DCAM classes

Dynamic Classes for Academic Mastery

Relations and Functions

1.1 Relation

A relation R from a set X to a set Y is defined as a subset of the cartesian product $X \times Y$, i.e. $R \subseteq X \times Y$.

Domain and Range of a Relation

The set of first elements of all ordered pairs in R, i.e. $\{x : (x, y) \in R\}$ is called the domain of relation R and the set of second elements of all ordered pairs in R, i.e. $\{y : (x, y) \in R\}$ is called the range of relation R.

NOTE If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$ and number of relations from set A to set $B = 2^{pq}$.

Types of Relation

1. **Empty** (or Void) **Relation** A relation R in a set X is called an empty relation, if no element of X is related to any element of X,
   i.e. $R = \phi \subset X \times X$.
2. **Universal Relation** A relation R in a set X is called universal relation, if each element of X is related to every element of X,
   i.e. $R = X \times X$.
3. **Reflexive Relation** A relation R defined on a set A is said to be reflexive, if
   $(x, x) \in R, \forall x \in A$
   or $xRx, \forall x \in A$.
4. **Symmetric Relation** A relation R defined on a set A is said to be symmetric, if
   $(x, y) \in R$
   $\Rightarrow (y, x) \in R, \forall x, y \in A$
   or $xRy \Rightarrow yRx, \forall x, y \in A$.
5. **Transitive Relation** A relation R defined on a set A is said to be transitive, if $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R, \forall x, y, z \in A$
   or xRy and yRz
   $\Rightarrow xRz, \forall x, y, z \in A$.
6. **Equivalence Relation** A relation R defined on a set A is said to be an equivalence relation, if R is reflexive, symmetric and transitive.
7. **Equivalence Classes** Let R be an equivalence relation in a set A and let $a \in A$. Then, the set of all those elements of A which are related to a under the relation R, is called the equivalence class determined by a and it is denoted by $[a]$.
   So, $[a] = \{b \in A : aRb\}$

NOTE (i) Two equivalence classes are either disjoint or identical.
(ii) The union of all equivalence classes gives the whole set.
(iii) Identity relation is always reflexive, symmetric and transitive.

1.2 Function

Let X and Y be two non-empty sets. A function or mapping f from X into Y written as $f : X \to Y$ is a rule by which each element $x \in X$ is associated to a unique element $y \in Y$.

Domain, codomain and Range of Function

The elements of X are called the **domain** of f and the elements of Y are called the **codomain** of f.
The images of the elements of X is called the range of f which is a subset of Y.

NOTE Every function is a relation but every relation is not a function.

Types of Function

1. **One-one** (or Injective) **and Many-one Function** A function $f: X \to Y$ is said to be a one-one function, if the images of distinct elements of X under f are distinct. Thus, f is one-one iff $f(x_1) = f(x_2)$
$\Rightarrow x_1 = x_2$ for all $x_1, x_2 \in X$
or f is one-one iff $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
for all $x_1, x_2 \in X$.
A function which is not one-one, is known as many-one function.
2. **Onto** (or Surjective) **and Into Function** A function $f: X \to Y$ is said to be an onto function, if every element of Y is image of some element of set X under f, i.e. for every $y \in Y$, there exists an element x in X such that $f(x) = y$.
In other words, a function is called an onto function, if its range is equal to codomain.
A function $f: X \to Y$ is said to be into function, if there exists atleast one element in Y, which do not have any pre-image in X.
3. **Bijective Function** A function $f: X \to Y$ is said to be a bijective function, if it is both one-one and onto.

Composition of Functions

Let $f: X \to Y$ and $g: Y \to Z$ be two functions. Then, composition of functions f and g is a function from X to Z, denoted by gof is defined as function $gof: X \to Z$ and given by
$gof(x) = g(f(x)), \forall x \in X$.

Similarly, if $f: X \to Y$, $g: Y \to Z$ and $h: Z \to S$ are three functions, then composition of f, g and h is defined as the function $hogof: X \to S$ given by

$$hogof(x) = ho(gof)(x) = h(gof)(x) = h(g(f(x)))$$

NOTE (i) fog may or may not be equal to gof.
(ii) If f and g are onto, then gof is also onto.
(iii) If f and g are one-one, then gof is also one-one.
(iv) In general, gof is one-one implies that f is one-one and gof is onto implies that f is onto.
(v) $ho(gof)$ is always same as $(hog)of$, i.e. $ho(gof)(x) = (hog)of(x), \forall x$ in X

Invertible Function

A function $f: X \to Y$ is said to be invertible, if there exists a function $g: Y \to X$ such that $gof = I_X$ and $fog = I_Y$. The function g is called inverse of function f and is denoted by f^{-1}.

NOTE (i) To prove a function is invertible, we need to prove that, it is both one-one and onto, i.e. bijective.
(ii) The inverse of a bijective function is also a bijective function.
(iii) If f is an invertible function, then $(f^{-1})^{-1} = f$.
(iv) If $f: X \to Y$ and $g: Y \to Z$ are two invertible functions, then gof is also invertible with $(gof)^{-1} = f^{-1}og^{-1}$.

Domain and Range of Some Useful Functions

S.No.	Function	Domain	Range
1.	Polynomial function	R	R (if degree is odd) and subset of R (if degree is even)
2.	Rational function	All real values except for which denominator = 0.	Depends on particular rational function.
3.	Exponential function, $a^x, a > 0$	R	$(0, \infty)$
4.	Logarithmic function, $\log_a x, a > 0$ and $a \neq 1$	$(0, \infty)$	R
5.	Identity function, $y = x$	R	R
6.	Modulus function, $\lvert x \rvert$	R	$[0, \infty)$
7.	Signum function, $\begin{cases} \frac{\lvert x \rvert}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$	R	$\{-1, 0, 1\}$
8.	Greatest integer function, $[x]$	R	Set of integers (I)