Differential Equations

[TOPIC 1] Formation of Differential Equations

1.1 Differential Equation

An equation involving independent variable, dependent variable, derivatives of dependent variable with respect to one (or more) independent variable and constant is called a differential equation. e.g. $x\frac{dy}{dx}+xy\frac{d^2y}{dx^2}+4=0$.

Ordinary Differential Equation

An equation involving derivatives of the dependent variable with respect to only one independent variable is called an ordinary differential equation,

e.g. $$\frac{dy}{dx}+\frac{d^2y}{dx^2}-2=0.$$

From any given relation between the dependent and independent variables, a differential equation can be formed by differentiating it with respect to the independent variable and eliminating arbitrary constants involved.

Order of a Differential Equation

Order of a differential equation is defined as the order of the highest order derivative of the dependent variable with respect to the independent variable involved in the given differential equation.

NOTE Order of the differential equation cannot be more than the number of arbitrary constants in the equation.

Degree of a Differential Equation

The highest exponent of the highest order derivative is called degree of a differential equation provided exponent of each derivative appearing in the differential equation is a non-negative integer.

NOTE
(i) Order and degree (if defined) of a differential equation are always positive integers.
(ii) The differential equation is a polynomial equation in derivatives.
(iii) If the given differential equation is not polynomial equation in its derivatives, then its degree is not defined.

e.g. $$\frac{d^2y}{dx^2}+\sin\frac{dy}{dx}=0,$$

here we cannot defined the degree because differential equation is not in a polynomial equation.

(iv) If the differential equation have radical (like $\sqrt{\ },\sqrt[3]{\ }$ etc) and fractions $\left(\text{like } \frac{1}{\frac{dy}{dx}}, \frac{1}{\sqrt{1+\frac{d^2y}{dx^2}}} \text{ etc}\right)$, then to find degree, first made it free from radicals and fractions by simplifying it.

Formation of a Differential Equation

To form a differential equation from a given relation, we use the following steps:

***Step* I** Write the given equation and find the number of arbitrary constants it has.

***Step* II** Differentiate the given equation with respect to the independent variable n times, where n is the number of arbitrary constants in the given equation.

***Step* III** Eliminate all arbitrary constants from the equations formed after differentiating in Step II and the given equation.

***Step* IV** The equation obtained without the arbitrary constants is the required differential equation.

[TOPIC 2] Solution of Different Types of Differential Equations

2.1. Solution of a Differential Equation

A function of the form $y = \phi(x) + C$, which satisfies given differential equation, is called the solution of the differential equation.

General solution

The solution which contains as many arbitrary constants as the order of the differential equation, is called the general solution of the differential equation, i.e. if the solution of a differential equation of order n contains n arbitrary constants, then it is the general solution.

Particular Solution

A solution obtained by giving particular values to arbitrary constants in the general solution of a differential equation, is called the particular solution.

2.2 Methods of Solving First Order and First Degree Differential Equation

Variable Separable Form

Suppose a differential equation is $\frac{dy}{dx} = F(x, y)$.

Here, we separate the variables and then integrate both sides to get the general solution, i.e. above equation may be written as $\frac{dy}{dx} = h(x) \cdot k(y)$.

Then, by separating the variables,

we get $\frac{dy}{k(y)} = h(x)\, dx$.

Now, integrate above equation and get the general solution as $K(y) = H(x) + C$

Here, $K(y)$ and $H(x)$ are the anti-derivatives of $\frac{1}{k(y)}$ and $h(x)$, respectively and C is an arbitrary constant.

Homogeneous Differential Equation

A differential equation $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ is said to be homogeneous, if $f(x, y)$ and $g(x, y)$ are homogeneous functions of same degree, i.e. it may be written as $\frac{dy}{dx} = \frac{x^n f\left(\frac{y}{x}\right)}{x^n g\left(\frac{y}{x}\right)} = \frac{f\left(\frac{y}{x}\right)}{g\left(\frac{y}{x}\right)}$

$$= F\left(\frac{y}{x}\right).$$

To check that given differential equation is homogeneous or not, we write differential equation as $\frac{dy}{dx} = F(x,y)$ or $\frac{dx}{dy} = F(x,y)$ and replace x by λx, y by λy to get the function $F(x,y) = \lambda F(x,y)$.

Here, if power of λ is zero, then differential equation is homogeneous otherwise not.

Solution of homogeneous differential equation

To solve homogeneous differential equation of the form

$$\frac{dy}{dx} = F(x, y) \quad \ldots \text{(i)}$$

we put $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

in Eq. (i) to reduce it into variable separable form. Then, solve it and in the final result put $v = \frac{y}{x}$ to get required solution.

NOTE If the homogeneous differential equation is in the form of $\frac{dx}{dy} = F(x, y)$, where $F(x, y)$ is homogeneous function of degree zero, then we make substitution $x = vy$ and we proceed further to find the general solution as mentioned above.

Linear Differential Equation

General form of linear differential equation

is $$\frac{dy}{dx} + Py = Q \quad \ldots(i)$$

where, P and Q are functions of x or constants.

or $$\frac{dx}{dy} + P'x = Q' \quad \ldots(ii)$$

where, P' and Q' are functions of y or constants.

Then, solution of Eq. (i) is given by the equation

$$y \times \text{IF} = \int (Q \times \text{IF})\, dx + C$$

where, IF = Integrating factor and IF = $e^{\int P\, dx}$.

Also, solution of Eq. (ii) is given by the equation

$$x \times \text{IF} = \int (Q' \times \text{IF})\, dy + C$$

where, IF = Integrating factor and IF = $e^{\int P'dy}$.