $\quad (4x - 7y + 2) + \tan\theta(3x + 2y + 1) = 0$ which is of the form $L_1 + \lambda L_2 = 0$

Hence each member of it will pass through a fixed point which is the intersection of

$$4x - 7y + 2 = 0 \text{ and } 3x + 2y + 1 = 0 \quad \text{i.e.} \quad \left(\frac{-11}{29}, \frac{2}{29}\right).$$

PAIR OF STRAIGHT LINES

(a) Homogeneous Equation of Second Degree

Let us consider the homogeneous equation of 2nd degree as

$$ax^2 + 2hxy + by^2 = 0 \qquad ......\text{(i)}$$

which represents pair of straight lines passing through the origin.

Now, we divide by x^2, we get

$$a + 2h\left(\frac{y}{x}\right) + b\left(\frac{y}{x}\right)^2 = 0$$

$\frac{y}{x} = m$ (say)

then $a + 2hm + bm^2 = 0$ ......(ii)

if m_1 & m_2 are the roots of equation (ii), then $m_1 + m_2 = -\frac{2h}{b}$, $m_1m_2 = \frac{a}{b}$ and also,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2}\right|$$

$$= \left|\frac{\sqrt{\frac{4h^2}{b} - \frac{4a}{b}}}{1 + \frac{a}{b}}\right| = \pm\frac{2\sqrt{h^2 - ab}}{a + b}$$

These line will be

(i) **Real and different**, if $h^2 - ab > 0$

(ii) **Real and coincident**, if $h^2 - ab = 0$

(iii) **Imaginary**, if $h^2 - ab < 0$

(iv) The condition that these lines are :

- **(i)** At **right angles** to each other is $\mathbf{a + b = 0}$. i.e. coefficient of $\mathbf{x^2}$ + coefficient of $\mathbf{y^2} = 0$.
- **(ii)** **Coincident** is $\mathbf{h^2 = ab}$.
- **(iii)** **Equally inclined** to the axes of x is $\mathbf{h = 0}$. i.e. coefficient of $\mathbf{xy = 0}$.

Homogeneous equation of 2nd degree $\mathbf{ax^2 + 2hxy + by^2 = 0}$ always represent a pair of straight lines whose equations are

$$y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right)x \equiv y = m_1x \ \& \ y = m_2x \ \text{ and } m_1 + m_2 = -\frac{2h}{b}\ ; m_1m_2 = \frac{a}{b}$$

These straight lines passes through the origin.

❖ A homogeneous equation of degree n represents n straight lines passing through **origin**.

(b) General Equation and Homogeneous Equation of Second Degree

(i) The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines,

if $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ i.e. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(ii) If θ be the angle between the lines, then $\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a + b}$

Obviously these lines are

- **(i)** Parallel, if $\Delta = 0$, $h^2 = ab$ or if $h^2 = ab$ and $bg^2 = af^2$
- **(ii)** Perpendicular, if $a + b = 0$ i.e. coeff. of x^2 + coeff. of $y^2 = 0$.

(iii) Pair of straight lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ and through origin are given by $bx^2 - 2hxy + ay^2 = 0$.

(iv) The product of the perpendiculars drawn from the point (x_1, y_1) on the lines $ax^2 + 2hxy + by^2 = 0$ is

$$\left|\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}\right|$$

(v) The product of the perpendiculars drawn from the origin to the lines

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\left|\dfrac{c}{\sqrt{(a-b)^2 + 4h^2}}\right|$

(c) The Combined Equation of Angle Bisectors

The combined equation of angle bisectors between the lines represented by homogeneous equation of 2^{nd} degree is given by $\dfrac{x^2 - y^2}{a - b} = \dfrac{xy}{h}$, $\mathbf{a \neq b, h \neq 0}$.

(i) If $a = b$, the bisectors are $x^2 - y^2 = 0$ i.e. $x - y = 0, x + y = 0$

(ii) If $h = 0$, the bisectors are $xy = 0$ i.e. $x = 0, y = 0$.

(iii) The two bisectors are always at **right angles**, since we have coefficient of x^2 + coefficient of $y^2 = 0$

Ex. Show that the equation $6x^2 - 5xy + y^2 = 0$ represents a pair of distinct straight lines, each passing through the origin. Find the separate equations of these lines.

Sol. The given equation is a homogeneous equation of second degree. So, it represents a pair of straight lines passing through the origin. Comparing the given equation with

$ax^2 + 2hxy + by^2 = 0$, we obtain $a = 6$, $b = 1$ and $2h = -5$.

$\therefore \quad h^2 - ab = \dfrac{25}{4} - 6 = \dfrac{1}{4} > 0 \quad \Rightarrow \quad h^2 > ab$

Hence, the given equation represents a pair of distinct lines passing through the origin.

Now, $6x^2 - 5xy + y^2 = 0 \quad \Rightarrow \quad \left(\dfrac{y}{x}\right)^2 - 5\left(\dfrac{y}{x}\right) + 6 = 0$

$\Rightarrow \quad \left(\dfrac{y}{x}\right)^2 - 3\left(\dfrac{y}{x}\right) - 2\left(\dfrac{y}{x}\right) + 6 = 0 \quad \Rightarrow \quad \left(\dfrac{y}{x} - 3\right)\left(\dfrac{y}{x} - 2\right) = 0$

$\Rightarrow \quad \dfrac{y}{x} - 3 = 0$ or $\dfrac{y}{x} - 2 = 0 \quad \Rightarrow \quad y - 3x = 0$ or $y - 2x = 0$

So the given equation represents the straight lines $y - 3x = 0$ and $y - 2x = 0$.

Ex. If $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines, then find the value of λ.

Sol. **Here** $a = \lambda$, $b = 12$, $c = -3$, $f = -8$, $g = 5/2$, $h = -5$

Using condition $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, we have

$\lambda(12)(-3) + 2(-8)(5/2)(-5) - \lambda(64) - 12(25/4) + 3(25) = 0$

$\Rightarrow \quad -36\lambda + 200 - 64\lambda - 75 + 75 = 0 \quad \Rightarrow \quad 100\lambda = 200$

$\therefore \quad \lambda = 2$

Ex. Find the angle between the pair of straight lines $4x^2 + 24xy + 11y^2 = 0$

Sol. Given equation is $4x^2 + 24xy + 11y^2 = 0$

Here a = coeff. of $x^2 = 4$, b = coeff. of $y^2 = 11$

and 2h = coeff. of xy = 24 $\therefore$ h = 12

Now $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\frac{2\sqrt{144-44}}{4+11}\right| = \frac{4}{3}$

Where θ is the acute angle between the lines.

$\therefore$ acute angle between the lines is $\tan^{-1}\left(\frac{4}{3}\right)$ and obtuse angle between them is $\pi - \tan^{-1}\left(\frac{4}{3}\right)$

Ex. If pairs of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that pq = –1.

Sol. According to the question, the equation of the bisectors of the angle between the lines

$x^2 - 2pxy - y^2 = 0$ ...... **(i)**

is $x^2 - 2qxy - y^2 = 0$ ...... **(ii)**

$\therefore$ The equation of bisectors of the angle between the lines **(i)** is $\frac{x^2 - y^2}{1-(-1)} = \frac{xy}{-p}$ .....**(iii)**

$\Rightarrow$ $-px^2 - 2xy + py^2 = 0$

Since **(ii)** and **(iii)** are identical, comparing **(ii)** and **(iii)**, we get $\frac{1}{-p} = \frac{-2q}{-2} = \frac{-1}{p}$ $\Rightarrow$ pq = –1

HOMOGENIZATION

This method is used to write the joint equation of two lines connecting origin to the points of intersection of a given line and a given second degree curve.

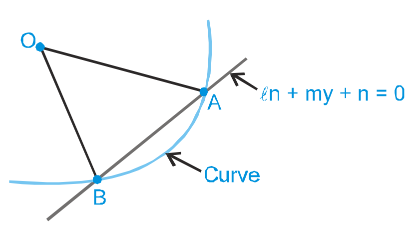
The equation of a pair of straight lines joining origin to the points of intersection of the line

$L \equiv \ell x + my + n = 0$ and a second degree curve

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

is $ax^2 + 2hxy + by^2 + 2gx\left(\frac{\ell x + my}{-n}\right) + 2fy\left(\frac{\ell x + my}{-n}\right) + c\left(\frac{\ell x + my}{-n}\right)^2 = 0.$

The equation is obtained by homogenizing the equation of curve with the help of equation of line.



(i) Here we have written 1 as $\frac{\ell x + my}{-n}$ and converted all terms of the curve to second degree expressions.

(ii) Equation of any curve passing through the points of intersection of two curves $C_1 = 0$ and $C_2 = 0$ is given by $\lambda C_1 + \mu C_2 = 0$, where λ & μ are parameters.

Ex. The chord $\sqrt{6}y = \sqrt{8}px + \sqrt{2}$ of the curve $py^2 + 1 = 4x$ subtends a right angle at origin then find the value of p.

Sol. $\sqrt{3}y - 2px = 1$ is the given chord. Homogenizing the equation of the curve, we get,

$$py^2 - 4x(\sqrt{3}\,y - 2px) + (\sqrt{3}\,y - 2px)^2 = 0$$

$\Rightarrow$ $(4p^2 + 8p)x^2 + (p + 3)y^2 - 4\sqrt{3}\,xy - 4\sqrt{3}\,pxy = 0$

Now, angle at origin is 90°

$\therefore$ coefficient of x^2 + coefficient of $y^2 = 0$

$\therefore$ $4p^2 + 8p + p + 3 = 0$

$\Rightarrow$ $4p^2 + 9p + 3 = 0$

$\therefore$ $p = \frac{-9 \pm \sqrt{81 - 48}}{8} = \frac{-9 \pm \sqrt{33}}{8}$.

0TIPS & FORMULAS

1. Relation Between Cartesian Co-ordinate & Polar Co-ordinate System

If (x, y) are cartesian co-ordinates of a point P, then : $x = r\cos\theta$, $y = r\sin\theta$

and $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

2. Distance Formula and its Applications

If A (x_1, y_1) and B(x_2, y_2) are two points, then

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note

(i) Three given points A, B and C are collinear, when sum of any two distance out of AB, BC, CA is equal to the remaining third otherwise the points will be the vertices of triangle.

(ii) Let A, B, C & D be the four given points in a plane. Then the quadrilateral will be :

- **(a)** Square if $AB = BC = CD = DA$ & $AC = BD$; $AC \perp BD$
- **(b)** Rhombus if $AB = BC = CD = DA$ and $AC \neq BD$; $AC \perp BD$
- **(c)** Parallelogram if $AB = DC$, $BC = AD$; $AC \neq BD$; $AC \not\perp BD$
- **(d)** Rectangle if $AB = CD$, $BC = DA$, $AC = BD$; $AC \not\perp BD$

3. Section Formula

The co-ordinates of a point dividing a line joining the points A(x_1, y_1) and B(x_2, y_2) in the ratio m : n is given by :

(a) For internal division : $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$

(b) For external division : $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$

(c) Line $ax + by + c = 0$ divides line joining points P(x_1y_1) & Q(x_2,y_2) in ratio $= \frac{(ax_1 + by_1 + c)}{(ax_2 + by_2 + c)}$

4. Co-ordinates of Some Particular Points

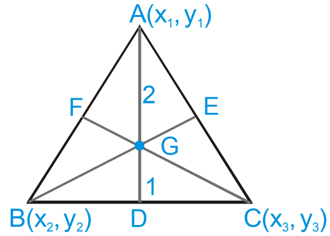
Let A(x_1, y_1), B(x_2,y_2) and C(x_3, y_3) are vertices of any triangle ABC, then

(a) Centroid

(i) The centroid is the point of intersection of the medians (line joining the mid point of sides and opposite vertices).

(ii) Centroid divides the median in the ratio of 2 : 1.

(iii) Co-ordinates of centroid $G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$



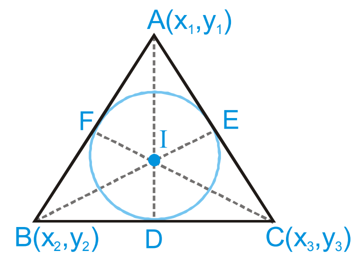
(iv) If P is any internal point of triangle such that area of ΔAPB, ΔAPC and ΔBPC are same then P must be centroid.

(b) Incentre

The incentre is the point of intersection of internal bisectors of the angles of a triangle. Also it is a centre of a circle touching all the sides of a triangle .

Co-ordinates of incentre $I = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

Where a, b, c are the sides of triangle ABC.



Note

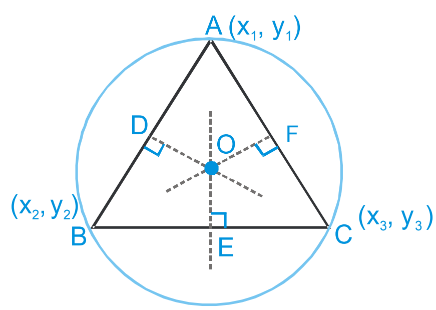
(i) Angle bisector divides the opposite sides in the ratio of remaining sides e.g. $\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$

(ii) Incentre divides the angle bisectors in the ratio (b + c) : a, (c + a) : b, (a + b) : c

(c) Circumcentre

It is the point of intersection of perpendicular bisectors of the sides of a triangle. If O is the circumcentre of any triangle ABC, then $OA^2 = OB^2 = OC^2$.

Also it is a centre of a circle touching all the vertices of a triangle.



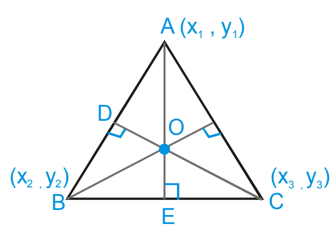
Note

(i) If a triangle is right angle, then its circumcentre is mid point of hypotenuse.

(ii) Find perpendicular bisector of any two sides and solve them to find circumcentre.

(d) Orthocentre

It is the point of intersection of perpendicular drawn from vertices on opposite sides of a triangle and can be obtained by solving the equation of any two altitudes.



Note

If a triangle is right angled triangle, then orthocentre is the point where right angle is formed.

Remarks

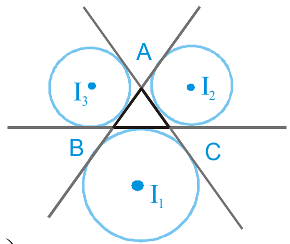
(i) If the triangle is equilateral, then centroid, incentre, orthocentre, circumcentre, coincides.

(ii) Orthocentre, centroid and circumcentre are always collinear and centroid divides the line joining. Orthocentre and circumcentre in the ratio 2 : 1

(iii) In an isosceles triangle centroid, orthocentre, incentre, circumcentre lies on the same line.

(e) Ex-centres

The centre of the circle which touches side BC and the extended portions of sides AB and AC is called the ex-centre of ΔABC with respect to the vertex A. It is denoted by I_1 and its coordinates are



$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

Similarly ex-centres of ΔABC with respect to vertices B and C are denoted by I_2 and I_3 respectively, and

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right)$$

$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

5. Area of Triangle

Let $A(x_1, y_1)$ $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of a triangle, then

$$\text{Area of } \Delta ABC = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| = \frac{1}{2}\left|\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]\right|$$

To remember the above formula, take the help of the following method :

$$= \frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{bmatrix} = \frac{1}{2}\left|\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\right]\right|$$

Remarks

(i) If the area of triangle joining three points is zero, than the points are collinear.

(ii) Area of Equilateral triangle

If altitude of any equilateral triangle, then its area = $\frac{P^2}{\sqrt{3}}$

If 'a' be the side of equilateral triangle, then its area = $\left(\frac{a^2\sqrt{3}}{4}\right)$

(iii) Area of quadrilateral whose consecutive vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) & (x_4, y_4) is $\frac{1}{2}\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_4 & y_2 - y_4 \end{vmatrix}$

6. Condition of Collinearity for Three Points

Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if any one of the given point lies on the line passing through the remaining two points.

Thus the required condition is -

(a) Area of triangle ABC is zero i.e. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \mathbf{0}$

(b) Slope of AB = slope of BC = slope of AC. i.e. $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_3 - y_2}{x_3 - x_2} = \dfrac{y_3 - y_1}{x_3 - x_1}$

(c) Find the equation of line passing through 2 given points, if the third point satisfies the given equation of the line, then three points are collinear.

7. Equation of Straight Line

A relation between x and y which is satisfied by co-ordinates of every point lying on a line is called equation of the straight line. Here, remember that every one degree equation in variable x and y always represents a straight line i.e. $\mathbf{ax + by + c = 0}$ **; a & b** $\mathbf{\neq 0}$ simultaneously.

(a) Equation of a line parallel to x-axis at a distance 'a' is $\mathbf{y = a}$ or $\mathbf{y = -a}$

(b) Equation of x-axis is $\mathbf{y = 0}$

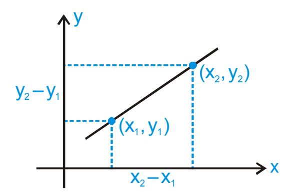
(c) Equation of a line parallel to y-axis at a distance 'b' is $\mathbf{x = b}$ or $\mathbf{x = -b}$

(d) Equation of y-axis is $\mathbf{x = 0}$

8. Slope of line

If a given line makes an angle $\theta(0° \le \theta < 180°, \theta \neq 90°)$ with the positive direction of x-axis, then slope of this line will be $\tan\theta$ and is usually denoted by the letter **m** i.e. $\mathbf{m = \tan\theta}$. If $A(x_1, y_1)$ and $B(x_2, y_2)$ & $x_1 \neq x_2$ then slope of line

AB $= \dfrac{y_2 - y_1}{x_2 - x_1}$



9. Standard Forms of Equations of a Straight Line

(a) **Slope Intercept Form :** Let m be the slope of a line and c its intercept on y-axis. Then the equation of this straight line is written as : $y = mx + c$

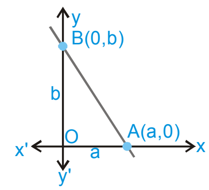
If the line passes through origin, its equation is written as $y = mx$

(b) **Point Slope Form :** If m be the slope of a line and it passes through a point (x_1,y_1), then its equation is written as : $y - y_1 = m(x - x_1)$

(c) **Two Point Form :** Equation of a line passing through two points (x_1,y_1) and (x_2,y_2) is written as :

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

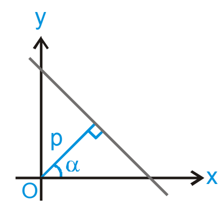
(d) **Intercept Form :** If a and b are the intercepts made by a line on the axes of x and y, its equation is written as : $\frac{x}{a}+\frac{y}{b}=1$



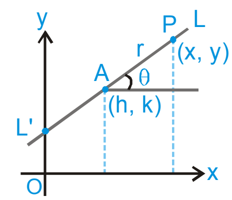
(i) Length of intercept of line between the coordinate axes $=\sqrt{a^2+b^2}$

(ii) Area of triangle AOB $=\frac{1}{2}OA.OB=\left|\frac{1}{2}ab\right|$

(e) **Normal Form :** If p is the length of perpendicular on a line from the origin, and α the angle which this perpendicular makes with positive x-axis, then the equation of this line is written as :
$\mathbf{x\cos\alpha + y\sin\alpha = p}$ (p is always positive) where $\mathbf{0 \le \alpha < 2\pi}$.



(f) **Parametric Form :** To find the equation of a straight line which passes through a given point A(h, k) and makes a given angle θ with the positive direction of the x-axis.
P(x, y) is any point on the line LAL'.



Let AP = r, then $\mathbf{x-h=r\cos\theta}$, $\mathbf{y-k=r\sin\theta}$ & $\frac{x-h}{\cos\theta}=\frac{y-k}{\sin\theta}=\mathbf{r}$ is the equation of the straight line LAL'.

Any point P on the line will be of the form $(h+r\cos\theta, k+r\sin\theta)$, where $|r|$ gives the distance of the point P from the fixed point (h, k).

(g) **General Form :** We know that a first degree equation in x and y, $ax+by+c=0$ always represents a straight line. This form is known as general form of straight line.

(i) Slope of this line $=\frac{-a}{b}=-\frac{\text{coeff. of } x}{\text{coeff. of } y}$

(ii) Intercept by this line on x-axis $=-\frac{c}{a}$ and intercept by this line on y-axis $=-\frac{c}{b}$

(iii) To change the general form of a line to normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{a^2+b^2}$.

10. Angle Between two Lines

(a) If θ be the angle between two lines : $y=m_1x+c$ and $y=m_2x+c_2$, then $\tan\theta=\pm\left(\frac{m_1-m_2}{1+m_1m_2}\right)$

(b) If equation of lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then these line are

(i) Parallel $\Leftrightarrow$ $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$

(ii) Perpendicular $\Leftrightarrow$ $a_1a_2+b_1b_2=0$

(iii) Coincide $\Leftrightarrow$ $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

11. Length of Perpendicular From a Point on a Line

Length of perpendicular from a point (x_1,y_1) on the line $ax + by + c = 0$ is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

In particular, the length of the perpendicular from the origin on the line $ax + by + c = 0$ is $P = \dfrac{|c|}{\sqrt{a^2 + b^2}}$

12. Distance Between two Parallel Lines

(a) The distance between two parallel lines $ax + by + c_1=0$ and $ax+by+c_2=0$ is $= \dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$

(**Note :** The coefficients of x & y in both equations should be same)

(b) The area of the parallelogram $= \dfrac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides . Note that area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\dfrac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.

13. Equation of Lines Parallel and Perpendicular to a Given Line

(a) Equation of line parallel to line $ax + by + c = 0$, $\quad$ $\mathbf{ax + by + \lambda = 0}$

(b) Equation of line perpendicular to line $ax + by + c = 0$, $\quad$ $\mathbf{bx - ay + k = 0}$

Here λ, k, are parameters and their values are obtained with the help of additional information given in the problem.

14. Straight Line Making a Given Angle with a Line

Equations of lines passing through a point (x_1,y_1) and making an angle α, with the line $y = mx + c$ is written as :

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

15. Position of two Points with Respect to a Given Line

Let the given line be $ax + by + c = 0$ and $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points. If the expressions $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs, then both the points P and Q lie on the same side of the line $ax + by + c = 0$. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite sides of the line.

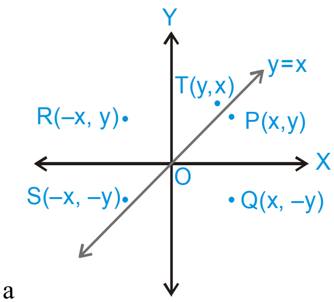
16. Concurrency of Lines

Three lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent, if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_2 & c_3 \end{vmatrix} = 0$

NOTE

If lines are concurrent then $\Delta = 0$ but if $\Delta = 0$ then lines may or may not be concurrent {lines may be parallel}

17. Reflection of a Point



Let P(x, y) be any point, then its image with respect to

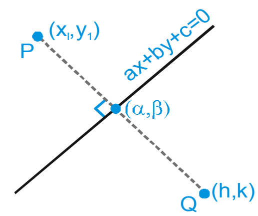
(a) x-axis is Q(x, –y)

(b) y-axis is R(–x, y)

(c) origin is S(–x,–y)

(d) line y = x is T(y, x)

(e) Reflection of a point about any arbitrary line : The image (h,k) of a point $P(x_1, y_1)$ about the line ax + by + c = 0 is given by following formula.



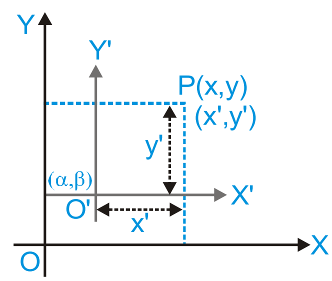
$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = -2\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$$

and the foot of perpendicular (α,β) from a point (x_1, y_1) on the line ax + by + c = 0 is given by following formula.

$$\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = -\frac{ax_1 + by_1 + c}{a^2 + b^2}$$

18. Transformation of Axes

(a) Shifting of Origin without Rotation of Axes



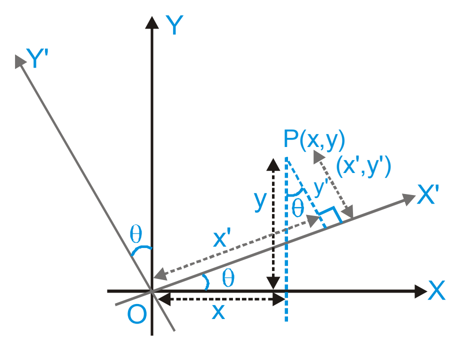
If coordinates of any points P(x, y) with respect to new origin (α, β) will be (x', y')

Then **x = x' + α, y = y' + β**

or **x' = x – α, y' = y – β**

Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting x + α in place of x and y + β in place of y.

(b) Rotation of Axes without Shifting the Origin



Let O be the origin. Let P (x, y) with respect to axes OX and OY and let P (x', y') with respect to axes OX' and OY' where ∠X'OX = ∠YOY' = θ, where θ is measured in anticlockwise direction.

then $x = x' \cos\theta - y' \sin\theta$

$y = x' \sin\theta + y' \cos\theta$

and $x' = x \cos\theta + y \sin\theta$

$y' = -x \sin\theta + y \cos\theta$

The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

New \ Old	x ↓	y ↓
x' →	cos	
y' →	–sin	

19. Equation of Bisectors of Angles Between two Lines

If equation of two intersecting lines are $a_1x+b_1y+c_1=0$ and $a_2x + b_2y+c_2=0$, then equation of bisectors of the angles between these lines are written as :

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \qquad \text{..........(i)}$$

(a) Equation of Bisector of Angle Containing Origin

If the equation of the lines are written with constant terms c_1 and c_2 positive, then the equation of the bisectors of the angle containing the origin is obtained by taking positive sign in (i)

(b) Equation of Bisector of Acute/obtuse Angles

See whether the constant terms c_1 and c_2 in the two equation are +ve or not. If not multiply both sides given equation by –1 to make the constant terms positive.

If sign of $a_1a_2 + b_1b_2$	For obtuse angle bisector	For acute angle bisector
+	use + sign in eq. (i)	use – sign in eq. (i)
–	use – sign in eq. (i)	use + sign in eq. (i)

i.e. if $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to + sign gives obtuse angle bisector

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

20. Family of Lines

If equation of two lines be $P \equiv a_1x + b_1y + c_1= 0$ and $Q \equiv a_2x + b_2y + c_2=0$, then the equation of the lines passing through the point of intersection of these lines is : $P + \lambda Q = 0$ or $a_1x + b_1y + c_1 + \lambda (a_2x + b_2y + c_2) = 0$. The value of λ is obtained with the help of the additional informations given in the problem.

21. General Equation and Homogeneous Equation of Second Degree

(a) The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, if

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \text{ i.e. } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

(b) If θ be the angle between the lines, then $\tan\theta = \pm\frac{2\sqrt{h^2-ab}}{a+b}$

Obviously these lines are

(1) Parallel, if $\Delta = 0$, $h^2 = ab$ or if $h^2 = ab$ and $bg^2 = af^2$

(2) Perpendicular, if $a + b = 0$ i.e. coeff. of x^2 + coeff. of $y^2 = 0$.

(c) Homogeneous equation of 2nd degree $ax^2 + 2hxy + by^2 = 0$ always represent a pair of straight lines whose equations are $y = \left(\frac{-h\pm\sqrt{h^2-ab}}{b}\right)x \equiv y = m_1x$ & $y = m_2x$

and $m_1 + m_2 = -\frac{2h}{b}$, $m_1m_2 = \frac{a}{b}$

These straight lines passes through the origin and for finding the angles between these lines same formula as given for general equation is used.

The condition that these lines are :

(i) At right angle to each other is $a + b = 0$ i.e,. coeff. of x^2 + co-efficient of $y^2 = 0$.

(ii) Coincident is $h^2 = ab$.

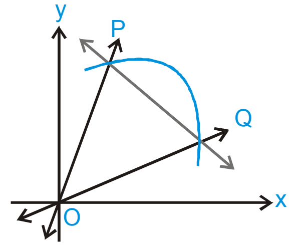
(iii) Equally inclined to the axis of x is $h = 0$. i.e., coeff. of $xy = 0$

(d) The combined equation of angle bisector between the lines represented by homogeneous equation of 2nd degree is given by $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$, $a \neq b$, $h \neq 0$.

(e) Pair of straight lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ and through origin are given by $bx^2 - 2hxy + ay^2 = 0$

(f) If lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are parallel then distance between them is $= 2\sqrt{\frac{g^2 - ac}{a(a+b)}}$

22. Equations of Lines Joining the Points of Intersection of a Line and a Curve to the Origin



Let the equation of curve be :

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...**(i)**

and straight line be

$lx + my + n = 0$ ...**(ii)**

Now joint equation of line OP and OQ joining the origin and points of intersection P and Q can be obtained by making the equation **(i)** homogenous with the help of equation of the line. Thus required equation is given by

$$ax^2 + 2hxy + by^2 + 2(gx + fy)\left(\frac{\ell x + my}{-n}\right) + c\left(\frac{\ell x + my}{-n}\right)^2 = 0$$

23. Standard Results

(a) Area of rhombus formed by lines $a|x| + b|y| + c = 0$

or $\pm\, ax \pm\, by + c = 0$ is $\frac{2c^2}{|ab|}$

(b) Area of triangle formed by line $ax + by + c = 0$ and axes is $\frac{c^2}{2|ab|}$

(c) Co-ordinates of foot of perpendicular (h, k) from (x_1, y_1) to the line $ax + by + c = 0$ is given by $\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}$

(d) Image of point (x_1, y_1) w.r. to the line $ax + by + c = 0$ is given by $\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$