CIRCULAR MOTION

Introduction

A car rounds a curve. A satelite circles earth. Electrons revolve around the nucleus. Since they are not in straight line, their velocites are changing either with direction or magnitude or with both i.e., they are accelerated. Newton`s law tell us that force acts on each. which is this force and how it does. So, will be discussed in this chapter.

Kinematics of Circular motion

Circular motion is that kind of motion in which a particle moves on the circumference of a circle. A particle moves on the circumference of a circle. Direction of velocity of the particle is always along the tangent to the circle and hence change continuously. But the speed (magnitude of the velocity) of the particle may or may not change during the motion.

If speed of a particle describing a circle is a constant, then it is said to be in uniform circular motion (abbreviated as UCM).

If speed of the particle change then it is said to be in non-uniform circular motion.

motion of a giant wheel. It takes few minutes to pick its maximum speed and retain that speed for few minutes and then slow down to come to a halt in the last few minutes. the wheel is in UCM in middle few minutes during which it retains the constant pace of rotation and in the rest of the time it is non-uniform circular motion.

Variables of motion

(a) Angular position :

To decide the angular position of a point in space we need to specify.

(i) Origin **(ii)** Reference line

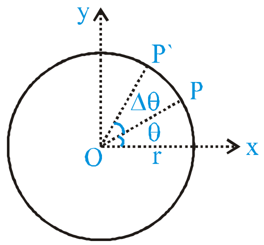
The angle made by the position. Vector w.r.t. origin, with the reference line is called angular position.

Clearly angular position dependes on the choice of the origin as well as the reference line.

Circular motion is a two dimensional motion or motion in a plane. Suppose a particle P is moving in a circle of radius r and curve O. The angular position of the particle P at a given instant may be described by the angle θ is called the angular position of the particle.

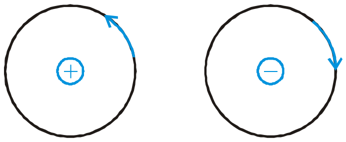
(b) Angular Displacement:

Angle through which the position vector of the moving particle rotates in a given time interval is called it angular displacement. Angular displacement depends on origin, but it does not depends on the reference line. As the particle moves on the above circle its angualr position changes. Suppose the point rotates through an angle $\Delta\theta$ in time Δt, then $\Delta\theta$ is angular displacement.



Sign Convention:

Let us take anticlockwise direction positive and clockwise direction negative



Anticlockwise (Positive) Clockwise direction (Negative)

Axis of Circle:

A line passing through the centre of the circle and perpendicular to the plane of the circle is known as axis.

(i) Angular displacement is a dimensionless quantity. Its SI unit is radian, some other unit are degree and revolution.
$2\pi \text{rad} = 360^{\circ} = 1 \text{ rev}$

(ii) Infinitesimally small angular displacement is a vector quantity, but finite angular displacement is a scalar, because while the displacement the addition of the infinitesimally.

Small angular displacements is commutaive, addition of finite angular displacement is not.

$$d\vec{\theta}_1 + d\vec{\theta}_2 = d\vec{\theta}_2 + d\vec{\theta}_1$$

But $\theta_1 + \theta_2 \neq \theta_2 + \theta_1$

(iii) Direction of small angualr displacement is decided by right hand thumb rule. When the fingers are directed along the motion of the point then thumb will represent the direction of angular displacement.

(c) Angular Velocity (ω)

(i) Average Angular Velocity

$$\omega_{av} = \frac{\text{Angular displacement}}{\text{Total time taken}} \quad \Rightarrow \quad \omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

Where θ_1 and θ_2 are angular position of the particle at time t_1 and t_2. Since angular displacement is a scalar, average angular velocity is also a scalar.

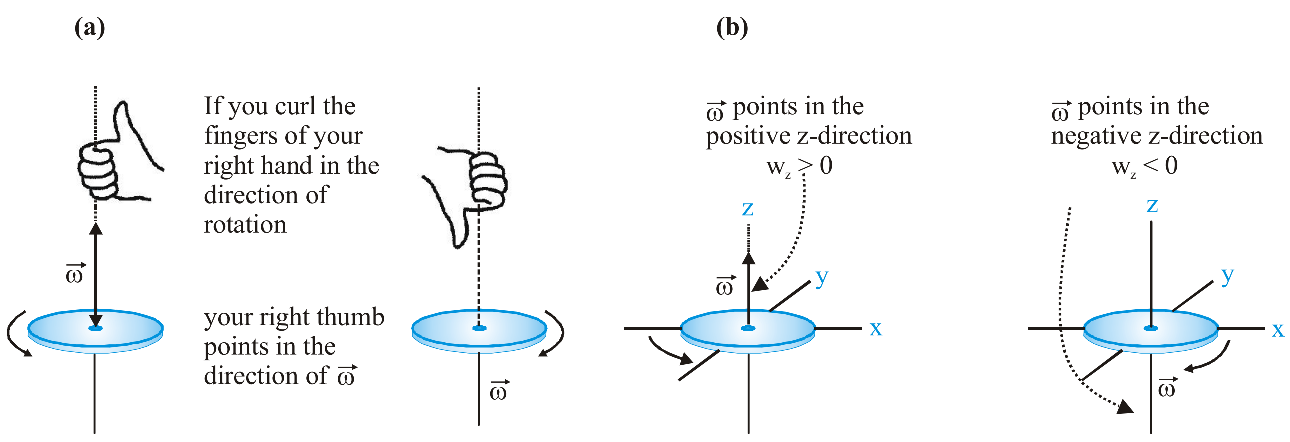
(ii) Instataneous Angular Velocity:

It is the limit of average velocity as Δt approaches zero i.e.,

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\theta}}{\Delta t} = \frac{d\vec{\theta}}{dt}$$

Since infinitesimally small angular displacement $d\vec{\theta}$ is a vector quantity, instantaneous angualr velocity $\vec{\omega}$ is also a vector, whose direction is given by rigth hand thumb rule.

Right hand thumb rule :



(i) Angular velocity has dimension of $[T^{-1}]$ and SI unit rad/s.

(ii) For a rigid body, as all points will rotate through same angle in same time, angular velocity is a characteristic of the body as a whole, e.g. angular velocity of all points of earth about earth`s axis is $(2\pi/24)$ rad/hr.

(iii) If a body makes'n' rotation in 't' second then average angular velocity in radian per second will be

$$\omega_{av} = \frac{2n\pi}{T}$$

If T is the period and 'f' the frequancy of uniform circular motion.

$$\omega_{av} = \frac{2\pi}{T} = 2nf$$

Ex. If angular displacement of a particle is given by $\theta = a - bt + ct^2$, then find its angular velocity.

Sol. $\omega = \frac{d\theta}{dt} = -b + 2ct$

Ex. Is the angular velocity of rotation of hour hand of a watch greater or smaller than the angular velocity of Earth's rotation about its own axis ?

Sol. Hour hand completes one rotation in 12 hours while Earth completes one rotation in 24 hours. So, angular velocity of hour hand is double the angular velocity of Earth. $\left(\omega = \frac{2\pi}{T}\right)$.

(d) Angular Acceleration (α) :

(i) Average Angular Acceleration:

Let w_1 and w_2 be the instantaneous angular speed at times t_1 and t_2 respectively, then average angular acceleration α_{av} is defined as

$$\vec{\alpha}_{av} = \frac{\vec{\omega}_2 - \vec{\omega}_1}{t_2 - t_1} = \frac{\Delta\vec{\omega}}{dt}$$

(ii) Instantaneous Angular Acceleration:

It is the limit of average angular acceleration as Δt approaches zero i.e.,

$$\vec{\alpha} \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{dt} = \frac{d\vec{\omega}}{dt}$$

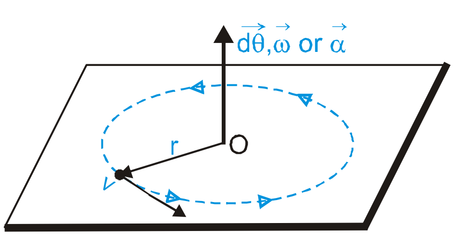
Since $\vec{\omega} = \frac{d\vec{\theta}}{dt}$, $\quad \therefore \; \vec{\alpha} = \frac{d\vec{\omega}}{dt} \; \therefore \; = \frac{d^2\vec{\theta}}{dt^2}$ $\qquad$ Also $\vec{\alpha} = \frac{\omega d\vec{\omega}}{d\theta}$

Note

- Both average and instantaneous angular acceleration are axial vector with dimension. $[T^{-2}]$ and unit rad/s^2.
- If $\alpha = 0$, circular motion is said to be uniform.

Motion with constant angular acceleration

Circular motion with constant angular acceleration is analogous to one dimensional translational motion with constant acceleration. Hence even here equation of motion have same form.



$$\omega = \omega_0 + \alpha t$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0{}^2 + 2\alpha\,\theta$$

$$\theta = \left(\frac{\omega + \omega_0}{2}\right) t$$

$$\theta_{n^{th}} = \omega_0 + \frac{\alpha}{2}(\theta_n - \theta_{n-1})$$

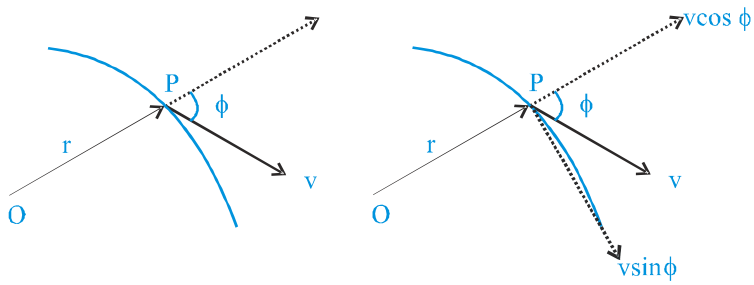
$\omega_0 \Rightarrow$ Initial angular velocity
$\omega \Rightarrow$ Final angular velocity
$\alpha \Rightarrow$ Constant angular acceleration
$\theta \Rightarrow$ Angular displacement

Relation between Linear Velocity (v) and angular Velocity (ω):

For a particle undergoing circular motion

$$v = \lim_{\Delta t \to 0}\left(\frac{\Delta s}{\Delta t}\right) = r \lim_{\Delta t \to 0}\left(\frac{\Delta\theta}{\Delta t}\right) \quad \Rightarrow \quad v = r\omega$$

Thus different points of second`s hand of a clock are rotating with same angular velocity but with different speed and its tip has greatest speed.
For any curvilinear motion (like the motion of particle P as shown below)



If the particle has only velocity component $v\cos\phi$ (along $\vec{r}$) the observer O need not turn his head to always look at the particle i.e., this component does not contribute in angular motion. Thus only the component $v\sin\phi$ is responsible for changing the angular displacement.

$$\therefore \qquad \omega = \frac{v\sin\phi}{r}$$

In general, $\omega = \dfrac{\text{velocity component perpendicular to the line joining the particle and the observer}}{\text{distance betwen the particle and observer}}$

$$\omega = \frac{v_\perp}{r}$$

Ex. A fan is rotating with angular velocity 100 rev/sec. Then it is switched off. It takes 5 minutes to stop.
(a) Find the total number of revolution made before it stops. (Assume uniform angular retardation)
(b) Find the value of angular retardation
(c) Find the average angular velocity during this interval.

Sol. **(a)** $\theta = \left(\frac{\omega + \omega_0}{2}\right)t = \left(\frac{100+0}{2}\right) \times 5 \times 60 = 15000$ revolution

(b) $\omega = \omega_0 + \alpha t \quad \Rightarrow \quad 0 = 100 - \alpha(5\times 60) \Rightarrow \alpha = \frac{1}{3}$ rev./sec^2

(c) $\omega_{av} = \dfrac{\text{Total Angle of Rotation}}{\text{Total time taken}} = \dfrac{15000}{50\times 60} = 50$ rev./sec.

Ex. A particle is moving with constant speed in a circular path. Find the ratio of average velocity to its instantaneous velocity when the particle describes an angle $\theta = \frac{\pi}{2}$

Sol. Time taken to describe angle θ, $t = \frac{\theta}{\omega} = \frac{\theta R}{v} = \frac{\pi R}{2v}$

$$\text{Average velocity} = \frac{\text{Total displacement}}{\text{Total time}} = \frac{\sqrt{2}R}{\pi R/2v} = \frac{2\sqrt{2}}{\pi}v$$

Instantaneous velocity = v

The ratio of average velocity to its instantaneous velocity $= \frac{2\sqrt{2}}{\pi}$ **Ans.**

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(b) $\omega = \omega_0 + \alpha t \quad \Rightarrow \quad 0 = 100 - \alpha(5 \times 60) \Rightarrow \alpha = \frac{1}{3}$ rev./sec²

(c) $\omega_{av} = \frac{\text{Total Angle of Rotation}}{\text{Total time taken}} = \frac{15000}{50 \times 60} = 50$ rev./sec.

Relative Angular Velocity:

Just as velocities are always relative, similarly angular velocity is alsi always relative, there is no such thing as absolute angular velocity.Angular velocity is defined with respect to origin, the point from which the position. Vector of the moving particle is drawn.

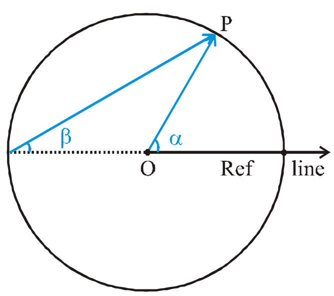
Consider a particle P moving along a circular path shown in the figure given below . Here angular velocity of the particle P w.r.t. 'O' and 'A' will be different.

Angular velocity of a particle P w.r.t. O.

$$\omega_{PO} = \frac{d\alpha}{dt}$$

Angular velocity of a particle P w.r.t. A,

$$\omega_{PA} = \frac{d\beta}{dt}$$

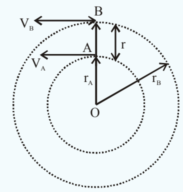


Definition:

Angular velocity of a particle 'A' with respect to the other moving particle 'B' is the rate at which position vector of 'A' with respect to 'B' rotates at that instant (or it is simply, angular velocity of A with origin fixed at B) Angular velocity of A w.r.t. B , W_{AB} is mathematically defined as.

$$\omega_{AB} = \frac{\text{Component of relative velocity of A w.r.t. B, perpendicualr to line}}{\text{separation between A and B}} = \frac{(V_{AB})_{\perp}}{r_{AB}}$$

(i) If two particles are moving on two different concentric circles with different velocites. Then angular velocity of B as observed by A will depend on thier position and velocities. Consider the case when A and B are closest to each other moving in same direction as shown in figure. In this situation

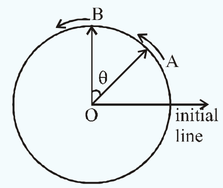


$$(V_{AB})_\perp = V_B - V_A$$

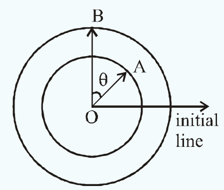
Separation between A andB is $r_{BA} = r_B - r_A$

so $$\omega_{AB} = \frac{(V_{AB})_\perp}{r_{AB}} = \frac{V_B - V_A}{r_{AB}}$$

(ii) If two particle are moving on the same circle or different coplanar concentric circles in same direction with different uniform angular speed W_A and W_B respectively, the rate of change of angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is $\frac{d\theta}{dt} = \omega_B - \omega_A$



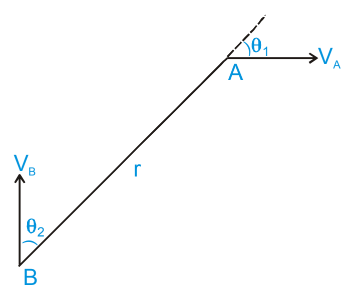
So the time taken by one to complete one revolution around point O w.r.t. the other.

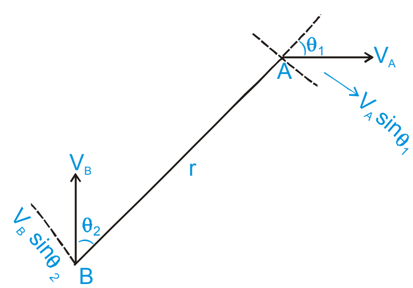


$$T = \frac{2\pi}{\omega_{rd}} = \frac{2\pi}{\omega_2 - \omega_1} = \frac{T_1T_2}{T_1 - T_2}$$

(iii) $\omega_B - \omega_A$ is rate of change of angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ this is not angular velocity of B w.r.t. A (which is rate at which line AB rotates)

Ex. Find the angular velocity of A with respect to B in the figure given below :





Sol. Angular velocity of A with respect to B ;

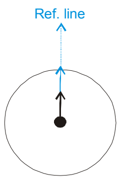
$$\omega_{AB} = \frac{(V_{AB})_\perp}{r_{AB}}$$

$\Rightarrow$ $(V_{AB})_\wedge = V_A\sin\theta_1 + V_B\sin\theta_2$

$\Rightarrow$ $r_{AB} = r$

$$\omega_{AB} = \frac{V_A\sin\theta_1 + V_B\sin\theta_2}{r}$$

Ex. Find the time period of meeting of minute hand and second hand of a clock.



$\omega_{min} = \frac{2\pi}{60}$ rad/min. ; $\omega_{sec} = \frac{2\pi}{1}$ rad/min.

$\theta_{sec} - \theta_{min} = 2\pi$ (for second and minute hand to meet again)

$(\omega_{sec} - \omega_{min})t = 2\pi$

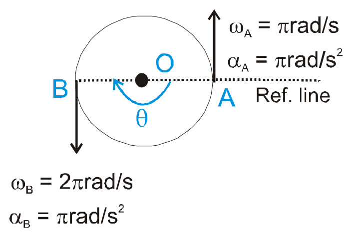
$2\pi\,(1 - 1/60)\,t = 2\pi \quad \Rightarrow \quad t = \frac{60}{59}$ min

Ex. Two particle A and B move on a circle. Initially Particle A and B are diagonally opposite to each other. Particle A move with angular velocity π rad/sec. , angular acceleration $\pi/2$ rad/sec^2 and particle B moves with constant angular velocity 2π rad/sec. Find the time after which both the particle A and B will collide.

Sol. Suppose angle between OA and OB = θ

then, rate of change of θ,

$$\dot{\theta} = \omega_B - \omega_A = 2\pi - \pi = \pi \text{ rad/sec}$$



$$\ddot{\theta} = \alpha_B - \alpha_A = -\frac{\pi}{2} \text{ rad/sec}^2$$

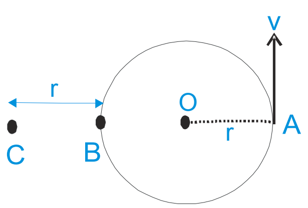
If angular displacement is $\Delta\theta$,

$$\Delta\theta = \dot{\theta}t + \frac{1}{2}\ddot{\theta}t^2$$

for A and B to colide angular displacement $\Delta\theta = \pi$

$$\Rightarrow \quad \pi = \pi t + \frac{1}{2}\left(\frac{-\pi}{2}\right)t^2 \quad \Rightarrow \quad t^2 - 4t + 4 = 0 \quad \Rightarrow \quad t = 2 \text{ sec.}$$

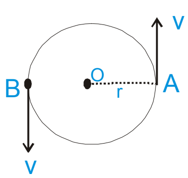
Ex. A particle is moving with constant speed in a circle as shown, find the angular velocity of the particle A with respect to fixedn point B and C if angular velocity with respect to O is ω.



Sol. Angular velocity of A with respect to O is ; $\omega_{AO} = \frac{(V_{AO})_\perp}{r_{AO}} = \frac{v}{r} = \omega$

$$\therefore\ \omega_{AB} = \frac{(V_{AB})_\perp}{r_{AB}} = \frac{v}{2r} = \frac{\omega}{2} \quad \text{and} \quad \therefore\ \omega_{AC} = \frac{(V_{AC})_\perp}{r_{AC}} = \frac{v}{3r} = \frac{\omega}{3}$$

Ex. Particles A and B move with constant and equal speeds in a circle as shown, find the angular velocity of the particle A with respect to B, if angular velocity of particle A w.r.t. O is ω.



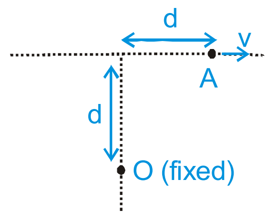
Sol. Angular velocity of A with respect to O is ; $\omega_{AO} = \frac{(V_{AO})_\perp}{r_{AO}} = \frac{v}{r} = \omega$

Now, $\omega_{AB} = \frac{(V_{AB})_\perp}{r_{AB}} \quad \Rightarrow \quad V_{AB} = 2V$

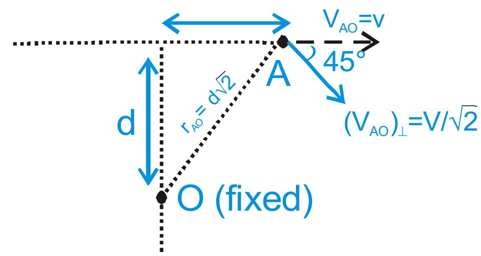
Since V_{AB} is perpendicular to r_{AB},

$$\therefore\ (V_{AB})_\perp = V_{AB} = 2v\ ;\ r_{AB} = 2r \quad \Rightarrow \quad \omega_{AB} = \frac{(V_{AB})_\perp}{r_{AB}} = \frac{2v}{2r} = \omega$$

Ex. Find angular velocity of A with respect to O at the instant shown in the figure.



Sol. Angular velocity of A with respect to O is ;



$$\omega_{AO} = \frac{(V_{AO})_{\perp}}{r_{AO}}$$

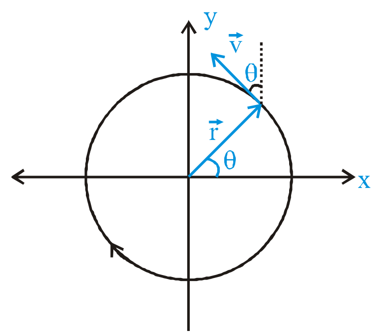
$$V_{AO} = V.(V_{AO})_{\perp} = \frac{V}{\sqrt{2}}$$

$$r_{AO} = d\sqrt{2}$$

$$\omega_{AO} = \frac{(V_{AO})_{\perp}}{r_{AO}} = \frac{V/\sqrt{2}}{d\sqrt{2}} = \frac{v}{2d}$$

Unit Vectors along Radical direction ($\hat{r}$) and tangential direction ($\hat{v}$):

Suppose a particle moving in a circular path of radius r in xy plane with origin as centre and makes angle θ with x-axis as shown, at any instant its position vector at this instant can be given by:



$$\vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j}$$

$$\Rightarrow \hat{r} = \frac{\vec{r}}{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$$

Also $\vec{v} = -v\sin\theta\hat{i} + v\cos\theta\hat{j}$

$\therefore$ $\hat{v} = \frac{\vec{v}}{v} = -\sin\theta i + \cos\theta\hat{j}$

Uniform Circular motion:

When a particle is moving with constant speed in circular path, its motion is called uniform circular motion. Although magnitude of velocity is constant but its direction, changes continuously. It means it is continuously having some acceleration. This acceleration is always directing towards the centre. This is called acceleration (a_r) or centripetal acceleration (a_c).

As we have discussed in previous topic

$$\vec{r} = r\left(\cos\theta i + \sin\theta\hat{j}\right)$$

and $\vec{v} = -v\sin\theta i + v\cos\theta\hat{j} = v\left(-\sin\theta i + \cos\theta\hat{j}\right)$

$$\vec{a} = \frac{d\vec{v}}{dt} = v\left(-\cos\theta i - \sin\theta j\right)\left(\frac{d\theta}{dt}\right) + v\left(\sin\theta i + \cos\theta j\right)\left(\frac{dv}{dt}\right)$$

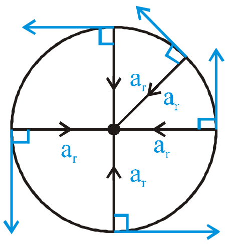
$\left|\vec{V}\right|$ is constant $\Rightarrow \frac{d\left|\vec{V}\right|}{dt} = 0$ i.e., $\frac{dv}{dt} = 0$

Also $\frac{d\theta}{dt} = \omega$

$\therefore$ $\vec{a} = -v\omega\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right) \Rightarrow \vec{a} = v\omega\hat{a}$

Here $\hat{a} = -(\cos\theta i + \sin\theta j) = \hat{r}$

Thus direction of this acceleration is opposite to $\hat{r}$ i.e., radially inwards.



Also $|\vec{a}| = v\omega \quad \Rightarrow a_r = v\omega$ or $\frac{v^2}{r}$ or $\omega^2 r$ [$\because$ $v = \omega r$]

magnitude of this acceleration is constant but direction changes continuously, always being normal to velocity as shown in the figure.

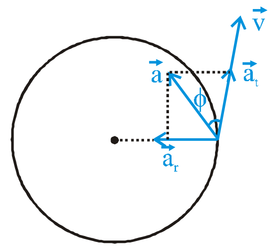
Non-uniform Circular motion:

For a particle moving in non-uniform circular motion both direction as well as magnitude of velocity change.

Now, $\vec{v} = v(-\sin\theta\hat{i} + \cos\theta\hat{j})$

$$\vec{a} = \frac{d\vec{v}}{dt} = -v(\sin\theta\hat{i} + \cos\theta\hat{j})\frac{d\theta}{dt} + (-\sin\theta\hat{i} + \cos\theta\hat{j})\frac{dv}{dt}$$

we can write $\vec{a} = \vec{a}_r + \vec{a}_t$



where $\vec{a}_r = v\omega\,(-\hat{r})$ i.e., radial acceleration and $\vec{a}_t = \left(\frac{dv}{dt}\right)\hat{v}$

having unit vector equal to $\hat{v}$ it is also in tangential direction and is called tangential acceleration.

Thus in this case acceleration has two components one along velocity i.e., tangential acceleration and another normal to velocity i.e., radial acceleration as shown.

If net acceleration $(\vec{a})$ makes angle ϕ with tangential direction, we may write

$$\tan\phi \frac{a_r}{a_t}$$

also $\quad |\vec{a}| = \sqrt{a_r^{\,2} + a_t^{\,2}}$

Some point to remember:

$$a_t = \frac{dv}{dt} = \frac{rd\omega}{dt} = r\alpha \qquad \Rightarrow \qquad |\vec{a}| = \sqrt{(\omega^2 r)^2 + (r^\alpha)^2}$$

Student need not to confused with $\left|\frac{d\vec{v}}{dt}\right|$ and $\frac{d|\vec{v}|}{dt}$. $\vec{v}$ contains both magnitude and direction. Thus $\frac{d\vec{v}}{dt}$ means $\vec{a}_{net}$

i.e., $\left|\frac{d\vec{v}}{dt}\right| = |\vec{a}_{net}|$

Also $|\vec{v}|$ means magnitude of velocity only which change because of tangential acceleration.

$\therefore \qquad \left|\frac{d\vec{v}}{dt}\right| = a_t$

Radial and tangential acceleration

There are two types of acceleration in circular motion ; Tangential acceleration and centripetal acceleration.

(a) Tangential acceleration :-

Component of acceleration directed along tangent of circle is called tangential acceleration. It is responsible for changing the speed of the particle. It is defined as,

$$a_t = \frac{dv}{dt} = \frac{d|\vec{v}|}{dt} = \text{Rate of change of speed.}$$

$a_t = \alpha r$

(b) **Centripetal acceleration :-**

It is responsible for change in direction of velocity. In circular motion, there is always a centripetal acceleration.
Centripetal acceleration is always variable because it changes in direction.
Centripetal acceleration is also called radial acceleration or normal acceleration.

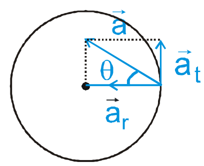
(c) **Total acceleration** :-

Total acceleration is vector sum of centripetal acceleration and tangential acceleration.

$$\vec{a} = \frac{d\vec{v}}{dt} = \vec{a}_r + \vec{a}_t$$

$$a = \sqrt{a_t^2 + a_r^2}$$

$$\tan\theta = \frac{a_r}{a_t}$$



(i) In vector form $\vec{a}_t = \vec{\alpha} \times \vec{r}$

(ii) If tangential acceleration is directed in direction of velocity then the speed of the particle increases.

(iii) If tangential acceleration is directed opposite to velocity then the speed of the particle decreases

(iv) Differentiation of speed gives tangential acceleration.

(v) Diffrentiation of velocity ($\vec{v}$) gives total acceleration.

(vi) $\left|\frac{d\vec{v}}{dt}\right|$ and $\frac{d|\vec{v}|}{dt}$ are not same physical quantity $\left|\frac{d\vec{v}}{dt}\right|$ is the magnitude of rate of change of velocity i.e., magnitude of total acceleration and $\frac{d|\vec{v}|}{dt}$ is a rate of change of speed i.e., tangential acceleration.

Ex. A particle is revoluting a circular of radius 0.2 m with angular velocity $\omega = 20\, t^{\circ}$ rad/s, where t is in seconds. Find its acceleration at t = 0.5 sec.

Sol. $a_r = \omega^2 r = (20t^2)^2 (0.2) = 80\, t^2$ m/s^2

$\therefore$ at $t = 0.5$ sec ; $a_r = 80\,(0.5)^4 = 5$ m/s^2

Also $\alpha = \frac{d\omega}{dt} = 40\, t$ rad/s^2

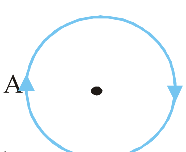
$\Rightarrow$ $a_t = r\alpha = 8t$ m/s^2

at $t = 0.5$ sec; $a_t = 8\,(0.5) = 4$ m/s^2

$\therefore$ $|\vec{a}| = \sqrt{a_t^2 + a_r^2} = \sqrt{(4)^2 + (5)^2} = \sqrt{41}$ m/s^2

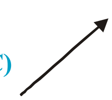
$\therefore$ $a = 6.4$ m/s^2

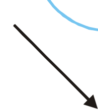
Ex. A particle is moving in circular path clockwise as shown with decreasing speed. When it is at point A, the direction its acceleration may be given as



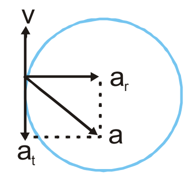
(A) 

(B) 

(C) 

(D) 

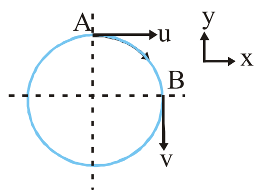
Sol. Since its speed is decreasing so its tangental acceleration is opposite to v. When it is at A



so (D) is correct

Ex. A particle is moving in a circular path of radius R with speed u, when it begins to speed up at a constant rate. After that when it completes and fourth revolution, change in its velocity vector has magnitude 2u. At that moment, find
(i) its radial acceleration
(ii) angle it makes with its velocity vector.

Sol. **(i)** Suppose initially it was at point A with speed u and now it is at point B as shown with speed v.



Now $\Delta\vec{v} = \vec{v}_f - \vec{v}_{in}$

$= (-v\hat{j}) - (u\hat{i})$

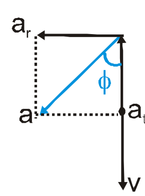
$\therefore$ $|\Delta\vec{v}| = \sqrt{v^2 + u^2}$

$\Rightarrow$ $\sqrt{v^2 + u^2} = 2u$

$\Rightarrow$ $v^2 = 3u^2$

$a_r = \dfrac{v^2}{R}$ $\Rightarrow$ $a_r = \dfrac{3u^2}{R}$

(ii) Also $\omega^2 = \omega_0^2 + 2\alpha(\Delta\theta)$



$\Rightarrow$ $\left(\dfrac{v}{R}\right)^2 = \left(\dfrac{u}{R}\right)^2 + 2\alpha\left(\dfrac{\pi}{2}\right)$

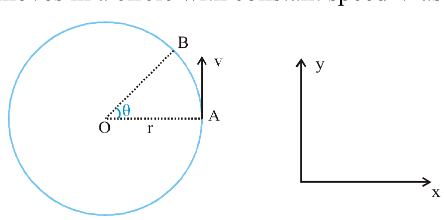
$\Rightarrow$ $\alpha = \dfrac{v^2 - u^2}{\pi R^2} = \dfrac{3u^2 - u^2}{\pi R^2} = \dfrac{2u^2}{\pi R^2}$

$\Rightarrow$ $a_t = R\alpha = \dfrac{2u^2}{\pi R}$

Now $\tan\phi = \dfrac{a_r}{a_t}$ $\Rightarrow$ $\phi = \tan^{-1}\left(\dfrac{3\pi}{2}\right)$

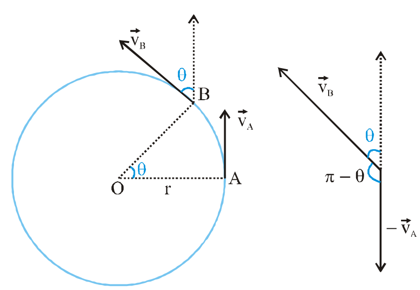
Calculation of Centripetal acceleration:

Consider a particle which moves in a circle with constant speed V as shown in figure.



$\therefore$ change in velocity between the point A and B is ;

$\Delta\vec{v} = \vec{v}_B - \vec{v}_A$



magnitude of change in velocity:

$|\Delta\vec{v}| = |\vec{v}_B - \vec{v}_A| = \sqrt{v_B^2 + v_A^2 + 2v_A v_B\cos(\pi - \theta)}$

($V_A = V_B = V$, since speed is same)

$\therefore$ $|\Delta\vec{V}| = 2v\sin\dfrac{\theta}{2}$

Distance travelled by particle between A and B = rθ

Hence time taken, $\Delta t = \frac{r\theta}{v}$

Net acceleration, $|\vec{a}_{net}| = \left|\frac{\Delta\vec{V}}{dt}\right| = \frac{2v\sin\frac{\theta}{2}}{r\frac{\theta}{v}} \quad = \frac{v^2}{r}\frac{2\sin\frac{\theta}{2}}{\theta}$

if $\Delta t \to 0$, then θ is small, $\sin(\theta/2) = \theta/2$

$$\lim_{\Delta t - 0}\left|\frac{\Delta\vec{v}}{\Delta t}\right| = \left|\frac{d\vec{v}}{dt}\right| = \frac{v^2}{r}$$

i.e., net acceleration is $\frac{v^2}{r}$ but speed is constant so that tangential acceleration,

$$a_t = \frac{dv}{dt} = 0 \qquad \therefore \qquad a_{net} = a_r = \frac{v^2}{r}$$

Through we have derived the formula of centripetal acceleration under condition of constant speed, the same formula is applicable ever when speed is variable.

Note : In vector form $\vec{a}_c = \vec{w} \times \vec{v}$

Ex. The speed of a particle traveling in a circle of radius 20 cm increases uniformly from 6.0 m/s to 8.0 m/s in 4.0 s, find theangular acceleration.

Sol : Since speed increases uniformly, average tangential acceleration is equal to instantaneous tangential acceleration
$\therefore$ The instantaneous tangential acceleration is given by

$$a_t = \frac{dv}{dt} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{8.0 - 6.0}{4.0} \text{ m/s}^2 = 0.5 \text{ m/s}^2.$$

The angular acceleration is $\alpha = a_t \quad = \frac{0.5\, m/s^2}{20\, cm} = 2.5 \text{ rad/s}^2$.

Ex. A particle is moving in a circle of radius 10 cm with uniform speed completing the circle in 4s, find the magnitude of its acceleration.

Sol. : The distance covered in completing the circle is $2\pi r = 2\pi \times 10$ cm. The linear speed is

$$v = 2\pi r/t = \frac{2\pi \times 10\,\text{cm}}{4\text{s}} = 5\pi \text{ cm/s}.$$

The acceleration is $a = \frac{v^2}{r} = \frac{(5\pi\,\text{cm/s})^2}{10\,\text{cm}} = 2.5\,\pi^2 \text{ cm/s}^2$.

Ex. A particle moves in a circle of radius 2.0 cm at a speed given by v = 4t, where v is in cm/s and t is in seconds.
(a) Find the tangential acceleration at t = 1s.
(b) Find total acceleration at t = 1s.

Sol. : **(a)** Tangential acceleration

$$a_t = \frac{dv}{dt} \qquad \text{or} \qquad a_t = \frac{d}{dt}(4t) = 4 \text{ cm/s}^2$$

$$a_C = \frac{v^2}{R} = \frac{(4)^2}{2} = 8 \text{ cm/s}^2 \Rightarrow \qquad a = \sqrt{a_t^2 + a_C^2} = \sqrt{(4)^2 + (8)^2} = 4\sqrt{5} \text{ cm/s}^2$$

Ex. A particle begins to move with a tangential acceleration of constant magnitude 0.6 m/s^2 in a circular path. If it slips when its total acceleration becomes 1 m/s^2, Find the angle through which it would have turned before it starts to slip.

Sol. :

$a_{Net} = \sqrt{a_t^2 + a_c^2}$ $\Rightarrow$ $\omega^2 = \omega_0{}^2 + 2\alpha\theta$

$\because \omega_0 = 0$ so $\omega^2 = 2\alpha\theta$

$\omega^2 R = 2(\alpha R\theta)$

$a_c = \omega^2 R = 2a_t\theta$

$1 = \sqrt{0.36 + (1.2\times\theta)^2}$ $\Rightarrow$ $1 - 0.36 = (1.2\,\theta)^2$

$\Rightarrow \dfrac{0.8}{1.2} = \theta$ $\Rightarrow$ $\theta = \dfrac{2}{3}$ radian **Ans.**

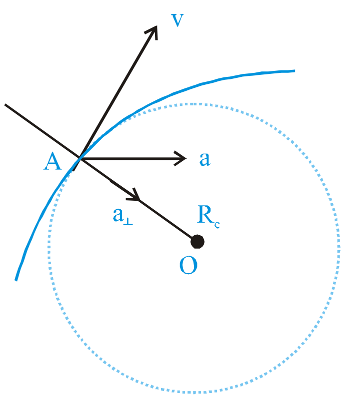
Radius of Curvature:

If a body is moving in any curvilinear path, then at different locations, the curvature would be different, thus the radius would be different.

For general curvilinear motion, when the particle crosses a point A, it is satisfying condition of moving on an imaginary circle, At this instant if

$a_\perp = \dfrac{v^2}{R_c}$ (where R_c is radius of curvature at this instant)

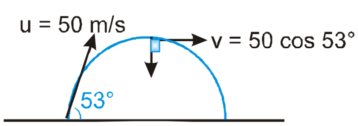
$R_c = \dfrac{v^2}{a_\perp}$ $\Rightarrow$ $R_c = \dfrac{(\text{speed})^2}{\text{comp. of acceleration perpendicular to velocity}}$



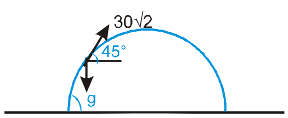
Ex. An object is projected with speed 50 m/s at an angle 53° with the horizontal from ground. Find radius of its trajectory **(i)** at the instant it is at heighest point **(ii)** at t = 1 sec. after projection.

Sol. **(i)** At any instant acceleration of the projectile is 'g' downward. At the highest point velocity has magnitude = 50 cos 53° = 30 m/s and is in horizontal direction. Thus acceleration perpendicular to velocity is 'g' itself.

$\Rightarrow$ $a_r = \dfrac{v^2}{R} = g$



$\Rightarrow$ $R = \dfrac{v^2}{a_r} = \dfrac{(30)^2}{10} = 90m$



(ii) at t = 1 sec $V_x = 50\cos 53° = 30$ m/s

and $V_y = 50\sin 53° - g(1) = 30$ m/s

$\therefore \quad V = \sqrt{V_x^2 + V_y^2} = 30\sqrt{2} m/s$

i.e. $\vec{v}$ is at angle 45° with the horizontal

$\therefore \quad a_\perp$ = componant of g perpendicular to velocity = $\dfrac{g}{\sqrt{2}}$

$$\therefore \quad \frac{(30\sqrt{2})^2}{R} = \frac{10}{\sqrt{2}}$$

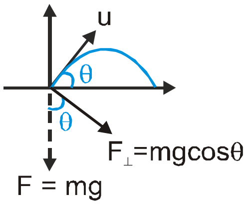
$$\Rightarrow \quad R = 180\sqrt{2}m$$

Note: If a particle moves in a trajectory given by y = f(x) then radius of curvature at any point(x, y) of the trajectory is given by

$$\Rightarrow \quad R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

Ex. A particle of mass m is projected with speed u at an angle θ with the horizontal. Find the radius of curvature of the path traced out by the particle at the point of projection and also at the highest point of trajectory.

Sol. at point of projection



$$R = \frac{mv^2}{F_\perp} = \frac{mu^2}{mg\cos\theta}$$

$$R = \frac{u^2}{g\cos\theta} \qquad \textbf{Ans.}$$

at highest point

$$a_\perp = g \;,\; v = u\cos\theta \;: \qquad R = \frac{v^2}{a_\perp} = \frac{u^2\cos^2\theta}{g} \qquad \textbf{Ans.}$$

Ex. A particle moves along the plane trajectory y(x) with constant speed v. Find the radius of curvature of the trajectory at the point x = 0 if the trajectory has the form of a parabola $y = ax^2$ where 'a' is a positive constant.

Sol. If the equation of the trajectory of a particle is given we can find the radius of trajectory of the instantaneous circle by using the formula

$$R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

As ; $y = ax^2 \quad \Rightarrow \quad \dfrac{dy}{dx} = 2ax = 0 \quad$ (at x = 0) and $\quad \dfrac{d^2y}{dx^2} = 2a$

Now radius of trajectory is given by

$$R = \frac{[1+0]^{3/2}}{2a} = \frac{1}{2a}$$

Aliter : This problem can also be solved by using the formula : $R = \frac{v^2}{a_\perp}$. $y = ax^2$,

differentiate with respect to time $\frac{dy}{dt} = 2ax\frac{dx}{dt}$ ....(1)

at x = 0, $v_y = \frac{dy}{dt} = 0$ hence $v_x = v$

since v_x is constant, $a_x = 0$

Now, differentiate (1) with respect to time $\frac{d^2y}{dt^2} = 2ax\frac{d^2x}{dt^2} + 2a\left(\frac{dx}{dt}\right)^2$

at x = 0, $v_x = v$

$\therefore$ net acceleration, $a = a_y = 2av^2$ (since $a_x = 0$)

this acceleration is perpendicular to velocity (v_x)

Hence it is equal to centripetal acceleration

$R = \frac{v^2}{a_\perp} = \frac{v^2}{2av^2} = \frac{1}{2a}$ **Ans.**

Dynamics of Circular Motion:

If there is no force acting on a body it will move in a straight line (with constant speed). Hence if an object moving in a circular path or any curved path, there must be some force acting on the body.

If speed of body is constant, the net force acting on the body is along the inside normal to the path of the body and it is called centripetal force.

Centripetal Force $= ma_c = \frac{mv^2}{r} = m\omega^2 r$

However if speed of the body varies then, in addition to above centripetal force which acts along inside normal, there is also a force acting along the tangent of the path of the body which is called tangential force.

Tangential Force $(F_t) = Ma_t = \frac{Mdv}{dt} = M\alpha r$;

(where α is the angular acceleration)

Note : Remember $\frac{mv^2}{r}$ is not a force itself. It is just the value of the net force acting along the inside normal which is responsible for circular motion. This force may be friction, normal, tension, spring force, gravitation force or a combination of them.

So to solve any problems in uniform circular motion we identify all the forces acting along the normal (towards center), calculate their resultant and equate it to $\frac{mv^2}{r}$

If circular motion is non-uniform then in addition to above step we also identify all the circular path, caculate their resultant and equate it to $\frac{mdv}{dt}$ or $\frac{md|\vec{v}|}{dt}$.

Example of Centripetal Force:

Combination of forces which provide centrepetal acceleration necessary for the revolution of a particle is called centripetal force or radial force for example

(a) For object tied to a string and is revolving on a smooth horizontal surface, tension is centripetal force.

(b) To revolve satelite around the earth, gravitational force provides centripetal. So gravitational force is centripetal force on it.

(c) For motion of electron around nucleus, the electrostatic force on electron is centripetal force on it.

(d) For an object placed on a rough rotating table, friction on the object due to table is centripetal force.

Problem Solving strategy:

(i) Identify the plane of circular motion

(ii) locate the centre of rotation and calculate the radius.

(iii) Make F.B.D.

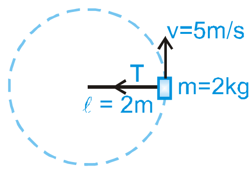
(iv) The net force along radial direction is mass times the raidal acceleration i.e., $m\left(\frac{v^2}{R}\right)$ or m ($\omega^2 R$)

Centripetal force $\left(\frac{mv^2}{R}\right)$ is no seperate force like tension, weight, spring force, normal reaction, friction etc. In fact anyone of these centripetal force.

CIRCULAR MOTION IN HORIZONTAL PLANE:

Ex. A block of mass 2kg is tied to a string of length 2m, the other end of which is fixed. The block is moved on a smooth horizontal table with constant speed 5 m/s. Find the tension in the string.

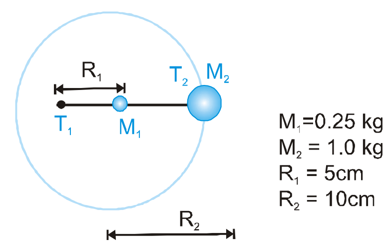
Sol.



here centripetal force is provided by tension.

$$T = \frac{mv^2}{r} = \frac{2 \times 5^2}{2} = 25\text{ N}$$

Ex. Two different masses are connected to two light and inextensible strings as shown in the figure. Both masses revolve about a central fixed point with constant angular speed of 10 rad s^{-1} on a smooth horizontal plane. Find the ratio of tensions $\frac{T_1}{T_2}$ in the strings.

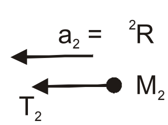


Sol. Both the masses are moving in horizontal plane with same angular speed 10 rad/s. Here forces radial direction can be tension only.

For M2

$F_{net} = T_2 = M_2 a_2$

F.B.D. of M_2

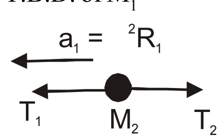


(1) $T_2 = M_2\,\omega^2 R_2$ .... (i)

For M1

$T_1 - T_2 = m_1 a_1$

F.B.D. of M_1



$T_1 = M_1 a_1 + T_2$

$T_1 = M_1\,\omega^2 R_1 + M_2\,\omega^2 R_2$ .... (ii)

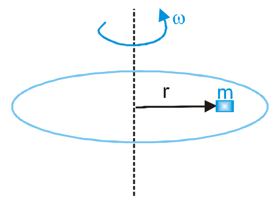
Dividing equation (ii) from (i), we get

$$\therefore \quad \frac{T_1}{T_2} = \frac{M_1 R_1 + M_2 R_2}{M_2 R_2} = \frac{M_1}{M_2} \times \frac{R_1}{R_2} + 1$$

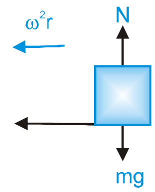
$$= \frac{0.25}{1} \times \frac{5}{10} + 1 \Rightarrow \frac{T_1}{T_2} = \frac{9}{8}$$

Here centripetal force on M_2 is "T_2" and on M_1 is "$T_1 - T_2$".

Ex. A rough horizontal table can rotate about its axes as shown. A small block is placed at r = 20 cm from its axis. The coefficient of friction between them is 0.5. Find the maximum angular speed that can be given to the block table system so that the block does not slip on the table.



Sol. The force acting on the block are as shown below. N and mg are in vertical direction, so the only force that can provide necessary centrepetal acceleration in horizontal plane is friction. It makes the block to revolve with the table without slipping.



$(F_{net})_y = 0 \Rightarrow N = mg$

& $(F_{net})_x = ma$

$\Rightarrow$ $f = m\omega^2 r$

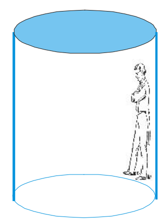
but $f \le \mu N$

$\Rightarrow$ $m\omega^2 r \le \mu\, mg$ $\Rightarrow$ $\omega \le \sqrt{\dfrac{\mu g}{r}}$

$\therefore$ $\omega_{max} = \sqrt{\dfrac{\mu g}{r}} = \sqrt{\dfrac{0.5 \times 10}{0.2}}$

$\Rightarrow$ $\omega_{max} = 5\ rad/s^2$

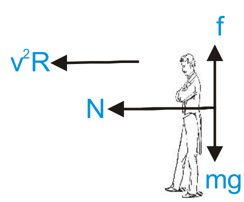
Ex. In a rotor, a hollow vertical cylindrical structure rotates about its axis and a person rests against the inner wall. A particular speed of the rotor, the floor below the person is removed and the person hangs resting against the wall without any floor. If the radius of the rotor is 2m and the coefficient of static friction between the wall and the person is 0.2 , find the minimum speed at which the floor may be removed. Take g = 10 m/s².



Sol. Here friction (f) is upward and opposes tendency to move down. Also normal force is radically towards the centre to provide centripetal acceleration.

$$\left(F_{net}\right)_x = \frac{mV^2}{R}$$

$$\Rightarrow \quad N = \frac{mV^2}{R}$$



Also as man doesnot fall down,

$f = mg \le \mu N$

$$\Rightarrow \quad mg \le \mu \frac{mV^2}{R}$$

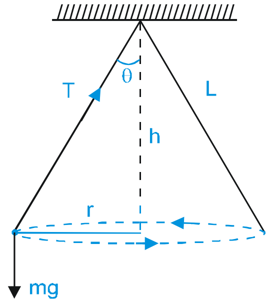
$$\Rightarrow \quad \sqrt{\frac{gR}{\mu}} \le V$$

$$\Rightarrow \quad V_{min} = \sqrt{\frac{gR}{\mu}} = \sqrt{\frac{10 \times 2}{0.2}} = 10 m/s$$

Ex. Consider a conical pendulum having bob of mass m is suspended from a ceiling through a string oflength L. The bob moves in a horizontal circle of radius r. Find

(a) the angular speed of the bob and

(b) the tension in the string.



Sol. The situation is shown in figure. The angle θ made by the string with the vertical is given by

$$\sin\theta = r/L,\ \cos\theta = h/L = \frac{\sqrt{L^2 - r^2}}{L} \quad ...(i)$$

The forces on the particle are

(a) the tension T along the string and

(b) the weight mg vertically downward.

The particle is moving in a circle with a constant speed v. Thus, the radial acceleration towards the centre has magnitude v^2/r. Resolving the forces along the radial direction and applying Newton's second law,

$$T\sin\theta = m(v^2/r) \quad ...(ii)$$

As there is no acceleration in vertical direction, we have from Newton's law,

$$T\cos\theta = mg \quad ...(iii)$$

Dividing (ii) by (iii),

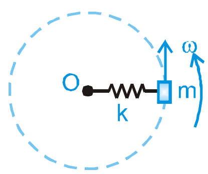
$$\tan\theta = \frac{v^2}{rg} \quad \text{or,} \quad v = \sqrt{rg\tan\theta}$$

$$\Rightarrow \quad \omega = \frac{v}{r} = \sqrt{\frac{g\tan\theta}{r}} = \sqrt{\frac{g}{h}} = \sqrt{\frac{g}{L\cos\theta}} = \sqrt{\frac{g}{(L^2 - r^2)^{\frac{1}{2}}}} \quad \textbf{Ans.}$$

And from (iii),

$$T = \frac{mg}{\cos\theta} = \frac{mgL}{(L^2 - r^2)^{\frac{1}{2}}} \quad \textbf{Ans.}$$

Ex. A block of mass m is tied to a spring of spring constant k, natural length ℓ, and the other end of spring is fixed at O. If the block moves in a circular path on a smooth horizontal surface with constant angular velocity ω, find tension in the spring.



Sol. Assume extension in the spring is x

Here centripetal force is provided by spring force.

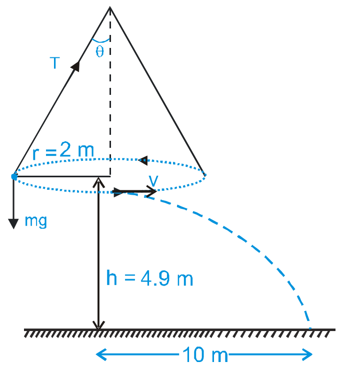
Centripetal force, $kx = m\omega^2(\ell + x)$

$$\Rightarrow \quad x = \frac{m\omega^2\ell}{k - m\omega^2}$$

therefore,

$$\text{Tension} = kx = \frac{km\omega^2\ell}{k - m\omega^2} \quad \textbf{Ans.}$$

Ex. A boy whirls a stone in a horizontal circle of radius 2 m and at height 4.9 m above level ground. The string breaks, and the stone files off horizontally and strikes the ground at a point which is 10 m away from the point on the ground directly below the point where the string had broken. What is the magnitude of the centripetal acceleration of the stone while in circular motion? ($g = 9.8\ m/s^2$)



Sol. : $t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 4.9}{9.8}} = 1\ s$

$$v = \frac{10}{t} = 10\ m/s$$

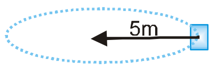
$$a = \frac{v^2}{R} = 50\ m/s^2$$

Ex. A block of mass 25 kg rests on a horizontal floor ($\mu = 0.2$). It is attached by a 5m long horizontal rope to a peg fixed on floor. The block is pushed along the ground with an initial velocity of 10 m/s so that it moves in a circle around the peg. Find

(a) Tangential acceleration of the block.

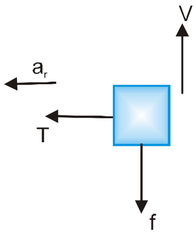
(b) Speed of the block at time t.

(c) Time when tension in rope becomes zero.



Sol. The block is pushed on a horizontal stationary rough surface, the friction here is kinetic and is always opposite to velocity i.e. it is tangential force and is in horizontal plane.

Also normal force is in vertical direction to balance mg



i.e. $N = mg = 250\ N$

$\Rightarrow$ $f = \mu N = 50\ N$

(a) $f = m a_t$

$\Rightarrow$ $a_t = \frac{f}{m} = -\frac{50}{25} = -2 m/s^2$

(Here –ve sign is taken to indicate that it is always opposite to velocity)

(b) $a_t = \frac{dV}{dt} = -2$ $\Rightarrow$ $\int_{10}^{V} dV = -2\int_{0}^{t} dt$

$\Rightarrow$ $V - 10 = -2t$

$\Rightarrow$ $V = 10 - 2t$

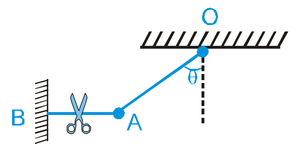
(c) Tension $T = \frac{mV^2}{R}$

$\therefore$ $T = 0$ when $V = 0$

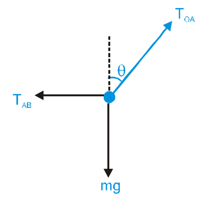
i.e. $10 - 2t = 0$

$\Rightarrow$ $t = 5$ sec.

Ex. Find tension in OA before and just after AB is cut. The mass of the particle is m.



Sol. Before AB is cut, the particle is in static equilibrium i.e. a = 0

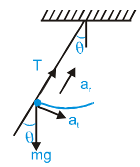


Resolving in vertical and horizontal direction,

$T_{OA} = \cos\theta = mg$

$\Rightarrow \quad T_{OA} = \dfrac{mg}{\cos\theta}$

After AB is cut, the particles moves in a circular path around O, with radius equal to the length of the string OA.



Now the acceleration cannot be directly taken zero, rather we have to consider acceleration in radial and tangential direction.

Resolving the force in radial and tangential direction

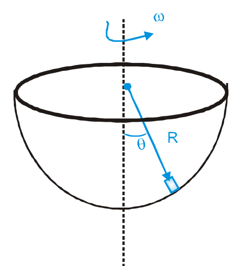
$\therefore \quad T - mg\cos\theta = m\,a_r$

but just after the string AB is cut, speed of the particle is zero $\Rightarrow a_r = \dfrac{V^2}{\ell} = 0$

$\therefore \quad T - mg\cos\theta = 0$

$\Rightarrow \quad T = mg\cos\theta$

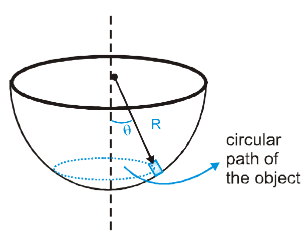
Ex. A hallow hemispherical bowl having radius of inner smooth surface R = 80 cm is rotated with angular velocity ω = 5 rad/s. A small object is placed at rest w.r.t. the bowl of position as shown. Find angle θ.



Sol. While the bowl is rotating, the plane of circular path of the particle is horizontal and its radius is r as shown.

$N\cos\theta = mg$

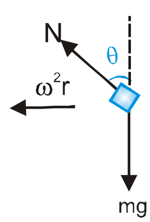
$\Rightarrow \quad N = \dfrac{mg}{\cos\theta}$

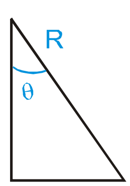


Also $\quad N\sin\theta = m\,\omega^2 r$

$\Rightarrow \quad \left(\dfrac{mg}{\cos\theta}\right)\sin\theta = m\omega^2(R\sin\theta)$

$\Rightarrow \quad \dfrac{g}{\cos\theta} = \omega^2 r$





r = R sinθ

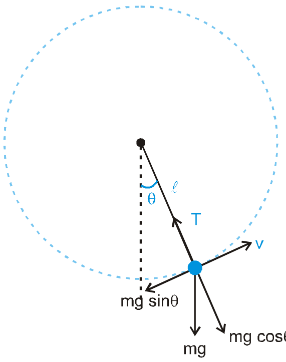
$\Rightarrow \quad \cos\theta = \dfrac{g}{\omega^2 R} = \dfrac{10}{(5)^2(0.8)}$

$\Rightarrow \quad \cos\theta = \dfrac{1}{2}$

$\Rightarrow \quad \theta = 60°$

Motion in a vertical circle :

Let us consider the motion of a point mass tied to a string of length ℓ and whirled in a vertical circle. If at any time the body is at angular position θ, as shown in the figure, the forces acting on it are tension T in the string along the radius towards the center and the weight of the body mg acting vertically down wards.



Applying Newton's law along radial direction

$$T - mg\cos\theta = m.a_c = \frac{mv^2}{\ell}$$

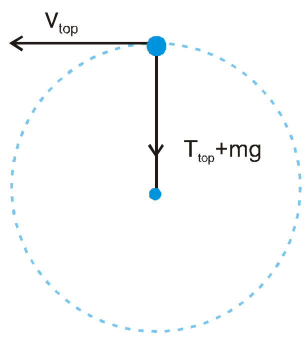
or $$T = \frac{mv^2}{\ell} + mg\cos\theta \qquad .....(1)$$

The point mass will complete the circle only and only if tension is never zero (except momentarily, if at all) if tension becomes zero at any point, string will go slack and subsequently, the only force acting on the body is gravity. Hence its subsequent motion will be similar to that of a projectile.

From equation ...(1), it is evident that tension decreases with increase in θ because $\cos\theta$ is a decreasing function and v decreases with height. Hence tension is minimum at the top most point. i.e. $T_{min} = T_{topmost}$.

$T > 0$ at all points. $\Rightarrow$ $T_{min} > 0$.

However if tension is momentarily zero at highest point the body would still be able to complete the circle.



Hence condition for completing the circle (or looping the loop) is

$T_{min} \geq 0$ or $T_{top} \geq 0$.

$$T_{top} + mg = \frac{mv_{top}^2}{\ell} \qquad .....(2)$$

Equation...(2) could also be obtained by putting $\theta = \pi$ in equation ..(1).

For looping the loop, $T_{top} \geq 0$.

$$\Rightarrow \quad \frac{mv_{top}^2}{\ell} \geq mg \quad \Rightarrow \quad v_{top} \geq \sqrt{g\ell} \qquad .....(3)$$

Condition for looping the loop is $v_{top} \geq \sqrt{g\ell}$.

If speed at the lowest point is u, then from conservation of mechanical energy between lowest point and top most point.

$$\frac{1}{2}mu^2 = \frac{1}{2}m\,v_{top}^2 + mg\,.\,2\ell$$

using equation ..(3) for v_{top} we get $u \geq \sqrt{5g\ell}$

i.e., for looping the loop, velocity at lowest point must be $\geq \sqrt{5g\ell}$.

If velocity at lowest point is just enough for looping the loop, value of various quantities. (True for a point mass attached to a string or a mass moving on a smooth vertical circular track.)

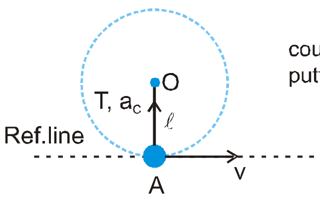


Fig. (1)

P.E. = 0

$v = \sqrt{5g\ell}$

$T - mg = \frac{mv^2}{\ell}$

could also be obtained by putting θ=0 in equation ..(1)

$\Rightarrow T = 6mg$

$a_c = 5g$

$a_t = 0$

$a_{net} = 5\ g$

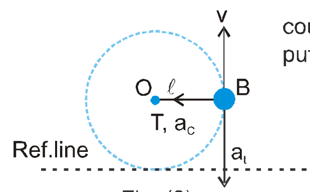


Fig. (2)

P.E.=mgℓ

By energy conservation, $v = \sqrt{3g\ell}$

$T = mv^2/\ell$

could also be obtained by putting θ =90° in equation ..(1)

T=3mg

$a_c = 3g$

$a_t = g$

$a_{net} = g\sqrt{10}$

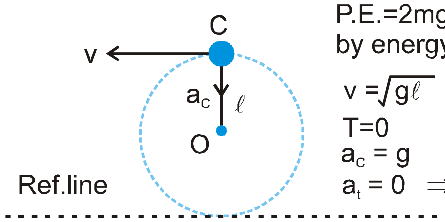


Fig. (3)

P.E.=2mgℓ

by energy conservation

$v = \sqrt{g\ell}$

T=0

$a_c = g$

$a_t = 0 \Rightarrow a_{net} = g$

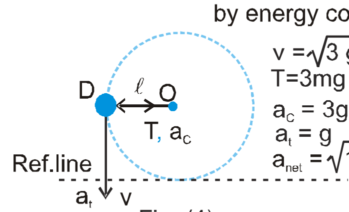


Fig. (4)

P.E.=mgℓ

by energy conservation

$v = \sqrt{3\ g\ell}$

T=3mg

$a_c = 3g$

$a_t = g$

$a_{net} = \sqrt{10}\ \ g$

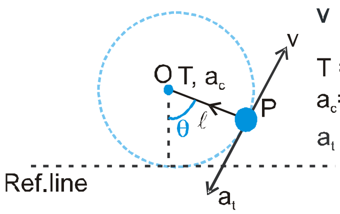


Fig. (5)

P.E.=mgℓ(1–cos θ)

by energy conservation

$v = \sqrt{g\ell(3+2\cos\theta)}$

T = 3mg (1+cosθ)

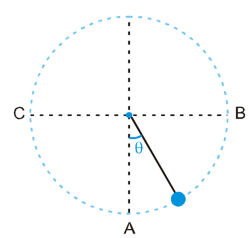
$a_c = g(3+2\cos\theta)$

$a_t = g\sin\theta$

		A	B,D	C	P(general point)
1	Velocity	$\sqrt{5g\ell}$	$\sqrt{3g\ell}$	$\sqrt{g\ell}$	$\sqrt{g\ell(3+2\cos\theta)}$
2	Tension	6mg	3mg	0	$3mg(1+\cos\theta)$
3	Potential Energy	0	$mg\ell$	$2mg\ell$	$mg\ell(1-\cos\theta)$
4	Radial acceleration	5g	3g	g	$g(3+2\cos\theta)$
5	Tangential acceleration	0	g	0	$g\sin\theta$

Note :- From above table we can see , $T_{bottom} - T_{top} = T_C - T_A = 6\ mg$, this difference in tension remain same even if $V > \sqrt{5g\ell}$

Ex. Find minimum speed at A so that the ball can reach at point B as shown in figure. Also discuss the motion of particle when T = 0, v = 0 simultaneously at θ = 90°.



Sol. From energy conservation

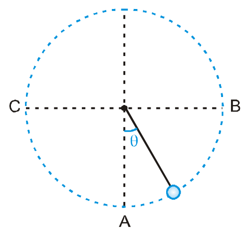
$$\frac{1}{2}mv_A^2 + 0 = 0 + mg\ell \qquad \text{(for minimum speed } v_B = 0)$$

$$v_{min} = \sqrt{2g\ell}$$

at the position B, v = 0 and T = 0 (putting $v_B = 0$ or $\theta = 90°$, in equation .....(1)

ball will return back, motion is oscillatory

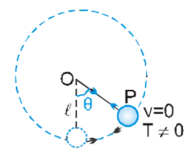
Condition for Oscillation or Leaving the Circle :



In case of non uniform circular motion in a vertical plane if velocity of body at lowest point is lesser than $\sqrt{5g\ell}$, the particle will not complete the circle in vertical plane. In this case, the motion of the point mass which depend on 'whether tension becomes zero before speed becomes zero or vice versa.

Case I (Speed becomes zero before tension)

In this case the ball never rises above the level of the center O i.e. the body is confined to move within C and B, (| θ | < 90°) for this the speed at A, $v < \sqrt{2g\ell}$ (as proved in above example)



(A)
For Oscillation
$0 < v_L < \sqrt{2g\ell}$
$0 < \theta < 90^\circ$

In this case tension cannot be zero, since a component of gravity acts radially outwards.

Hence the string will not go slack, and the ball will reverse back as soon as its speed becomes zero.

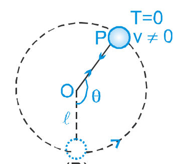
Its motion will be oscillatory motion.

Case II (Tension becomes zero before speed)

In this case the ball rises above the level of center O i.e. it goes beyond point B (θ > 90°) for this $v > \sqrt{2g\ell}$ (as proved in above example)

In this case a component of gravity will always act towards center, hence centripetal acceleration or speed will remain nonzero. Hence tension becomes zero first.

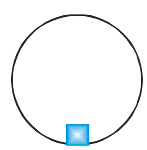
As soon as, Tension becomes zero at any point, string will go slack and subsequently, the only force acting on the body is gravity. Hence its subsequent motion will be similar to that of a projectile. In this case motion is a combination of circular and projectile motion.



(B)
For Leaving the circular path after which motion converts into projectile motion.
$\sqrt{2g\ell} < v_L < \sqrt{5g\ell}$
$90^\circ < \theta < 180^\circ$

Condition for looping the loop in some other cases

Case 1 : A mass moving on a smooth vertical circular track.

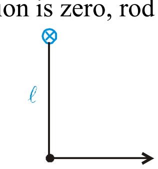


(B)
Mass moving along a smooth vertical circular loop.
condition for just looping the loop, normal at highest point = 0
By calculation similar to article (motion in vertical circle)
Minimum horizontal velocity at lowest point = $\sqrt{5g\ell}$

Case 2 : A particle attached to a light rod rotated in vertical circle.

Condition for just looping the loop, velocity v = 0 at highest point (even if tension is zero, rod won't slack, and a compressive force can appear in the rod).

By energy conservation,



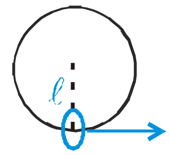
velocity at lowest point = $\sqrt{4g\ell}$

$V_{min} = \sqrt{4g\ell}$ (for completing the circle)

Case 3 : A bead attached to a ring and rotated.

Condition for just looping the loop, velocity v = 0 at highest point (even if normal is zero, the bead will not lose contact with the track, normal can act radially outward).

By energy conservation,



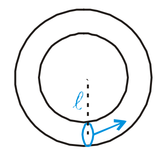
velocity at lowest point = $\sqrt{4g\ell}$

$V_{min} = \sqrt{4g\ell}$ (for completing the circle)

Case 4 : A block rotated between smooth surfaces of a pipe.

Condition for just looping the loop, velocity v = 0 at highest point (even if normal is zero, the bead will not lose contact with the track, normal can act radially outward).

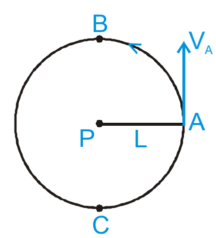
By energy conservation,



velocity at lowest point = $\sqrt{4g\ell}$

$V_{min} = \sqrt{4g\ell}$ (for completing the circle)

Ex. If a particle of mass M is tied to a light inextensible string fixed at point P and particle is projected at A with velocity $V_A = \sqrt{4gL}$ as shown. Find :



(i) velocity at points B and C

(ii) tension in the string at B and C

Assume particle is projected in the vertical plane.

Sol. : $V_B = \sqrt{2gL}$ (from energy conservation)

$V_C = \sqrt{6gL}$

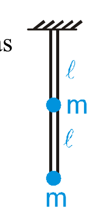
$T_B + Mg = \dfrac{Mv_B^2}{L}$

$T_B = Mg$

$T_C - Mg = \dfrac{Mv_C^2}{L}$

$T_C = 7Mg$ (where M $\Rightarrow$ Mass of the particle)

Ex. Two point mass m are connected the light rod of length ℓ and it is free to rotate in vertical plane as shown. Calculate the minimum horizontal velocity is given to mass so that it completes the circular motion in vertical lane.



Sol. Here tension in the rod at the top most point of circle can be zero or negative for completing the loop. So velocity at the top most point is zero.

From energy conservation

$$\frac{1}{2}mv^2 + \frac{1}{2}m\frac{v^2}{4} = mg(2\ell) + mg(4\ell) + 0 \quad \Rightarrow \quad v = \sqrt{\frac{48g\ell}{5}}$$ **Ans.**

Ex. You may have seen in a circus a motorcyclist driving in vertical loops inside a 'death well' (a hollow spherical chamber with holes, so that the cyclist does not drop down when he is at the uppermost point, with no support from below. What is the minimum speed required at the uppermost position to perform a vertical loop if the radius of the chamber is 25 m?

Sol. : When the motorcyclist is at the highest point of the death-well, the normal reaction R on the motorcyclist by the ceiling of the chamber acts downwards. His weight mg also act downwards.

$$F_{net} = ma_c \quad \Rightarrow \quad \therefore \quad R + mg = \frac{mv^2}{r}$$

Here v is the speed of the motorcyclist and m is the mass of the motorcyclist (including the mass of the motor cycle). Because of the balancing of the forces, the motorcyclist does not fall down.

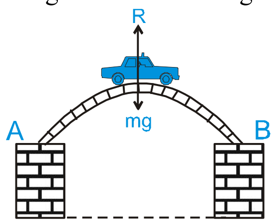
The minimum speed required to perform a vertical loop is given by equation (1) when R = 0.

$$\therefore \quad mg = \frac{mv_{min}^2}{r} \text{ or } v^2_{min} = gr \quad \text{or} \quad v_{min} = \sqrt{gr} = \sqrt{9.8 \times 25} \text{ m s}^{-1} = 15.65 \text{ ms}^{-1}.$$

So, the minimum speed, at the top, required to perform a vertical loop is 15.65 m s^{-1}.

Ex. Prove that a motor car moving over a convex bridge is lighter than the same car resting on the same bridge.

Sol. The motion of the motor car over a convex bridge AB is the motion along the segment of a circle AB (Figure) ;



The centripetal force is provided by the difference of weight mg of the car and the normal reaction R of the bridge.

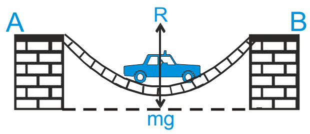
$$\therefore \quad mg - R = \frac{mv^2}{r} \quad \text{or} \quad R = mg - \frac{mv^2}{r}$$

Clearly R < mg, i.e., the weight of the moving car is less than the weight of the stationary car.

Ex. Prove that a motor car moving over a concave bridge is heavier than the same car resting on the same bridge.

Sol. The motion of the motor car over a concave bridge AB is the motion along the segment of a circle AB (Figure) ;

The centripetal force is provided by the difference of normal reaction R of the bridge and weight mg of the car.

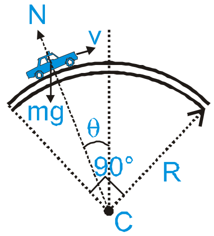


$$\therefore \quad R - mg = \frac{mv^2}{r} \quad \text{or} \quad R = mg + \frac{mv^2}{r}$$

Clearly R > mg, i.e., the weight of the moving car is greater than the weight of the stationary car.

Ex. A car is moving with uniform speed over a circular bridge of radius R which subtends an angle. of 90° at its centre. Find the minimum possible speed so that the car can cross the bridge without losing the contact any where.

Sol. Let the car losses the contact at angle θ with the vertical



$$mg\cos\theta - N = \frac{mv^2}{R}$$

$$N = mg\cos\theta - \frac{mv^2}{R} \quad \text{..........(1)}$$

for losing the contact N = 0 ,

$\Rightarrow v = \sqrt{Rg\cos\theta}$ from (1)

for minimum speed, cosθ should be minimum so that θ should be maximum.

$$\theta_{max} = 45° \Rightarrow \cos 45° = \frac{1}{\sqrt{2}}$$

$$v_{min} = \left(\frac{Rg}{\sqrt{2}}\right)^{1/2}$$ **Ans.**

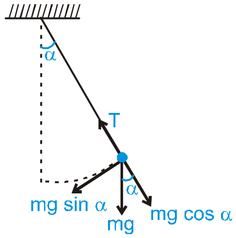
So that if car cannot lose the contact at initial or final point, car cannot be lose the contact anywhere.

Ex. Consider a simple pendulum having a bob of mass m suspended by string of length L fixed at its upper end. The bob is oscillating in a vertical circle. It is found that the speed of the bob is v when the string makes an angle α with the vertical. Find **(i)** tension in the string and **(ii)** magnitude of net force on the bob at the instant.

Sol. **(i)** The forces acting on the bob are :

(a) the tension T

(b) the weight mg



As the bob moves in a circle of radius L with centre at O. A centripetal force of magnitude $\frac{mv^2}{L}$ is required towards O. This force will be provided by the resultant of T and mg cos α. Thus,

or $$T - mg\cos\alpha = \frac{mv^2}{L} \qquad T = m\left(g\cos\alpha + \frac{v^2}{L}\right)$$

(ii) $$a_{net} = \sqrt{a_t^2 + a_r^2} = \sqrt{(g\sin\alpha)^2 + \left(\frac{v^2}{l}\right)^2} \Rightarrow |\vec{F}_{net}| = ma_{net} = m\sqrt{g^2\sin^2\alpha + \frac{v^4}{L^2}}$$ **Ans.**

CIRCULAR TURNING ON ROADS:

When vehicles go through turnings, they travel along a nearly circular arc. There must be some force which will produce the required centripetal acceleration. If the vehicles travel in a horizontal circular path, this resultant force is also horizontal. The necessary centripetal force is being provided to the vehicles by following three ways.

1. **By friction only**
2. **By banking of roads only.**
3. **By friction and banking of roads both.**

In real life the necessary centripetal force is provided by friction and banking of roads both. Now let us write equations of motion in each of the three cases separately and see what are the constant in each case.

By Friction Only

Suppose a car of mass m is moving at a speed v in a horizontal circular arc of radius r. In this case, the necessary centripetal force to the car will be provided by force of friction f acting towards center

Thus, $$f = \frac{mv^2}{r}$$

Further, limiting value of f is μN

or $$f_L = \mu N = \mu mg \quad (N = mg)$$

Therefore, for a safe turn without sliding $\frac{mv^2}{r} \le f_L$

or $$\frac{mv^2}{r} \le \mu mg$$ or $$\mu \ge \frac{v^2}{rg} \text{ or } v \le \sqrt{\mu rg}$$

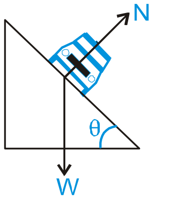
Here, two situations may arise. If μ and r are known to us, the speed of the vehicle should not exceed $\sqrt{\mu rg}$ and if v and r are known to us, the coefficient of friction should be greater than $\frac{v^2}{rg}$.

Ex. A bend in a level road has a radius of 100 m. Calculate the maximum speed which a car turning this bend may have without skidding. Given : $\mu = 0.8$.

Sol. $V_{max} = \sqrt{\mu rg} = \sqrt{0.8 \times 100 \times 10} = \sqrt{800} = 28$ m/s

By Banking of Roads Only

Friction is not always reliable at circular turns if high speeds and sharp turns are involved to avoid dependence on friction, the roads are banked at the turn so that the outer part of the road is some what lifted compared to the inner part. Applying Newton's second law along the radius and the first law in the vertical direction.



$$N \sin\theta = \frac{mv^2}{r}$$

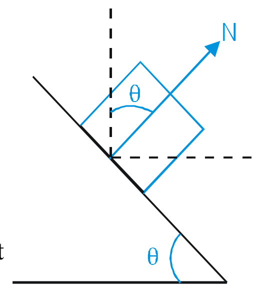
or $$N \cos\theta = mg$$

from these two equations, we get

$$\tan\theta = \frac{v^2}{rg} \quad \text{or} \quad v = \sqrt{rg\tan\theta}$$

Ex. What should be the angle of banking of a circular track of radius 600 m which is designed for cars at an average speed of 180 km/hr ?

Sol. Let the angle of banking be θ. The forces on the car are (figure)



(a) weight of the car Mg downward and

(b) normal force N.

For proper banking , static frictional force is not needed.

For vertical direction the acceleration is zero. So,

$N\cos\theta = Mg$ .....(i)

For horizontal direction , the acceleration is v^2 / r towards the centre , so that

$N\sin\theta = Mv^2 / r$ .....(ii)

From (i) and (ii), $\tan\theta = v^2 / rg$

Putting the values , $\tan\theta = \dfrac{180\,(\text{km/h})^2}{(600\,\text{m})(10\,\text{m/s}^2)} = 0.4167 \Rightarrow \theta = 22.6^\circ$.

By Friction and Banking of Road Both

If a vehicle is moving on a circular road which is rough and banked also, then three forces may act on the vehicle, of these the first force, i.e., weight (mg) is fixed both in magnitude and direction.

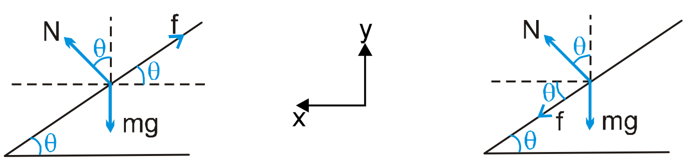


Figure (i) Figure (ii)

The direction of second force, i.e., normal reaction N is also fixed (perpendicular to road) while the direction of the third force i.e., friction f can be either inwards or outwards while its magnitude can be varied upto a maximum limit ($f_L = \mu N$). So the magnitude of normal reaction N and directions plus magnitude of friction f are so adjusted that the resultant of the three forces mentioned above is $\dfrac{mv^2}{r}$ towards the center. Of these m and r are also constant.

Therefore, magnitude of N and directions plus magnitude of friction mainly depends on the speed of the vehicle v. Thus, situation varies from problem to problem. Even though we can see that :

(i) Friction f will be outwards if the vehicle is at rest v = 0. Because in that case the component of weight mg $\sin\theta$ is balanced by f.

(ii) Friction f will be inwards if

$$v > \sqrt{rg\tan\theta}$$

(iii) Friction f will be outwards if

$$v < \sqrt{rg\tan\theta} \text{ and}$$

(iv) Friction f will be zero if

$$v = \sqrt{rg\tan\theta}$$

(v) For maximum safe speed (figure (ii)

$$N\sin\theta + f\cos\theta = \frac{mv^2}{r} \qquad \text{........(i)}$$

$$N\cos\theta - f\sin\theta = mg \qquad \text{........(ii)}$$

As maximum value of friction

$$f = \mu N$$

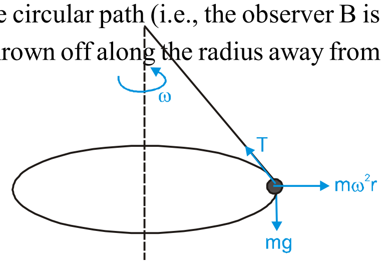
$$\therefore \quad \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \frac{v^2}{rg} \qquad \therefore \quad v_{max} = \sqrt{\frac{rg(\tan\theta + \mu)}{(1-\mu\tan\theta)}}$$

(i) The expression $\tan\theta = \frac{v^2}{rg}$ also gives the angle of banking for an aircraft, i.e., the angle through which it should tilt while negotiating a curve, to avoid deviation from the circular path.

(ii) The expression $\tan\theta = \frac{v^2}{rg}$ also gives the angle at which a cyclist should lean inward, when rounding a corner. In this case, θ is the angle which the cyclist must make with the vertical which will be discussed in chapter rotation.

Centrifugal Force :

When a body is rotating in a circular path and the centripetal force vanishes, the body would leave the circular path. To an observer A who is not sharing the motion along the circular path, the body appears to fly off tangentially at the point of release. To another observer B, who is sharing the motion along the circular path (i.e., the observer B is also rotating with the body which is released, it appears to B, as if it has been thrown off along the radius away from the centre by some force. This inertial force is called centrifugal force.)



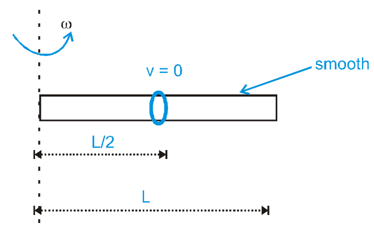
Its magnitude is equal to that of the centripetal force. $= \frac{mv^2}{r} = m\omega^2 r$

Direction of centrifugal force, it is always directed radially outward.

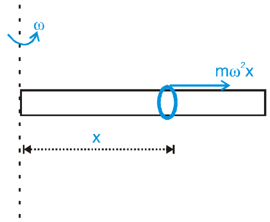
Centrifugal force is a fictitious force which has to be applied as a concept only in a rotating frame of reference to apply Newton's law of motion in that frame. FBD of ball w.r.t. non inertial frame rotating with the ball.

Suppose we are working from a frame of reference that is rotating at a constant, angular velocity ω with respect to an inertial frame. If we analyses the dynamics of a particle of mass m kept at a distance r from the axis of rotation, we have to assume that a force $mr\omega^2$ react radially outward on the particle. Only then we can apply Newton's laws of motion in the rotating frame. This radially outward pseudo force is called the centrifugal force.

Ex. A ring which can slide along the rod are kept at mid point of a smooth rod of length L. The rod is rotated with constant angular velocity ω about vertical axis passing through its one end. Ring is released from mid point. Find the velocity of the ring when it just leave the rod.



Sol. Centrifugal force



$$m\omega^2 x = ma$$

$$\omega^2 x = \frac{vdv}{dx}$$

$$\int_{L/2}^{L} \omega^2 x\, dx = \int_0^v v\, dv \text{ (integrate both side.)}$$

$$\omega^2 \left(\frac{x^2}{2}\right)_{L/2}^{L} = \left(\frac{v^2}{2}\right)_0^v$$

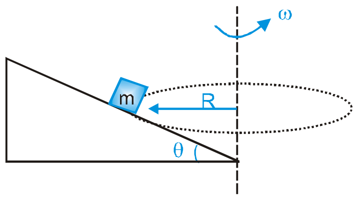
$$\omega^2 \left(\frac{L^2}{2} - \frac{L^2}{8}\right) = \frac{v^2}{2}$$

$$v = \frac{\sqrt{3}}{2}\omega L.$$

Velocity at time of leaving the rod

$$v' = \sqrt{(\omega L)^2 + \left(\frac{\sqrt{3}}{2}\omega L\right)^2} = \frac{\sqrt{7}}{2}\omega L$$ **Ans.**

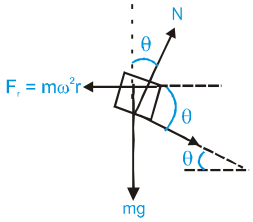
Ex. A small block is placed on a rough triangular shaped wedge which is revolving as shown such that the block undergoes circular path of radius R. The coefficient of friction between the block and the wedge is μ. Find the range of angular speed w so that the block does not slip with respect to the wedge.



Sol. If we select the frame of reference as the point at which the small block is placed. We take centrifugal force $m\omega^2 r$ outward.

(i) Maximum angular speed (ω_{max})

More is ω, more is centrifugal force to balance which more horizontal force is required radially inwards. The friction force acts down the incline as shown so that its horizontal componant helps horizontal componant of Normal force to balance centrifugal force.



$(F_{net})_y = 0 \Rightarrow N\cos\theta - f\sin\theta - mg = 0$

Under limiing condition i.e. when the block is about to slip, f = μN

$\therefore$ $N(\cos\theta - \mu\sin\theta) = mg$

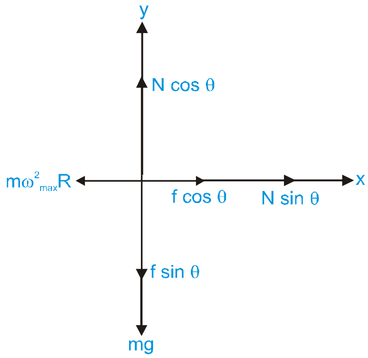
$$\Rightarrow \quad N = \frac{mg}{\cos\theta - \mu\sin\theta}$$

$$(F_{net})_x = 0 \quad \Rightarrow \quad N\sin\theta + (\mu N)\cos\theta - m\omega_{max}^2 R = 0$$

$$\Rightarrow \quad N(\sin\theta + \mu\cos\theta) = m\omega_{max}^2 R$$

$$\Rightarrow \quad mg\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right) = m\omega_{max}^2 R$$

$$\therefore \quad \omega_{max} = \sqrt{\frac{g}{R}\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right)}$$



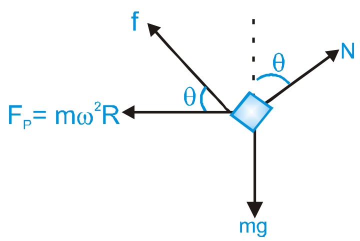
(ii) For minimum angular speed (ω_{min})

Lesser is w, lesser is centrifugal force thus friction now acts up the incline so as to provide horizontal componant along centrifugal force to balance horizontal componant of Normal force.

$(F_{net})_y = 0 \Rightarrow N\cos\theta + f\sin\theta - mg = 0$

$\Rightarrow$ $N(\cos\theta + \mu\sin\theta) = mg$

$$N = \frac{mg}{\cos\theta + \mu\sin\theta}$$



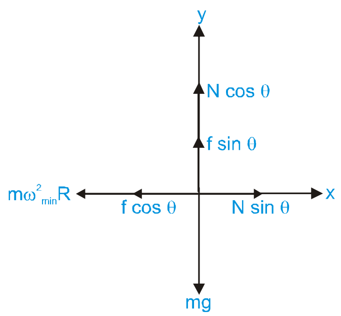
$$(Fnet)_x = 0 \quad \Rightarrow N\sin\theta - f\cos\theta - m\omega_{min}^2 R = 0$$

$$\Rightarrow N(\text{sinq} - \mu\,\text{cosq}) = m\,w_{min}^2\,R$$

$$\Rightarrow mg\left(\frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}\right) = m\omega_{min}^2 R$$

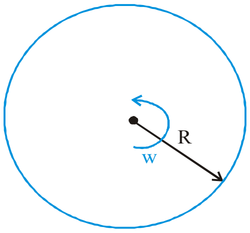
$$\Rightarrow \quad \omega_{min} = \sqrt{\frac{g(\sin\theta - \mu\cos\theta)}{R(\cos\theta + \mu\sin\theta)}}$$

$$\therefore \quad \sqrt{\frac{g(\sin\theta - \mu\cos\theta)}{R(\cos\theta + \mu\sin\theta)}} \le \omega \le \sqrt{\frac{g(\sin\theta + \mu\cos\theta)}{R(\cos\theta - \mu\sin\theta)}}$$

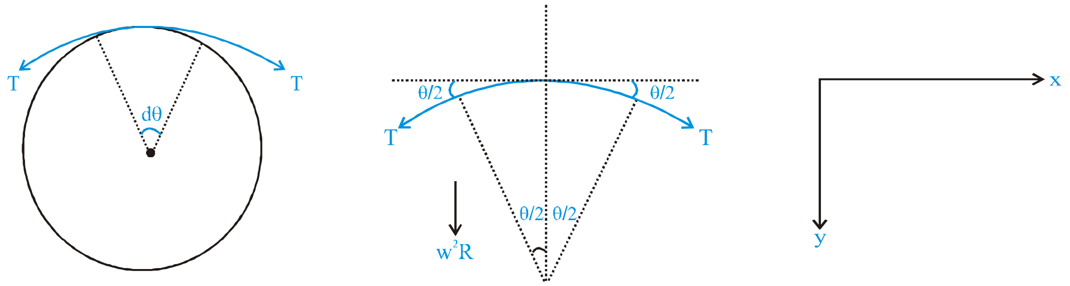


Tension in rotating ring:

Diagram shown top view of a uniforming of mass m rotating in horizontal plane.



Here tension inside the ring is an internal force for the ring so to find it, we take a very small part of the ring which subtends angle $d\theta$ as shown.



We take y-axis towards the centre and x-axis in tangential direction and origin as the mid-point of the element

$\therefore \quad (F_{net}) = (dm)\omega^2 R$

$$2T\sin\left(\frac{d\theta}{2}\right) = (dm)\omega^2 R \qquad ....(i)$$

but $\quad \sin\left(\frac{d\theta}{2}\right) \simeq \frac{d\theta}{2} \quad (\because d\theta \text{ is very small})$

Also dm = (mass per unit length of ring) × length of element

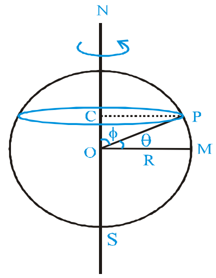
$\therefore \quad dm = \left(\frac{M}{2\pi R}\right)(Rd\theta) = \left(\frac{M}{2\pi}\right)d\theta$

Putting these values in equation (i), we get

$$2T = \left(\frac{d\theta}{2}\right) = \left(\frac{M}{2\pi}\right)d\theta \times \omega^2 R$$

$\therefore \quad T = \frac{M\omega^2 R}{2\pi}$

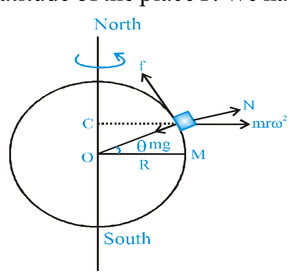
Effect of earths rotation on apparent weight:



The earth rotates about its axis at an angular speed of one revolution per 24 hours. The line joining the north and the south poles is the axis of rotation. Every point on the moves in a circle. A point at equator moves in a circle of radius equal to the radius of the earth and the centre of the circle of the circle is same as the centre of the earth. For any other point on the earth, the circle of rotation is smaller than this. Consider a place P on the earth.

Draw a perpendicular PC from P to the axis SN. The places P rotates in a circle with the centre at C. The radius of this circle is CP. The angle between the line OM and the radius OP through P is called the latitude of the place P. We have

$$CP = OP\cos\theta \qquad \text{or,} \qquad r = R\cos\theta$$

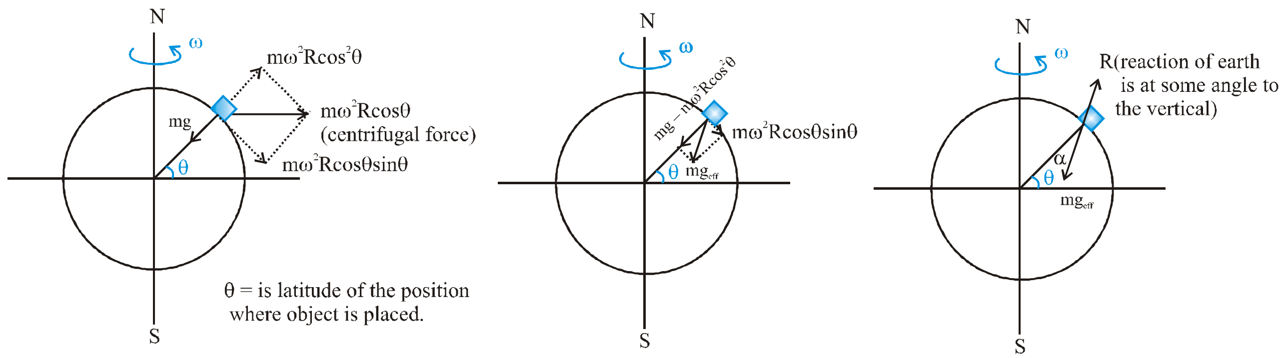


where R is the radius of the earth and ϕ is colatitude angle. If we work from the frame of reference of the earth, we shall have to assume the existence of pseudo force. In particular, a centrifugal force mw^2r has to be assumed on any particle of mass m placed at P.

If we consider a block of mass m at point P then this block is atrest with respect to earth. If resolve the forces along and perpendicular the centre of earth then

$$N + mr\omega^2\cos\theta = mg$$

$\Rightarrow$ $N = mg - mr\omega^2 \cos\theta$ $\Rightarrow$ $N = mg - mR\omega^2 \cos^2\theta$



Note: At equator $(\theta = 0)$ Wapp. is minimum and at pole $(\theta = \pi/2)$ Wapp.. is maximum .

This apparent weight is not along normal but at some angle α w.r.t. it. At all point except poles and equator

$(\alpha = 0$ at poles and equator$)$

Ex. A body weighs 98N on a spring balance at the north pole. What will be the reading on the same scale if it is shifted to the equator? Use $g = GM/R^2 = 9.8$ m/s^2 and $R_{earth} = 6400$ km.

Sol. At poles , the apparent weight is same as the true weight.

Thus, $98N = mg = m(9.8 \text{ m/s}^2)$

At the equator , the apparent weight is

$$mg' = mg - m\omega^2 R$$

The radius of the earth is 6400 km and the angular speed is

$$\omega = \frac{2\pi \text{ rad}}{24 \times 60 \times 60\text{s}} = 7.27 \times 10^{-6} \text{ rad/s}$$

$$mg' = 98N - (10 \text{ kg})(7.27 \times 10^{-5} \text{ s}^{-1})^2 (6400 \text{ km})$$

$= 97.66$N **Ans.**

Ex. A fan rotating with $\omega = 100$ rad/s, is switched off. After 2n rotation its angular velocity becomes 50 rad/s. Find the angular velocity of the fan after n rotations.

Sol. $\omega^2 = \omega_0^2 + 2\alpha\,\theta$

$50^2 = (100)^2 + 2\alpha\,(2\pi \,.\, 2n)$ ...(1)

If angular velocity after n rotation is ω_n

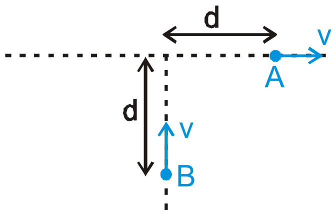
$\omega_n^2 = (100)^2 + 2\alpha\,(2\pi \,.\, n)$ ...(2)

from equation (1) and (2)

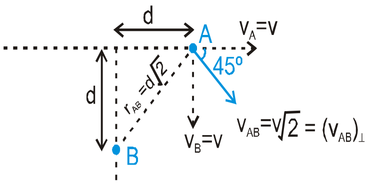
$$\frac{50^2 - 100^2}{\omega_n^2 - 100^2} = \frac{2\alpha(2\pi.2n)}{2\alpha 2\pi n} = 2$$

$\Rightarrow$ $\omega_n^2 = \dfrac{50^2 + 100^2}{2}$ $\Rightarrow$ $\omega = 25\sqrt{10}$ rad/s **Ans.**

Ex. Find angular velocity of A with respect to B at the instant shown in the figure.



Sol. Angular velocity of A with respect to B is ;



$$\omega_{AB} = \frac{(v_{AB})_\perp}{r_{AB}}$$

$\Rightarrow$ $v_{AB} = \sqrt{2}\,v = (v_{AB})_\perp$

$\Rightarrow$ $r_{AB} = \sqrt{2}\,d$

$\Rightarrow$ $\omega_{AB} = \dfrac{(v_{AB})_\perp}{r_{AB}} = \dfrac{v\sqrt{2}}{d\sqrt{2}} = \dfrac{v}{d}$

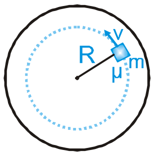
Ex. A particle is moving with a constant angular acceleration of 4 rad./sec^2 in a circular path. At time t = 0 particle was at rest. Find the time at which the magnitudes of centripetal acceleration and tangential acceleration are equal.

Sol. $a_t = \alpha R \Rightarrow v = 0 + \alpha R t$

$$a_c = \frac{v^2}{R} = \frac{\alpha^2 R^2 t^2}{R}$$

$\because$ $|a_t| = |a_c| \Rightarrow \alpha R = \dfrac{\alpha^2 R^2 t^2}{R} \Rightarrow t^2 = \dfrac{1}{\alpha} = \dfrac{1}{4} \Rightarrow t = \dfrac{1}{2}$ sec. **Ans.**

Ex. The coefficient of friction between block and table is μ. Find the tension in the string if the block moves on the horizontal table with speed v in circle of radius R.



Sol. : The magnitude of centripetal force is $\dfrac{mv^2}{R}$.

(i) If limiting friction is greater than or equal to $\dfrac{mv^2}{R}$, then static friction alone provides centripetal force, so tension is equal to zero.

T = 0 **Ans.**

(ii) If limiting friction is less than $\dfrac{mv^2}{R}$, then friction as well as tension both combine to provide the necessary centripetal force.

$$T + f_r = \frac{mv^2}{R}$$

In this case friction is equal to limiting friction, $f_r = \mu mg$

$\therefore$ Tension $= T = \dfrac{mv^2}{R} - \mu mg$ **Ans.**

Ex. A block of mass m is kept on rough horizontal turn table at a distance r from centre of table. Coefficient of friction between turn table and block is μ . Now turn table starts rotating with uniform angular acceleration α.

(i) Find the time after which slipping occurs between block and turn table.

(ii) Find angle made by friction force with velocity at the point of slipping.

Sol.: **(i)** $a_t = \alpha r$

speed after t time

$$\frac{dv}{dt} = \alpha r \qquad \Rightarrow \qquad v = 0 + \alpha rt$$

Centripetal acceleration

$$a_c = \frac{v^2}{r} = \alpha^2 rt^2$$

Net acceleration $a_{net} = \sqrt{a_t^2 + a_c^2} = \sqrt{\alpha^2 r^2 + \alpha^4 r^2 t^4}$

block just start slipping

$$\mu mg = ma_{net} = m\sqrt{\alpha^2 r^2 + \alpha^4 r^2 t^4}$$

$$t = \left(\frac{\mu^2 g^2 - \alpha^2 r^2}{\alpha^4 r^2}\right)^{1/4} \Rightarrow t = \left[\left(\frac{\mu g}{\alpha^2 r}\right)^2 - \left(\frac{1}{\alpha}\right)^2\right]^{1/4}$$ **Ans.**

(ii) $\tan\theta = \frac{a_c}{a_t} \Rightarrow \tan\theta = \frac{\alpha^2 rt^2}{\alpha r}$

$\Rightarrow \theta = \tan^{-1}(\alpha t^2)$ **Ans.**