Application of Derivatives

[TOPIC 1] Rate Measure, Increasing-Decreasing Functions and Approximation

1.1 Rate of Change of Quantities

Let $y = f(x)$ be a function of x. Then, $\frac{dy}{dx}$ represents the rate of change of y with respect to x. Also, $\left(\frac{dy}{dx}\right)_{\text{at } x=x_0}$ represents the rate of change of y with respect to x at $x = x_0$.

Rate of Change of Two Variables

If two variables x and y are varying with respect to another variable t, i.e. $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}, \text{ where } \frac{dx}{dt} \neq 0$$

In other words, the rate of change of y with respect to x can be calculated using the rate of change of y and that of x both with respect to t.

NOTE $\frac{dy}{dx}$ is positive, if y increases as x increases and it is negative, if y decreases as x increases.

1.2 Marginal Cost and Marginal Revenue

Marginal Cost

Marginal cost represents the instantaneous rate of change of the total cost at any level of output.

If $C(x)$ represents the cost function for x units produced, then marginal cost (MC) is given by $MC = \frac{d}{dx}\{C(x)\}$.

Marginal Revenue

Marginal revenue represents the rate of change of total revenue with respect to the number of items sold at an instant.

If $R(x)$ is the revenue function for x units sold, then marginal revenue (MR) is given by

$$MR = \frac{d}{dx}\{R(x)\}.$$

1.3 Increasing and Decreasing Functions

1. Let I be an open interval contained in the domain of a real valued function f. Then, f is said to be

   (i) **increasing on** I, if $x_1 < x_2$ in I $\Rightarrow f(x_1) \le f(x_2), \forall\ x_1, x_2 \in I$.

   (ii) **strictly increasing on** I, if $x_1 < x_2$ in I $\Rightarrow f(x_1) < f(x_2), \forall\ x_1, x_2 \in I$.

   (iii) **decreasing on** I, if $x_1 < x_2$ in $I \Rightarrow f(x_1) \ge f(x_2), \forall\ x_1, x_2 \in I$.

   (iv) **strictly decreasing on** I, if $x_1 < x_2$ in I $\Rightarrow f(x_1) > f(x_2), \forall\ x_1, x_2 \in I$.

NOTE For a given interval $I \subseteq R$, function f increase for some values in I and decrease for other values in I, then we say function is neither increasing nor decreasing.

2. Let f be continuous on $[a, b]$ and differentiable on the open interval (a, b). Then,

   (i) f is increasing in $[a, b]$, if $f'(x) > 0$ for each $x \in (a, b)$.

   (ii) f is decreasing in $[a, b]$, if $f'(x) < 0$ for each $x \in (a, b)$.

   (iii) f is a constant function in $[a, b]$, if $f'(x) = 0$ for each $x \in (a, b)$.

NOTE (i) f is strictly increasing in (a,b), if $f'(x) > 0$ for each $x \in (a,b)$.

(ii) f is strictly decreasing in (a,b), if $f'(x) < 0$ for each $x \in (a,b)$.

Monotonic Function

A function which is either increasing or decreasing in a given interval I, is called monotonic function.

1.4 Approximation

Let $y = f(x)$ be any function of x. Let Δx be the small change in x and Δy be the corresponding change in y, i.e. $\Delta y = f(x + \Delta x) - f(x)$.

Then, $dy = f'(x)\,dx$ or $dy = \frac{dy}{dx} \cdot \Delta x$ is a good approximation of Δy, when $dx = \Delta x$ is relatively small, we denote it by $dy = \Delta y$.

Thus, $f(x + \Delta x) = f(x) + f'(x) \cdot \Delta x$

NOTE The differential of the dependent variable is not equal to the increment of the variable whereas the differential of independent variable is equal to the increment of the variable.

Some Important Terms

(i) **Absolute error** The error Δx in x, is called the absolute error in x.

(ii) **Relative error** If Δx is an error in x, then $\frac{\Delta x}{x}$ is called the relative error in x.

(iii) **Percentage error** If Δx is an error in x, then $\frac{\Delta x}{x} \times 100$ is called percentage error in x.

[TOPIC 2] Tangents and Normals

2.1 Tangent and Normal

A line which touches the curve at a single point is called tangent at a point and if a line is perpendicular to the tangent at the point of contact, then it is called normal.

Slope of Tangent and Normal

1. The slope of a tangent to the curve $y = f(x)$ at the point (x_1, y_1) is given by
$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} \quad \text{or} \quad f'(x_1).$$
2. The slope of a normal to the curve $y = f(x)$ at the point (x_1, y_1) is given by
$$\frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}.$$

NOTE If a tangent line to the curve $y = f(x)$ makes an angle θ with X-axis in the positive direction, then $\frac{dy}{dx}$ = Slope of the tangent = $\tan\theta$.

Equations of Tangent and Normal

1. The equation of tangent to the curve $y = f(x)$ at the point $P(x_1, y_1)$ is given by
$$y - y_1 = m(x - x_1),$$
where $m = \frac{dy}{dx}$ at point (x_1, y_1).
2. The equation of normal to the curve $y = f(x)$ at the point $Q(x_1, y_1)$ is given by
$$y - y_1 = \frac{-1}{m}(x - x_1),$$
where $m = \frac{dy}{dx}$ at point (x_1, y_1).

Some Important Terms

1. If slope of the tangent line is zero, then $\tan\theta = 0$, so $\theta = 0$, which means that tangent line is parallel to the X-axis and then equation of tangent at the point (x_1, y_1) is $y = y_1$.
2. If $\theta \to \frac{\pi}{2}$, then $\tan\theta \to \infty$, which means that tangent line is perpendicular to the X-axis, i.e. parallel to the Y-axis and then equation of the tangent at the point (x_1, y_1) is $x = x_1$.
3. Suppose m_1 and m_2 are slopes of tangent to the curves. The condition for two curves to be perpendicular (orthogonal), is $m_1 m_2 = -1$.
4. The angle of intersection between two curves is the angle between the tangents to the curves at the point of intersection.

[TOPIC 3] Maxima and Minima

3.1 Maximum and Minimum Value

Let f be a function define on an interval I. Then,

1. f is said to have a **maximum value** in I, if there exists a point c in I such that $f(c) \geq f(x)$, $\forall\ x \in I$. The number $f(c)$ is called the maximum value of f in I and the point c is called a **point of maximum value** of f in I.
2. f is said to have a **minimum value** in I, if there exists a point c in I such that $f(c) \leq f(x)$, $\forall x \in I$. The number $f(c)$ is called the minimum value of f in I and the point c is called a **point of minimum value** of f in I.
3. f is said to have an extreme value in I, if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I. The number $f(c)$ is called an **extreme value** of f in I and the point c is called an **extreme point**.

Local Maxima and Local Minima

1. A point a in the interior of the domain of f, is called local maxima, if there exists a $\delta > 0$ such that $f(x) \leq f(a)$, $\forall\ x \in (a-\delta, a+\delta)$. Here, $f(a)$ is called the local maximum value of $f(x)$ at the point $x = a$.

2. A point a in the interior of the domain of f, is called local minima, if there exists a $\delta > 0$ such that $f(x) \geq f(a)$, $\forall x \in (a-\delta, a+\delta)$. Here, $f(a)$ is called the local minimum value of $f(x)$ at the point $x = a$.

Some Important Points

1. The points at which a function changes its nature, from decreasing to increasing or *vice-versa*, are called **turning points**.

NOTE (i) Through the graphs, we can even find maximum/minimum value of a function at a point at which it is not even differentiable.

(ii) Every monotonic function assumes its maximum/minimum value at the end points of the domain of definition of the function.

2. Every continuous function on a closed interval has a maximum and a minimum value.

3. Let f be a function defined on an open interval I. Suppose $c \in I$ is any point. If f has a local maxima or a local minima at $x = c$, then either $f'(c) = 0$ or f is not differentiable at c.

4. **Critical Point** A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable, is called a critical point of f.

3.2 First Derivative Test

Let f be a function defined on an open interval I and f be continuous at a critical point c in I. Then,

1. if $f'(x)$ changes sign from positive to negative as x increases through c, then c is a point of **local maxima.**
2. if $f'(x)$ changes sign from negative to positive as x increases through c, then c is a point of **local minima.**
3. if $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Such a point is called a **point of inflection.**

3.3 Second Derivative Test

Let $f(x)$ be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then,

1. $x = c$ is a point of local maxima, *if* $f'(c) = 0$ and $f''(c) < 0$.
2. $x = c$ is a point of local minima, if $f'(c) = 0$ and $f''(c) > 0$.
3. the test fails, if $f'(c) = 0$ and $f''(c) = 0$.

NOTE (i) If the test fails, then we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflection.

(ii) If we say that f is twice differentiable at a, then it means second order derivative exist at a.

3.4 Maximum and Minimum Value of Function in a Closed Interval

If a function $f(x)$ is continuous on a closed interval $[a, b]$, then it attains the maximum value (minimum value) which is called the absolute maximum value or global maximum (absolute minimum value or global minimum), at critical points or at the end points of the interval $[a, b]$.

Let f be a continuous function on a closed interval $I = [a, b]$. Then, f has the absolute maximum value and f attains it atleast once in I. Also, f has the absolute minimum value and attains it atleast once in I.

Working Rule for Finding Absolute Maxima and Absolute Minima

***Step* I** Find all the critical points of f in the given interval.

***Step* II** At all these points and at the end points of the interval, calculate the values of f.

***Step* III** Identify the maximum and minimum values of f out of the values calculated in Step II. The maximum value will be the absolute maximum value of f and the minimum value will be the absolute minimum value of f.