Motion in a Plane

(Projectile and Circular Motion)

Motion in a Plane

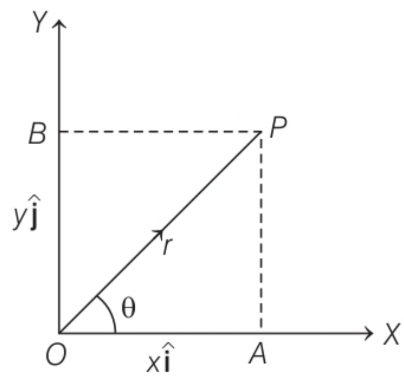
Motion in plane is called as motion in two dimensions, *e.g.* projectile motion, circular motion. For the analysis of such motion our reference will be made of an origin and two co-ordinate axes X and Y.

Terms Related to Motion in Plane

Few terms related to motion in plane are given below

1. **Position Vector**

A vector that extends from a reference point to the point at which particle is located is called position vector.



Position vector is given by $\mathbf{r} = \hat{x}\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$

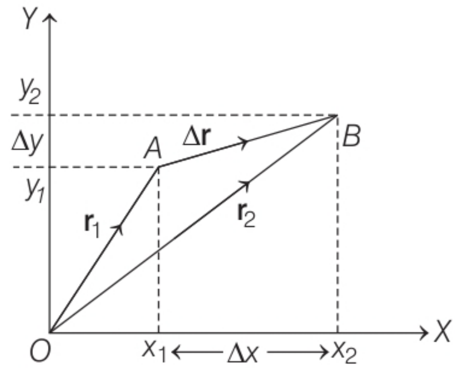
Direction of this position vector $\mathbf{r}$ is given by the angle θ with X-axis, where, $\tan\theta = \dfrac{y}{x}$

In three dimensions, the position vector is represented as

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

2. **Displacement Vector**

The displacement vector is a vector which gives the position of a point with reference to a point other than the origin of the co-ordinate system.



Magnitude of displacement vector

$$|\Delta \mathbf{r}| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Direction of the displacement vector $\Delta \mathbf{r}$ is given by

$$\tan \theta = \frac{\Delta y}{\Delta x}$$

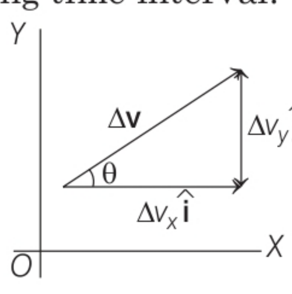
In three dimensions, the displacement can be represented as

$$\Delta \mathbf{r} = (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} + (z_2 - z_1)\hat{\mathbf{k}}$$

3. **Velocity Vector**

Velocity of an object in motion is defined as the ratio of displacement and the corresponding time interval taken by the object.

(i) **Average Velocity** It is defined as the ratio of the displace- ment and the corresponding time interval.



Average velocity, $v_{av} = \dfrac{\Delta r}{\Delta t} = \dfrac{r_2 - r_1}{t_2 - t_1}$

Average velocity can be expressed in the component forms as

$$v_{av} = \frac{\Delta x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta y}{\Delta t}\hat{\mathbf{j}} = \Delta v_x \hat{\mathbf{i}} + \Delta v_y \hat{\mathbf{j}}$$

The magnitude of v_{av} is given by

$$\tan\theta = \frac{\Delta v_y}{\Delta v_x}$$

(ii) **Instantaneous Velocity** The velocity at an instant of time (t) is known as instantaneous velocity.

Instantaneous velocity, $v = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \frac{dr}{dt}$

$$v = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}}$$

$$v = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$$

Magnitude of instantaneous velocity

$$|v| = \sqrt{v_x^2 + v_y^2}$$

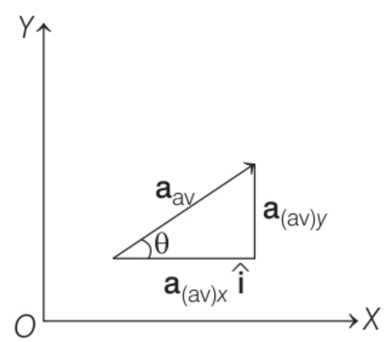
Direction of v is given by

$$\tan\theta = \left(\frac{v_y}{v_x}\right)$$

4. Acceleration Vector

It is defined as the rate of change of velocity.

(i) **Average Acceleration** It is defined as the change in velocity (Δv) divided by the corresponding time interval (Δt).



Average acceleration, $a_{av} = \frac{\Delta v}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta v_y}{\Delta t}\hat{\mathbf{j}}$

$$a_{av} = a_{(av)\,x}\,\hat{\mathbf{i}} + a_{(av)\,y}\,\hat{\mathbf{j}}$$

Magnitude of average acceleration is given by

$$|a_{av}| = \sqrt{(a_{(av)\,x})^2 + (a_{(av)\,y})^2}$$

Angle θ made by average acceleration with X-axis is

$$\tan\theta = \frac{a_y}{a_x}$$

(ii) **Instantaneous Acceleration** It is defined as the limiting value of the average acceleration as the time interval approaches to zero.

Instantaneous acceleration, $a = \lim_{\Delta t \to 0} \frac{dv}{dt}$

$$a = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}$$

If acceleration a makes an angle θ with X-axis

then $$\tan\theta = \left(\frac{a_y}{a_x}\right)$$

Motion in Plane with Uniform Acceleration

A body is said to be moving with uniform acceleration, if its velocity vector suffers the same change in the same interval of time however small.

According to definition of average acceleration, we have

$$a = \frac{v - v_0}{t - 0} = \frac{v - v_0}{t}$$

$$v = v_0 + at$$

In terms of rectangular component, we can express it as

$$v_x = v_{0x} + a_x t$$

and $$v_y = v_{0y} + a_y t$$

Path of Particle Under Constant Acceleration

Now, we can also find the position vector ($\mathbf{r}$). Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of the particle at time $t = 0$ and $t = t$ and their velocities at these instants be $\mathbf{v}_0$ and $\mathbf{v}$ respectively. Then, the average velocity is given by

$$\mathbf{v}_{\text{av}} = \frac{\mathbf{v}_0 + \mathbf{v}}{2}$$

Displacement is the product of average velocity and time interval.

It is expressed as

$$\mathbf{r} - \mathbf{r}_0 = \left(\frac{\mathbf{v} + \mathbf{v}_0}{2}\right)t = \left[\frac{(\mathbf{v}_0 + \mathbf{a}t) + \mathbf{v}_0}{2}\right]t$$

$$\Rightarrow \quad \mathbf{r} - \mathbf{r}_0 = \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$$

$$\Rightarrow \quad \mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$$

In terms of rectangular components, we have

$$x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = x_0\hat{\mathbf{i}} + y_0\hat{\mathbf{j}} + (v_{0x}\hat{\mathbf{i}} + v_{0y}\hat{\mathbf{j}})\,t + \frac{1}{2}(a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}})\,t^2$$

Now, equating the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$,

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \text{ .... along } x\text{-axis}$$

and

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \text{ .... along } y\text{-axis}$$

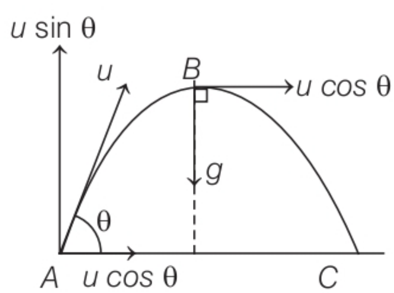
Note Motion in a plane (two-dimensional motion) can be treated as two separate simultaneous one-dimensional motions with constant acceleration along two perpendicular directions.

Projectile Motion and Circular Motion

Projectile Motion

When any object is thrown from horizontal at an angle θ except 90°, then it moves on a parabolic known as its **trajectory,** the object is called **projectile** and its motion is called **projectile motion.**

If any object is thrown with velocity u, making an angle θ, from horizontal, then



Horizontal component of initial velocity $= u \cos\theta$.

Vertical component of initial velocity $= u \sin\theta$.

Horizontal component of velocity ($u \cos\theta$) remains same during the whole journey as no force is acting horizontally.

Vertical component of velocity ($u \sin\theta$) decreases gradually and becomes zero at highest point of the path.

At highest point, the velocity of the body is $u \cos\theta$ in horizontal direction and the angle between the velocity and acceleration is 90°.

Time of flight It is defined as the total time for which the projectile remains in air.

$$T = \frac{2u \sin\theta}{g}$$

Maximum height It is defined as the maximum vertical height covered by projectile.

$$H = \frac{u^2 \sin^2 \theta}{2g}$$

Horizontal range It is defined as the maximum distance covered in horizontal distance.

$$R = \frac{u^2 \sin 2\theta}{g}$$

Important Points and Formulae of Projectile Motion

(i) At highest point, the linear momentum is $mu \cos \theta$ and the kinetic energy is $\frac{1}{2} m(u \cos \theta)^2$.

(ii) The horizontal displacement of the projectile after t seconds,

$$x = (u \cos \theta) t$$

(iii) The vertical displacement of the projectile after t seconds,

$$y = (u \sin \theta) t - \frac{1}{2} gt^2$$

(iv) Equation of the path of projectile,

$$y = x \tan \theta - \frac{g}{2u^2 \cos^2 \theta} x^2$$

(v) The path of a projectile is parabolic.

(vi) Velocity of the projectile at any instant t,

$$|\mathbf{v}| = \sqrt{u^2 + g^2t^2 - 2ugt \sin \theta}$$

(vii) Kinetic energy at the lowest point $= \frac{1}{2} mu^2$

(viii) Linear momentum at lowest point $= mu$

(ix) Acceleration of projectile is constant throughout the motion and it acts vertically downwards being equal to g.

(x) Angular momentum of projectile $= mu \cos \theta \times h$, where h denotes the height.

(xi) In case of angular projection, the angle between velocity and acceleration varies from $0° < \theta < 180°$.

(xii) The projectile attains maximum height when it covers a horizontal distance equal to half of the horizontal range, *i.e.* $R/2$.

(xiii) When the maximum range of projectile is R, then its maximum height is $R/4$.

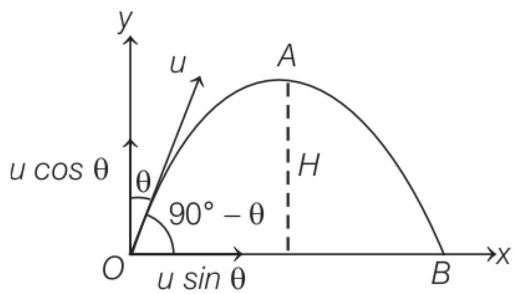
(i) Horizontal range is maximum when it is thrown at an angle of 45° from the horizontal

$$R_{max} = \frac{u^2}{g}$$

(ii) For angle of projection θ and (90° – θ), the horizontal range is same.

Projectile Projected at an Angle θ with the Vertical

Let a particle be projected vertically with an angle θ with vertical and speed of projection is u.



$$\text{Time of flight, } T = \frac{2u\sin(90° - \theta)}{g} = \frac{2u\cos\theta}{g}$$

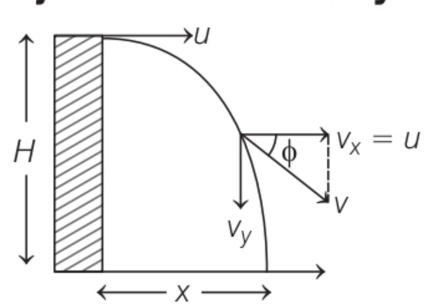
$$\text{Maximum height, } H = \frac{u^2\sin^2(90° - \theta)}{2g} = \frac{u^2\cos^2\theta}{2g}$$

$$\text{Horizontal range, } R = \frac{u^2\sin(180° - 2\theta)}{g} = \frac{u^2\sin 2\theta}{g}$$

Equation of path of projectile, $y = x\cot\theta - \dfrac{gx^2}{2u^2\sin^2\theta}$

Projectile Projected from Some Height

1. When Projectile Projected Horizontally



Initial velocity in vertical direction = 0

Time of flight, $T = \sqrt{\frac{2H}{g}}$

Horizontal range, $x = uT = u\sqrt{\frac{2H}{g}}$

Vertical velocity after t seconds,

$$v_y = gt \qquad (\because u_y = 0)$$

Velocity of projectile after t seconds,

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + (gt)^2}$$

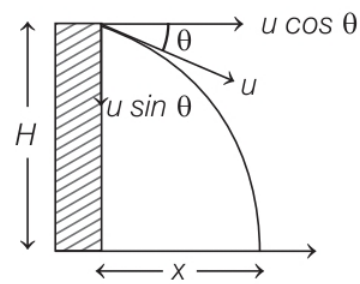
If velocity makes an angle ϕ from horizontal, then

$$\tan\phi = \frac{v_y}{v_x} = \frac{gt}{u}$$

Equation of the path of the projectile,

$$y = \frac{g}{2u^2}x^2$$

2. **When Projectile Projected Downward at an Angle θ with Horizontal**



Initial velocity in horizontal direction $= u\cos\theta$

Initial velocity in vertical direction $= -u\sin\theta$

$$\text{Time of flight, } T = -\frac{2u\sin\theta}{2g} \pm \frac{\sqrt{4u^2\sin^2\theta + 8gh}}{2g}$$

Horizontal range, $x = (u\cos\theta)T$

Vertical velocity after t seconds,

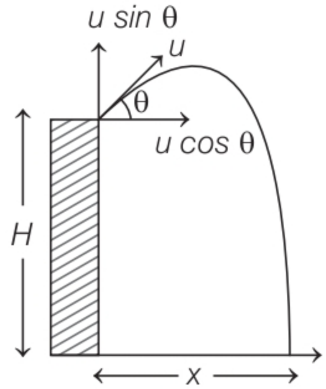
$$v_y = u\sin\theta + gt$$

Velocity of projectile after t seconds,

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(u\cos\theta)^2 + (u\sin\theta + gt)^2}$$

$$= \sqrt{u^2 + (gt)^2 + 2ugt\sin\theta}$$

3. When Projectile Projected Upward at an Angle θ with Horizontal



Initial velocity in horizontal direction $= u \cos\theta$

Initial velocity in vertical direction $= u \sin\theta$

$$\text{Time of flight, } T = \frac{u \sin\theta}{g} \pm \sqrt{\frac{u^2 \sin^2\theta}{g^2} + \frac{2h}{g}}$$

Horizontal range, $x = (u \cos\theta) T$

Vertical velocity after t seconds, $v_y = (-u \sin\theta) + gt$

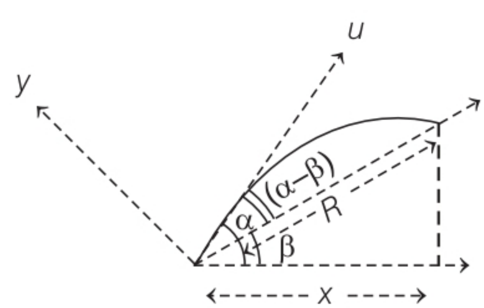
Velocity of projectile after t seconds,

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + (gt - u \sin\theta)^2}$$

$$= \sqrt{u^2 + (gt)^2 - 2ugt \sin\theta}$$

Projectile Motion on an Inclined Plane

When any object is thrown with velocity u making an angle α from horizontal, at a plane inclined at an angle β from horizontal, then



Initial velocity along the inclined plane $= u \cos(\alpha - \beta)$

Initial velocity perpendicular to the inclined plane $= u \sin(\alpha - \beta)$

Acceleration along the inclined plane $= g \sin\beta$

Acceleration perpendicular to the inclined plane $= g \cos\beta$

Time of flight, $T = \dfrac{2u \sin(\alpha - \beta)}{g \cos \beta}$

Maximum height, $H = \dfrac{u^2 \sin^2(\alpha - \beta)}{2g \cos \beta}$

Horizontal range, $x = \dfrac{2u^2 \sin(\alpha - \beta) \cos \alpha}{g \cos \beta}$

Range on inclined plane,

$$R = \frac{x}{\cos \beta} = \frac{2u^2 \sin(\alpha - \beta) \cos \alpha}{g \cos^2 \beta}$$

Range on inclined plane will be maximum, when

$$\alpha = 45^\circ + \frac{\beta}{2}$$

$$R_{\max} = \frac{u^2}{g(1 + \sin \beta)}$$

For angle of projections α and $(90^\circ - \alpha + \beta)$, the range on inclined plane are same.

If the projectile is thrown downwards, then maximum range is

$$R_{\max} = \frac{u^2}{g(1 - \sin \beta)}$$