Integrals

[TOPIC 1] Indefinite Integrals

1.1 Integration is the Inverse Process of Differentiation

In the differential calculus, we are given a function and we have to find the derivative or differential of this function, but in the integral calculus, we have to find a function whose differential is given. Thus, integration is the inverse process of differentiation.

Suppose $\frac{d}{dx}[F(x)] = f(x)$, then $\int f(x)dx = F(x) + C$.

These integrals are called indefinite integrals or general integrals and C is an arbitrary constant by varying which one gets different anti-derivatives of the given function.

NOTE Derivative of a function is unique but a function can have infinite anti-derivatives or integrals.

1.2 Properties of Indefinite Integral

1. $\int [f(x) \pm g(x)]\, dx = \int f(x)dx \pm \int g(x)dx$.
2. For any real number k, $\int k\, f(x)dx = k\int f(x)dx$.
3. In general, if $f_1, f_2, \ldots, f_n$ are functions and $k_1, k_2, \ldots, k_n$ are real numbers, then
$\int [k_1 f_1(x) + k_2 f_2(x) + \ldots + k_n f_n(x)]\, dx$
$= k_1 \int f_1(x)dx + k_2 \int f_2(x)dx + \ldots + k_n \int f_n(x)dx.$

1.3 Basic Formulae of Integrals

1. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
2. $\int e^{ax}\, dx = \frac{e^{ax}}{a} + C$
3. $\int a^x\, dx = \frac{a^x}{\log a} + C$
4. $\int \frac{1}{x} dx = \log|x| + C$
5. $\int \sin x\, dx = -\cos x + C$
6. $\int \cos x\, dx = \sin x + C$
7. $\int \tan x\, dx = -\log|\cos x| + C = \log|\sec x| + C$
8. $\int \cot x\, dx = \log|\sin x| + C = -\log|\operatorname{cosec} x| + C$
9. $\int \sec x\, dx = \log|\sec x + \tan x| + C$
$= \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$
10. $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$
$= \log\left|\tan\frac{x}{2}\right| + C$
11. $\int \sec x \tan x\, dx = \sec x + C$
12. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$
13. $\int \sec^2 x\, dx = \tan x + C$

14. $\int \operatorname{cosec}^2 x\, dx = -\cot x + C$

15. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

16. $\int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1} x + C$

17. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

18. $\int \frac{-1}{1+x^2} dx = \cot^{-1} x + C$

19. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$

20. $\int \frac{-1}{x\sqrt{x^2-1}} dx = \operatorname{cosec}^{-1} x + C$

21. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + C$

22. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log | x + \sqrt{x^2-a^2} | + C$

23. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log | x + \sqrt{x^2+a^2} | + C$

24. $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a}\right) + C$

25. $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$

26. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$

27. $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2} \sqrt{x^2-a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2-a^2} \right| + C$

28. $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$

29. $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2} \sqrt{x^2+a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2+a^2} \right| + C$

30. $\int (ax+b)^n dx = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + C, n \neq -1$

1.4 Integration by Substitutions

Substitution method is used, when a suitable substitution of variable leads to simplification of integral.

If $I = \int f(x)dx$, then by putting $x = g(z)$, we get

$I = \int f[g(z)]g'(z)dz$.

Substitutions to evaluate integrals of the form $\int \frac{\phi(x)}{P\sqrt{Q}} dx$

1. If P and Q both are linear functions of x, put $Q = t^2$
2. If P is quadratic and Q is a linear function of x, put $Q = t^2$
3. If P is linear and Q is quadratic function of x, put $P = \frac{1}{t}$
4. If P and Q both are pure quadratic of the form $P = ax^2 + b$ and $Q = cx^2 + d$, then substitute $x = \frac{1}{t}$ and then $c + dt^2 = u^2$.

Integration using trigonometric identities

Sometimes when the integrand involves some trigonometric functions, we use the following identities to find the integral:

1. $2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$
2. $2\cos A \cdot \sin B = \sin(A+B) - \sin(A-B)$
3. $2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$
4. $2\sin A \cdot \sin B = \cos(A-B) - \cos(A+B)$
5. $2\sin A \cdot \cos A = \sin 2A = \frac{2\tan A}{1+\tan^2 A}$
6. $\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$

7. $\sin^2 A = \frac{1-\cos 2A}{2}$ 8. $\cos^2 A = \frac{1+\cos 2A}{2}$

9. $\sin^3 A = \frac{3\sin A - \sin 3A}{4}$

10. $\cos^3 A = \frac{3\cos A + \cos 3A}{4}$

Some Special Substitutions

$a^2 + x^2$	$x = a\tan\theta$ or $a\cot\theta$
$a^2 - x^2$	$x = a\sin\theta$ or $a\cos\theta$
$x^2 - a^2$	$x = a\sec\theta$ or $a\,\text{cosec}\,\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{x-a}{b-x}}$ or $\sqrt{(x-a)(b-x)}$	$x = a\cos^2\theta + b\sin^2\theta$
$\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$	$x = a\sin^2\theta$ or $x = a\cos^2\theta$
$\sqrt{\frac{x}{a+x}}$ or $\sqrt{\frac{a+x}{x}}$	$x = a\tan^2\theta$ or $x = a\cot^2\theta$

1.5 Integration by Partial Fractions

A rational function is a ratio of two polynomials i.e. of the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials in x and $q(x) \neq 0$. If degree of $p(x) >$ degree of $q(x)$, then we may divide $p(x)$ by $q(x)$, so that $\frac{p(x)}{q(x)} = t(x) + \frac{p_1(x)}{q(x)}$, where $t(x)$ is a polynomial in x which can be integrated easily and degree of $p_1(x)$ is less than the degree of $q(x)$. Now, $\frac{p_1(x)}{q(x)}$ can be integrated by expressing $\frac{p_1(x)}{q(x)}$ as the sum of partial fractions of the following types:

1. $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}, a \neq b$
2. $\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$
3. $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4. $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
5. $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$, where x^2+bx+c cannot be factorised further.

- Integrals of the types $\int \frac{dx}{ax^2+bx+c}$ or $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ can be transformed into standard form by expressing

$$ax^2+bx+c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$
$$= a\left[\left(x+\frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right].$$

Integrals of the types $\int (px+q)\sqrt{ax^2+bx+c}\,dx$ or $\int \frac{px+q}{ax^2+bx+c}dx$ or $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ can be transformed into standard form by expressing

$$px+q = A\frac{d}{dx}(ax^2+bx+c) + B$$
$$= A(2ax+b) + B,$$

where A and B are determined by comparing coefficients and constants of both sides.

1.6 Integration by Parts

For given functions $f(x)$ and $g(x)$, we have

$$\int [\underset{\text{I}}{f(x)} \cdot \underset{\text{II}}{g(x)}]dx = f(x)\cdot \int g(x)dx - \int \{f'(x)\cdot \int g(x)dx\}\,dx$$

Here, we can choose first function according to its position in ILATE, where

I = Inverse trigonometric function
L = Logarithmic function
A = Algebraic function
T = Trigonometric function
E = Exponential function

[The function which comes first in ILATE should taken as first function and other as second function.]

NOTE (i) Keep in mind, ILATE is not a rule, as all questions of integration by parts cannot be solve by above method.

(ii) It is worth mentioning that integration by parts is not applicable to product of functions in all cases. For instance, the method does not work for $\int \sqrt{x} \sin x \, dx$. The reason is that there does not exist any function whose derivative is $\sqrt{x} \sin x$.

Integral of the type $\int e^x [f(x) + f'(x)] \, dx$

If the given integrand is of the form $e^x [f(x) + f'(x)]$, then we can directly write the integral as

$$\int e^x [f(x) + f'(x)] dx = e^x f(x) + C.$$

[TOPIC 2] Definite Integrals

2.1 Definite Integral

A definite integral is denoted by $\int_a^b f(x)dx$, where a is called the **lower limit** of the integral and b is called the **upper limit** of the integral. The definite integral has a unique value.

Also, we define $\int_a^b f(x)dx$ as the area of the region bounded by the curve $y = f(x), a \le x \le b$, the X-axis and the ordinates $x = a$ and $x = b$.

2.2 Limit of a Sum

The integral $\int_a^b f(x)\,dx$ may be calculated by using limit of a sum as

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h[f(a) + f(a+h) + \ldots + f\{a + (n-1)h\}],$$

$$\text{where } h = \frac{b-a}{n} \to 0 \text{ as } n \to \infty$$

NOTE Definite integral is used to determine either area of a function or the limit of a sum.

2.3 Fundamental Theorem of Integral Calculus

1. **First Fundamental Theorem of Integral Calculus** Let the area function be defined by $A(x) = \int_a^x f(x)\,dx$ for all $x \ge a$, where function f is assumed to be continuous on $[a, b]$, then $A'(x) = f(x)$ for all $x \in [a,b]$.
2. **Second Fundamental Theorem of Integral Calculus** Let f be a continuous function of x defined on the closed interval $[a, b]$ and F is another function, such that $\frac{d}{dx}F(x) = f(x)$ for all x in the domain of f, then $\int_a^b f(x)dx = [F(x) + C]_a^b = F(b) - F(a)$.

   This is called definite integral of f over the range $[a, b]$.

NOTE In $\int_a^b f(x)\,dx$, the function f needs to be well-defined and continuous in $[a, b]$. For instance, the consideration of definite integral $\int_{-2}^3 x(x^2-1)^{1/2}dx$ is erroneous since the function f expressed by $f(x) = x(x^2-1)^{1/2}$ is not defined in a portion $-1 < x < 1$ of the closed interval $[-2, 3]$.

2.4 Evaluation of Definite Integrals by Substitution

To evaluate $\int_a^b f(x)dx$ by substitution, the steps could be as follows

Step I Consider the integral without limits and substitute $x = g(y)$ to reduce the given integral to a known form.

Step II Integrate the new integrand with respect to the new variable without mentioning the constant of integration.

Step III Resubstitute for the new variable and write the integral in terms of the original variable.

Step IV Find the values of integral obtained in step III at the given limits of integral and find the difference of the values at the upper and lower limits.

2.5 Properties of Definite Integral

There are following properties of definite integrals:

1. $\int_a^b f(x)dx = \int_a^b f(t)dt$
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$
5. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
6. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
7. $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$
8. $\int_{-a}^a f(x)\,dx$
$$= \begin{cases} 0, \text{ if } f(x) \text{ is an odd function, i.e. } f(-x) = -f(x) \\ 2\int_0^a f(x)\,dx, \text{ if } f(x) \text{ is an even function, i.e. } f(-x) = f(x) \end{cases}$$
9. $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

NOTE The value of the definite integral of a function over any particular interval depends on the function and the interval, but not on the variable of integration that we choose to represent the independent variable.

If the independent variable is denoted by t or u instead of x, we simply write the integral as $\int_a^b f(t)dt$ or $\int_a^b f(u)du$ instead of $\int_a^b f(x)dx$. So, the variable of integration is a dummy variable.

Some Useful Results

1. $1 + 2 + 3 + \cdots + (n-1) = \dfrac{n(n-1)}{2}$
2. $1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 = \dfrac{n(n-1)(2n-1)}{6}$
3. $1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 = \left[\dfrac{n(n-1)}{2}\right]^2$
4. Sum to n terms of a GP is
$$S_n = \begin{cases} \dfrac{a(r^n-1)}{r-1}, & \text{if } r > 1 \\ \dfrac{a(1-r^n)}{1-r}, & \text{if } r < 1 \end{cases},$$
where a = first term and r = common ratio.