Inverse Trigonometric Functions

1.1 Inverse Trigonometric Functions

Trigonometric functions are many-one functions but we know that inverse of a function exists, if function is bijective (one-one and onto). So, if we restrict the domain and range of trigonometric functions, then these functions become bijective and inverse of trigonometric functions are defined within the restricted domain and range. Inverse of f is denoted by 'f^{-1}'.

Domain and Range of Inverse Trigonometric Functions

The range of trigonometric functions become the domain of inverse trigonometric functions and restricted domain of trigonometric function becomes range or principal value branch.

Function	Domain	Range (Principal value branch)
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	R	$(0, \pi)$
$\sec^{-1} x$	$R-(-1,1)$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$
$\text{cosec}^{-1} x$	$R-(-1,1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$

NOTE

(i) $\sin^{-1} x$ should not be confused with $(\sin x)^{-1} = \frac{1}{\sin x}$ or $\sin x^{-1} = \sin\left(\frac{1}{x}\right)$. Similarly, for other inverse trigonometric functions.

(ii) When $y = f(x) = \sin x$, then $x = \sin^{-1} y$

(iii) The value of an inverse trigonometric functions which lies in its principal value branch is called the Principal Value of that inverse trigonometric functions.

(iv) Whenever no branch of an inverse trigonometric function is mentioned, it means we have to consider the principal value branch of that function.

Properties of Inverse Trigonometric Functions

1. (i) $\sin^{-1}(\sin x) = x; \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

   (ii) $\cos^{-1}(\cos x) = x; \forall x \in [0,\pi]$

   (iii) $\tan^{-1}(\tan x) = x; \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

   (iv) $\text{cosec}^{-1}(\text{cosec}\, x) = x; \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$

   (v) $\sec^{-1}(\sec x) = x; \forall\, x \in [0,\pi]-\left\{\frac{\pi}{2}\right\}$

   (vi) $\cot^{-1}(\cot x) = x; \forall\, x \in (0,\pi)$

2. (i) $\sin(\sin^{-1} x) = x; \forall\, x \in [-1,1]$

   (ii) $\cos(\cos^{-1} x) = x;\ \forall x \in [-1,1]$

   (iii) $\tan(\tan^{-1} x) = x; \forall\, x \in R$

(iv) $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x; \forall x \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec(\sec^{-1} x) = x; \forall x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot(\cot^{-1} x) = x; \forall x \in R$

3. (i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x; x \geq 1 \text{ or } x \leq -1$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x; \ x \geq 1 \text{ or } x \leq -1$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x; x > 0 \\ -\pi + \cot^{-1} x; x < 0 \end{cases}$

4. (i) $\sin^{-1}(-x) = -\sin^{-1} x; x \in [-1, 1]$

(ii) $\tan^{-1}(-x) = -\tan^{-1} x; \ x \in R$

(iii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x; |x| \geq 1$

5. (i) $\cos^{-1}(-x) = \pi - \cos^{-1} x; \ x \in [-1, 1]$

(ii) $\sec^{-1}(-x) = \pi - \sec^{-1} x; |x| \geq 1$

(iii) $\cot^{-1}(-x) = \pi - \cot^{-1} x; \ x \in R$

6. (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}; x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}; x \in R$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}; |x| \geq 1$

7. (i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right); xy < 1$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right); xy > -1$

8. (i) $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right); |x| \leq 1$

(ii) $2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right); x \geq 0$

(iii) $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right); -1 < x < 1$

Substitutions in inverse trigonometric functions to make simplest form

S.No.	Expression	Substitution
1.	$\sqrt{a^2 - x^2}$	$x = a \sin\theta$ or $x = a \cos\theta$
2.	$\sqrt{a^2 + x^2}$	$x = a \tan\theta$ or $x = a \cot\theta$
3.	$\sqrt{x^2 - a^2}$	$x = a \sec\theta$ or $x = a \operatorname{cosec}\theta$
4.	$\sqrt{a+x}$ or $\sqrt{a-x}$	$x = a \cos\theta$ or $x = a \cos 2\theta$
5.	$\sqrt{1+x^2} \pm \sqrt{1-x^2}, \sqrt{\frac{1+x^2}{1-x^2}}, \sqrt{\frac{1-x^2}{1+x^2}}$	$x^2 = \cos 2\theta$
6.	$\sqrt{a^2+x^2} \pm \sqrt{a^2-x^2}, \sqrt{\frac{a^2+x^2}{a^2-x^2}}, \sqrt{\frac{a^2-x^2}{a^2+x^2}}$	$x^2 = a^2 \cos 2\theta$
7.	$\sqrt{1+x} \pm \sqrt{1-x}, \sqrt{\frac{1-x}{1+x}}, \sqrt{\frac{1+x}{1-x}}$	$x = \cos 2\theta$
8.	$\sqrt{a+x} \pm \sqrt{a-x}, \sqrt{\frac{a+x}{a-x}}, \sqrt{\frac{a-x}{a+x}}$	$x = a \cos 2\theta$

Points to Remember

(i) Sometimes, it may happen that some of the finding values of unknown does not satisfy the given equation.

(ii) While solving an equation, do not cancel the common factors from both sides, otherwise we may lose some values of unknown.