Matrices

[TOPIC 1] Matrix and Operations on Matrices

1.1 Matrix

A matrix is an ordered **rectangular array** of numbers or functions. The numbers (real or complex) or functions are called the elements or the entries of the matrix. It is denoted by the symbol [ ] or ( ).

Order of a Matrix

If a matrix has m rows and n columns, then its order is written as $m \times n$. If a matrix has order $m \times n$, then it has mn elements.
In general, $m \times n$ matrix has the following rectangular array:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

or $[a_{ij}]_{m \times n}$, where $1 \le i \le m, 1 \le j \le n; i, j \in N$.

NOTE We shall consider only those matrices, whose elements are real numbers or functions taking real values.

Types of Matrices

1. **Column Matrix** A matrix which has only one column, is called a column matrix.
   e.g. $\begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$
   In general, $A = [a_{ij}]_{m \times 1}$ is a column matrix of order $m \times 1$.
2. **Row Matrix** A matrix which has only one row, is called a row matrix.
   e.g. $[1 \;\; 5 \;\; 9]$
   In general, $A = [a_{ij}]_{1 \times n}$ is a row matrix of order $1 \times n$.
3. **Zero or Null Matrix** A matrix is said to be a zero or null matrix, if its all elements are zero.
   e.g. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We denote zero matrix by O.
4. **Square Matrix** A matrix which has equal number of rows and columns, is called a square matrix.
   e.g. $\begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix}$
   In general, $A = [a_{ij}]_{m \times m}$ is a square matrix of order m.

NOTE If $A = [a_{ij}]$ is a square matrix of order n, then elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are said to constitute the diagonal of the matrix A. We call these elements as diagonal elements.

5. **Diagonal Matrix** A square matrix whose all the elements except the diagonal elements are zero, is called a diagonal matrix.
   e.g. $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -8 \end{bmatrix}$ is a diagonal matrix and it can be written as $A = \text{diag}(3, -3, -8)$
   In general, $A = [a_{ij}]_{m \times m}$ is a diagonal matrix, if $a_{ij} = 0$, when $i \ne j$.

6. **Scalar Matrix** A diagonal matrix whose all diagonal elements are equal (non-zero), is called a scalar matrix. e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

In general, $A = [a_{ij}]_{n \times n}$ is a scalar matrix, if $a_{ij} = 0$, when $i \neq j$ and $a_{ij} = k$ (constant), when $i = j$.

NOTE A scalar matrix is a diagonal matrix but a diagonal matrix may or may not be a scalar matrix.

7. **Unit or Identity Matrix** A diagonal matrix in which all diagonal elements are '1', is called an identity matrix. It is denoted by I.

e.g. $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

In general, $A = [a_{ij}]_{n \times n}$ is an identity matrix, if $a_{ij} = 1$, when $i = j$ and $a_{ij} = 0$, when $i \neq j$.

Equality of Matrices

Two matrices A and B are said to be equal, if

(*i*) order of A and B are same.

(*ii*) corresponding elements of A and B are same i.e. $a_{ij} = b_{ij}$, $\forall i$ and j.

Symbolically, if two matrices A and B are equal, then we write $A = B$

e.g. $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ are equal matrices, but $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are not equal matrices, because all corresponding elements are not equal.

1.2 Operations on Matrices

Between two matrices, following operations can be applied as discussed below:

1. **Addition and Subtraction of Matrices** Addition and subtraction of two matrices is defined, if order of both the matrices are same.

Addition of Matrices

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then $A + B = [a_{ij} + b_{ij}]_{m \times n}, 1 \leq i \leq m, 1 \leq j \leq n$; $i, j \in N$.

Subtraction of Matrices

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then $A - B = [a_{ij} - b_{ij}]_{m \times n}, 1 \leq i \leq m, 1 \leq j \leq n$, $i, j \in N$.

Properties of Addition of Matrices

(*i*) **Commutative law** If $A = [a_{ij}]$ and $B = [b_{ij}]$ are any two matrices of the same order say $m \times n$, then $A + B = B + A$.

(*ii*) **Associative law** For any three matrices $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ of the same order say $m \times n$, $A + (B + C) = (A + B) + C$.

(*iii*) **Existence of additive identity** Let $A = [a_{ij}]$ be a $m \times n$ matrix and O be a $m \times n$ zero matrix, then $A + O = O + A = A$.

In other words, O is the additive identity for matrix addition.

(*iv*) **Existence of additive inverse** Let $A = [a_{ij}]_{m \times n}$ be any matrix, then we have another matrix as $-A = [-a_{ij}]_{m \times n}$ such that $A + (-A) = (-A) + A = O$. So, matrix $(-A)$ is called additive inverse of A or negative of A.

NOTE (i) If A and B are not of the same order, then $A + B$ is not defined.

(ii) Addition of matrices is an example of binary operation on the set of matrices of the same order.

2. **Multiplication of a matrix by scalar number** Let $A = [a_{ij}]_{m \times n}$ be a matrix and k is scalar, then kA is another matrix obtained by multiplying each element of A by the scalar k, i.e. if $A = [a_{ij}]_{m \times n}$, then $kA = [k(a_{ij})]_{m \times n}$.

e.g. $k\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2 \times 2} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}_{2 \times 2}$

Properties of Scalar Multiplication of a Matrix

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of the same order say $m \times n$, then

(*i*) $k(A + B) = kA + kB$, where k is a scalar.

(*ii*) $(k + l)A = kA + lA$, where k and l are scalars.

3. **Multiplication of Matrices** Let A and B be two matrices. Then, their product AB is defined, if the number of columns in matrix A is equal to the number of rows in matrix B.

[TOPIC 2] Transpose of a Matrix, Symmetric and Skew-Symmetric Matrices

2.1 Transpose of a Matrix

Let A be any matrix. Then, the matrix obtained by interchanging its rows and columns, is called the transpose of matrix A. e.g. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 3 & 5 \end{bmatrix}$, then

the transpose of $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 5 \end{bmatrix}$.

Transpose of A is written as A' or A^T. Also, if matrix $A = [a_{ij}]_{m \times n}$, then its transpose is $A^T = [a_{ji}]_{n \times m}$.

Properties of Transpose of a Matrices

Let A and B be any two matrices. Then, we have

- $(A')' = A$
- $(A \pm B)' = A' \pm B'$
- $(AB)' = B'A'$ (Reversal law)
- $(kA)' = k \cdot A'$, where k is any constant.
- $(-A)' = -A'$

NOTE $(A^n)^T = (A^T)^n$, where n is a positive integer.

2.2 Symmetric Matrix

A square matrix $A = [a_{ij}]$ is said to be a symmetric matrix, if $A' = A$, i.e. if $[a_{ij}] = [a_{ji}]$, for all possible value of i and j.

e.g. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ is a symmetric matrix as

$$A' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = A.$$

Skew-symmetric Matrix

A square matrix $A = [a_{ij}]$ is said to be a skew-symmetric matrix, if $A' = -A$, i.e. if $[a_{ji}] = -[a_{ij}]$, for all possible value of i and j.

e.g. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a skew-symmetric matrix as

$$A' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A.$$

NOTE (i) Diagonal elements of a skew-symmetric matrix are always zero.

(ii) Only zero square matrix is both symmetric and skew-symmetric matrices.

Some Important Theorems

(*i*) For every square matrix A, $A + A'$ is a symmetric matrix and $A - A'$ is a skew-symmetric matrix.

(*ii*) Any square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrices. i.e. Let A be a square matrix, then it can be written as $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$.

[TOPIC 3] Inverse of a Matrix by Elementary Operations

3.1 Elementary Operations (or Transformations) of a Matrix

There are six operations (or transformations) on a matrix, three of which are due to rows and three due to columns, which are known as elementary operations or transformations.

(i) *Interchange of any two rows or two columns*

The interchange of ith and jth rows or ith and jth columns are denoted by

$$R_i \leftrightarrow R_j \quad \text{or} \quad C_i \leftrightarrow C_j, \text{ respectively.}$$

(ii) *Multiplication of the elements of any row or column by a non-zero number*

The multiplication of each element of the ith row or ith column by k, where $k \neq 0$, is denoted by

$$R_i \to kR_i \quad \text{or} \quad C_i \to kC_i.$$

(iii) *The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non-zero number*

The addition to the elements of ith row or column, the corresponding elements of jth row or column multiplied by k, where $k \neq 0$ is denoted by

$$R_i \to R_i + kR_j \quad \text{or} \quad C_i \to C_i + kC_j.$$

3.2 Inverse of a Matrix

If A and B are two square matrices of same order such that $AB = BA = I$, then B is called the inverse matrix of A and is denoted by A^{-1}, i.e. $B = A^{-1}$. Here, A is said to be invertible. Inverse of a matrix can be obtained by using elementary row transformations or column transformations.

NOTE (i) Inverse of a square matrix, if exists, is unique.

(ii) A rectangular matrix does not possess inverse matrix.

(iii) If B is the inverse of A, then A is also the inverse of B.

(iv) $(AB)^{-1} = B^{-1}A^{-1}$

Steps to Find Inverse of a Matrix by Elementary Transformations

***Step* I** Suppose the given matrix is A. Then, first write $A = IA$ (for applying row operations only) or $A = AI$ (for applying column operations only), where I is the identity matrix of same order as that of

***Step* II** Now, apply various elementary transformations (either row operations or column operations, not both), on LHS and apply the same operations on I of RHS, so that LHS reduces to I.

***Step* III** From Step II, we get new matrix equation $I = BA$ (if row operations are applied) or $I = AB$ (if column operations are applied). Then, matrix B will be the inverse of A.

Hence, A^{-1} is obtained.

NOTE If after applying one or more elementary row (or column) operations on $A = IA (A = AI)$, we obtain all zeros in one or more rows (or columns) of the matrix A on LHS, then we say that A^{-1} does not exist.