Determinants

[TOPIC 1] Expansion of Determinants

1.1 Determinant

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) is called a determinant of the square matrix A. It is denoted by det A or $|A|$. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$.

NOTE (i) For matrix A, $|A|$ is read as determinant of A and not modulus of A.
(ii) Determinant gives numerical value but matrix do not give numerical value.
(iii) Only square matrix have determinants.

Value of a Determinant

1. Determinant of a matrix $A = [a]$ of order 1 is given by $|A| = |a| = a$

2. Determinant of a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ of order 2 is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

3. Determinant of a matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ of order 3 is given by expressing it in terms of second order determinants. This is known as expansion of a determinant along a row or column.

$$|A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

($\because$ expansion along first row, i.e. R_1)

NOTE (i) There are six ways of expanding a determinant of order 3 corresponding to each of three rows (R_1, R_2 and R_3) and each of three columns (C_1, C_2 and C_3).
(ii) For easier calculations we will expand the determinant along that row (or column) which contains maximum number of zeroes.
(iii) While expanding, instead of multiplying by $(-1)^{i+j}$, we can multiply by $+1$ or -1 according as $(i + j)$ is even or odd.

1.2 Minor

Minor of an element a_{ij} of a determinant is a determinant obtained by deleting the ith row and jth column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

NOTE Minor of an element of a determinant of order $n (n \geq 2)$ is a determinant of order $(n-1)$.

1.3 Cofactor

Cofactor of an element a_{ij} of a determinant, denoted by A_{ij} or C_{ij}, is defined as $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is a minor of an element a_{ij}.

NOTE (i) For expanding the determinant, we can use minors and cofactors as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

and $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

(ii) If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero.

1.4 Singular and Non-singular Matrices

If the value of determinant corresponding to a square matrix is zero, then matrix is said to be a singular matrix, otherwise it is non-singular matrix. i.e. For a square matrix A, if $|A| = 0$, then it is said to be a singular matrix and if $|A| \neq 0$, then it is said to be a non-singular matrix.

Theorems

1. If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.
2. The determinant of the product of matrices is equal to product of their respective determinants, i.e. $|AB| = |A||B|$, where A and B are square matrices of same order.
3. The area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

NOTE (i) Since, area is a positive quantity, so we always take the absolute value of the determinant.

(ii) If area is given, then take both positive and negative values of the determinant for calculation.

(iii) The area of the triangle formed by three collinear points is zero.

[TOPIC 2] Properties of Determinants

To find the value of the determinant, we try to make the maximum possible zeroes in a row (or column) by using properties given below and then expand the determinant along that row (or column).

2.1 Properties of Determinants

1. The value of a determinant remains unchanged on changing rows into columns and columns into rows. It follows that, if A is a square matrix, then $|A'| = |A|$.

e.g. $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$

2. If in a determinant any two rows or columns are interchanged, then the value of the determinant obtained is negative of the value of the given determinant. If we make n such changes of rows (or columns) in determinant Δ and obtain determinant Δ_1, then $\Delta_1 = (-1)^n \Delta$.

e.g. $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = - \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} \quad (R_1 \leftrightarrow R_3)$

3. If corresponding elements of any two rows or columns of a determinant are identical or proportional, then the value of the determinant is zero.

e.g. $\begin{vmatrix} a & b & c \\ d & e & f \\ a & b & c \end{vmatrix} = 0 \quad [\because R_1 \text{ and } R_3 \text{ are identical}]$

4. If we multiply each element of any one row or column of a determinant by a constant k, then the value of the determinant is multiplied by k.

e.g. $k \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} ka & kb & kc \\ d & e & f \\ g & h & i \end{vmatrix}$

5. Multiplying a determinant by k means multiplying the elements of only one row (or one column) by k.

6. If A is a square matrix of order n, then $|kA| = k^n |A|$, where $n \in N$.

7. If all the elements of any row or column of a determinant are zero, then the value of such determinant is zero.

8. If some or all elements of a row or column of a determinant are expressed as a sum of two (or more) terms, then the determinant can be expressed as the sum of two (or more) determinants.

e.g. $\begin{vmatrix} a+a' & b+b' & c+c' \\ d & e & f \\ g & h & i \end{vmatrix}$

$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{vmatrix}.$

9. In the elements of any row or column of a determinant, if we add or subtract the multiples of corresponding elements of any other row or column, then the value of determinant remains unchanged,

e.g. $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+kb & b & c \\ d+ke & e & f \\ g+kh & h & i \end{vmatrix}$

$(C_1 \to C_1 + kC_2)$

In other words, the value of determinants remain same, if we apply the operation $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$.

NOTE Above properties are true for determinants of any order.

[TOPIC 3] Adjoint and Inverse of a Matrix

3.1 Adjoint of a Matrix

The adjoint of a square matrix $A = [a_{ij}]_{n\times n}$ is defined as the transpose of the matrix formed by cofactors of the elements a_{ij} and it is denoted by adj (A).

i.e. Adjoint of a matrix $A = [a_{ij}]_{n\times n}$ is a matrix $[A_{ij}]^T_{n\times n}$, where A_{ij} is a cofactor of element a_{ij}.

NOTE For a square matrix of order 2 given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the adj (A) can also be obtained by interchanging a_{11} and a_{22} and by changing signs of a_{12} and a_{21}, i.e. adj $(A) = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$.

Properties of Adjoint of a Matrix

If A is a square matrix of order $n \times n$, then

1. $A(\text{adj}A) = (\text{adj}A)A = |A| I_n$
2. $|\text{adj } A| = |A|^{n-1}$, provided $|A| \neq 0$
3. $\text{adj } (A^T) = (\text{adj } A)^T$

3.2 Inverse of a Square Matrix

If A is a non-singular matrix (i.e. $|A| \neq 0$), then

$$A^{-1} = \frac{1}{|A|}\text{adj}(A).$$

For non-singular matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the inverse is $A^{-1} = \frac{[A_{ij}]^T}{|A|} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$,

where A_{ij} is the cofactor of element a_{ij}.

NOTE Inverse of a square matrix, if exists, it is unique.

Properties of a Inverse Matrix

Let A and B be two square invertible matrices of same order, then

(i) $(A^{-1})^{-1} = A$

(ii) $(A^T)^{-1} = (A^{-1})^T$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$

(iv) $|A^{-1}| = |A|^{-1}$

(v) $AA^{-1} = A^{-1}A = I$

(vi) $\text{adj}(A^{-1}) = (\text{adj}A)^{-1}$

NOTE (i) If A, B and C are invertible matrices of the same order, then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

(ii) Only square matrices have adjoint or inverse.

Solution of System of Linear Equations by Using Inverse of a Matrix (or by matrix method)

Let the system of linear equations be
$a_1x + b_1y + c_1z = d_1$; $a_2x + b_2y + c_2z = d_2$ and
$a_3x + b_3y + c_3z = d_3$.

We can write the above system of linear equations in matrix form as $AX = B$, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

***Case* I** If $|A| \neq 0$, then system is consistent and has a unique solution which is given by $X = A^{-1}B$.

***Case* II** If $|A| = 0$ and $(\text{adj } A)\, B \neq O$, then system is inconsistent and has no solution.

***Case* III** If $|A| = 0$ and $(\text{adj } A)\, B = O$, then system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solution.

NOTE A system of equations is consistent or inconsistent according as its solution exists or not.