DCAM classes

Dynamic Classes for Academic Mastery

# Straight Line

A straight line is the locus of all those points which are collinear with two given points.

General equation of a line is $ax + by + c = 0$

**Note**

- We can have one and only one line through a fixed point in a given direction.
- We can have infinitely many lines through a given point.

## Slope (Gradient) of a Line

The trigonometrical tangent of the angle that a line makes with the positive direction of the $X$-axis in anti-clockwise sense is called the slope or gradient of the line.

So, slope of a line, $\boldsymbol{m = \tan\theta}$

where, $\theta$ is the angle made by the line with positive direction of $X$-axis.

**Important Results on Slope of Line**

(i) Slope of a line parallel to $X$-axis, $m = 0$.

(ii) Slope of a line parallel to $Y$-axis, $m = \infty$.

(iii) Slope of a line equally inclined with axes is 1 or $-1$ as it makes an angle of $45^\circ$ or $135^\circ$, with $X$-axis.

(iv) Slope of a line passing through $(x_1, y_1)$ and $(x_2, y_2)$ is given by

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}.$$

## Angle between Two Lines

The angle $\theta$ between two lines having slopes $m_1$ and $m_2$, is

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|.$$

(i) Two lines are parallel, iff $m_1 = m_2$.

(ii) Two lines are perpendicular to each other, iff $m_1 m_2 = -1$.

## Equation of a Straight Line

General equation of a straight line is $Ax + By + C = 0$.

(i) The equation of a line parallel to $X$-axis at a distance $b$ from it, is given by

$$y = b$$

(ii) The equation of a line parallel to $Y$-axis at a distance $a$ from it, is given by

$$x = a$$

(iii) Equation of $X$-axis is

$$y = 0$$

(iv) Equation of $Y$-axis is

$$x = 0$$

### Different Forms of the Equation of a Straight Line

(i) **Slope Intercept Form** The equation of a line with slope $m$ and making an intercept $c$ on $Y$-axis, is

$$y = mx + c$$

If the line passes through the origin, then its equation will be

$$y = mx$$

(ii) **One Point Slope Form** The equation of a line which passes through the point $(x_1, y_1)$ and has the slope $m$ is given by

$$(y - y_1) = m(x - x_1)$$

(iii) **Two Points Form** The equation of a line passing through the points $(x_1, y_1)$ and $(x_2, y_2)$ is given by

$$(y - y_1) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

This equation can also be determined by the determinant method, that is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(iv) **Intercept Form** The equation of a line which cuts off intercept $a$ and $b$ respectively on the $X$ and $Y$-axes is given by

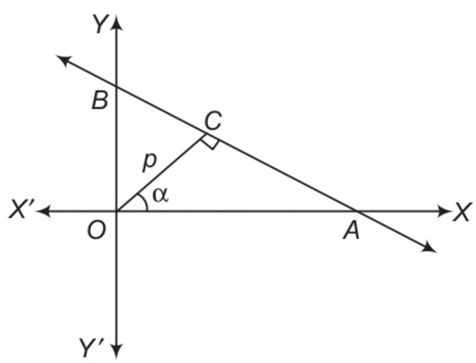
$$\frac{x}{a} + \frac{y}{b} = 1$$

The general equation $Ax + By + C = 0$ can be converted into the intercept form, as

$$\frac{x}{-(C/A)} + \frac{y}{-(C/B)} = 1$$

(v) **Normal Form** The equation of a straight line upon which the length of the perpendicular from the origin is $p$ and angle made by this perpendicular to the $X$-axis is $\alpha$, is given by

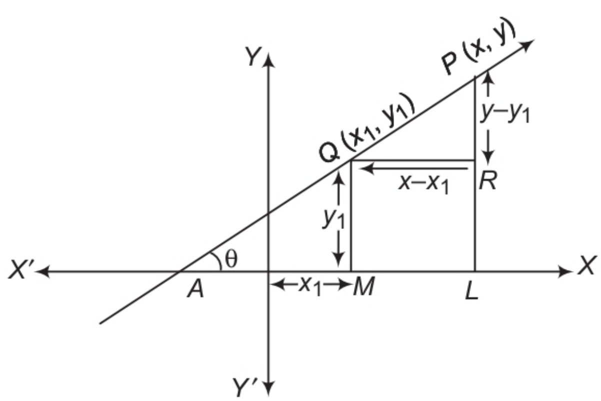
$$x\cos\alpha + y\sin\alpha = p$$




(vi) **Distance** (Parametric) **Form** The equation of a straight line passing through $(x_1, y_1)$ and making an angle $\theta$ with the positive direction of $X$-axis, is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

where, $r$ is the distance between two points $P(x, y)$ and $Q(x_1, y_1)$.



Thus, the coordinates of any point on the line at a distance $r$ from the given point $(x_1, y_1)$ are $(x_1 + r\cos\theta, y_1 + r\sin\theta)$. If $P$ is on the right side of $(x_1, y_1)$, then $r$ is positive and if $P$ is on the left side of $(x_1, y_1)$, then $r$ is negative.

## Position of Point(s) Relative to a Given Line

Let the equation of the given line be $ax + by + c = 0$ and let the coordinates of the two given points be $P(x_1, y_1)$ and $Q(x_2, y_2)$.

(i) The two points are on the same side of the straight line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same sign.

(ii) The two points are on the opposite side of the straight line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite sign.

(iii) A point $(x_1, y_1)$ will lie on the side of the origin relative to a line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and $c$ have the same sign.

(iv) A point $(x_1, y_1)$ will lie on the opposite side of the origin relative to a line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and $c$ have the opposite sign.

### Condition of Concurrency

Condition of concurrency for three given lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ is

$$a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - a_1c_2) + c_3(a_1b_2 - a_2b_1) = 0$$

or

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

## Distance of a Point from a Line

The distance of a point from a line is the **length of perpendicular** drawn from the point to the line. Let $L: Ax + By + C = 0$ be a line, whose perpendicular distance from the point $P(x_1, y_1)$ is $d$. Then,

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

**Note** The distance of origin from the line $Ax + By + C = 0$ is

$$d = \frac{|C|}{\sqrt{A^2 + B^2}}$$

## Distance between Two parallel Lines

The distance between two parallel lines

$$y = m\,x + c_1 \qquad \ldots\text{(i)}$$

$$y = m\,x + c_2 \qquad \ldots\text{(ii)}$$

is given by $$d = \frac{|c_1 - c_2|}{\sqrt{1 + m^2}}$$

## Point of Intersection of Two Lines

Let equation of lines be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then their point of intersection is $\left(\dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$.

## Line Parallel and Perpendicular to a Given Line

(i) The equation of a line parallel to a given line $ax + by + c = 0$ is $ax + by + \lambda = 0$, where $\lambda$ is a constant.

(ii) The equation of a line perpendicular to a given line $ax + by + c = 0$ is $bx - ay + \lambda = 0$, where $\lambda$ is a constant.

## Image of a Point with Respect to a Line

Let the image of a point $(x_1, y_1)$ with respect to $ax + by + c = 0$ be $(x_2, y_2)$, then

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{-2\,(ax_1 + by_1 + c)}{a^2 + b^2}$$

(i) The image of the point $P(x_1, y_1)$ with respect to $X$-axis is $Q(x_1, -y_1)$.

(ii) The image of the point $P(x_1, y_1)$ with respect to $Y$-axis is $Q(-x_1, y_1)$.

(iii) The image of the point $P(x_1, y_1)$ with respect to mirror $y = x$ is $Q(y_1, x_1)$.

(iv) The image of the point $P(x_1, y_1)$ with respect to the line mirror $y = x\tan\theta$ is

$$x = x_1\cos 2\theta + y_1\sin 2\theta$$

$$y = x_1\sin 2\theta - y_1\cos 2\theta$$

(v) The image of the point $P(x_1, y_1)$ with respect to the origin is the point $(-x_1, -y_1)$.

## Equation of the Bisectors

The equation of the bisectors of the angle between the lines

$$a_1x + b_1y + c_1 = 0$$

and

$$a_2x + b_2y + c_2 = 0$$

are given by

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

To find acute and obtuse angle bisectors, first make constant terms in the equations of given straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ positive, if it is required, then find $a_1a_2 + b_1b_2$.

(i) If $a_1a_2 + b_1b_2 > 0$, then we take positive sign for obtuse and negative sign for acute.

(ii) If $a_1a_2 + b_1b_2 < 0$, then we take negative sign for obtuse and positive sign for acute.

## Pair of Lines

General equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

It will represent a pair of straight line iff

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

or

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

## Homogeneous Equation of Second Degree

An equation in two variables $x$ and $y$ (whose RHS is zero) is said to be a homogeneous equation of the second degree, if the sum of the indices of $x$ and $y$ in each term is equal to 2. The general form of homogeneous equation of the second degree in $x$ and $y$ is $ax^2 + 2hxy + by^2 = 0$.

**Note** Any homogeneous equation of second degree in $x$ and $y$ represents two straight lines through the origin.

### Important Properties

(i) Let $ax^2 + 2hxy + by^2 = 0$ be an equation of pair of straight lines. Then,

(a) Slope of first line, $m_1 = \dfrac{-h + \sqrt{h^2 - ab}}{b}$

and slope of the second line, $m_2 = \dfrac{-h - \sqrt{h^2 - ab}}{b}$

$\therefore \qquad m_1 + m_2 = -\dfrac{2h}{b} = -\dfrac{\text{Coefficient of } xy}{\text{Coefficient of } y^2}$

and $\qquad m_1 m_2 = \dfrac{a}{b} = \dfrac{\text{Coefficient of } x^2}{\text{Coefficient of } y^2}$

Here, $m_1$ and $m_2$ are

(1) real and distinct, if $h^2 > ab$. (2) coincident, if $h^2 = ab$.

(3) imaginary, if $h^2 < ab$.

(b) Angle between the pair of lines is given by

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{|a + b|}$$

(1) If lines are coincident, then $h^2 = ab$.

(2) If lines are perpendicular, then $a + b = 0$.

**Note** The angle between the lines represented by

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$=$ angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$

(c) The joint equation of bisector of the angles between the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h} \quad \Rightarrow \quad hx^2 - (a - b)\,xy - hy^2 = 0.$$

(d) The equation of the pair of lines through the origin and perpendicular to the pair of lines given by $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$.

(ii) If the equation of a pair of straight lines is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then the point of intersection is given by $\left(\dfrac{hf - bg}{ab - h^2}, \dfrac{gh - af}{ab - h^2}\right)$.

(iii) The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will represent two parallel lines, if $g^2 - ac > 0$ and $\dfrac{a}{h} = \dfrac{h}{b} = \dfrac{g}{f}$ and the distance between them is $2\sqrt{\dfrac{g^2 - ac}{a\,(a + b)}}$ or $2\sqrt{\dfrac{f^2 - bc}{b\,(a + b)}}$.

(iv) The equation of the bisectors of the angles between the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are given by

$$\frac{(x - x_1)^2 - (y - y_1)^2}{a - b} = \frac{(x - x_1)(y - y_1)}{h},$$

where, $(x_1, y_1)$ is the point of intersection of the lines represented by the given equation.

(v) Equation of the straight lines joining the origin to the points of intersection of a second degree curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and a straight line $lx + my + n = 0$ is

$$ax^2 + 2hxy + by^2 + 2gx\left(\frac{lx + my}{-n}\right) + 2fy\left(\frac{lx + my}{-n}\right) + c\left(\frac{lx + my}{-n}\right)^2 = 0.$$

## Important Points to be Remembered

(i) A triangle is an isosceles, if any two of its median are equal.

(ii) In an equilateral triangle, orthocentre, centroid, circumcentre, incentre coincide.

(iii) The circumcentre of a right angled triangle is the mid-point of the hypotenuse.

(iv) Orthocentre, centroid, circumcentre of a triangle are collinear. Centroid divides the line joining orthocentre and circumcentre in the ratio 2 : 1.

(v) If $D$, $E$ and $F$ are the mid-point of the sides $BC$, $CA$ and $AB$ of $\Delta ABC$, then the centroid of $\Delta ABC =$ centroid of $\Delta DEF$.

(vi) Orthocentre of the right angled $\Delta ABC$, right angled at $A$ is $A$.

(vii) The distance of a point $(x_1, y_1)$ from the $ax + by + c = 0$ is

$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|.$$

(viii) Distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is given by

$$d = \left|\frac{c_2 - c_1}{\sqrt{a^2 + b^2}}\right|.$$

(ix) The area of the triangle formed by the lines $y = m_1x + c_1, y = m_2x + c_2$ and $y = m_3x + c_3$ is

$$\Delta = \frac{1}{2}\left|\Sigma \frac{(c_1 - c_2)^2}{m_1 - m_2}\right|.$$

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(x) Three given points $A, B, C$ are collinear i.e. lie on the same straight line, if any of the three points (say $B$) lie on the straight line joining the other two points.

$\Rightarrow \qquad AB + BC = AC$

(xi) Area of the triangle formed by the line $ax + by + c = 0$ with the coordinate axes is $\Delta = \dfrac{c^2}{2|ab|}$.

(xii) The foot of the perpendicular $(h, k)$ from $(x_1, y_1)$ to the line $ax + by + c = 0$ is given by $\dfrac{h - x_1}{a} = \dfrac{k - y_1}{b} = -\dfrac{(ax_1 + by_1 + c)}{a^2 + b^2}$.

(xiii) Area of rhombus formed by $ax \pm by \pm c = 0$ is $\left|\dfrac{2c^2}{ab}\right|$.

(xiv) Area of the parallelogram formed by the lines

$$a_1x + b_1y + c_1 = 0,\ a_2x + b_2y + c_2 = 0,\ a_1x + b_1y + d_1 = 0$$

and $\quad a_2x + b_2y + d_2 = 0$ is

$$\left|\frac{(d_1 - c_1)(d_2 - c_2)}{a_1b_2 - a_2b_1}\right|.$$

(xv) (a) Foot of the perpendicular from $(a, b)$ on $x - y = 0$ is

$$\left(\frac{a+b}{2}, \frac{a+b}{2}\right).$$

(b) Foot of the perpendicular from $(a, b)$ on $x + y = 0$ is

$$\left(\frac{a-b}{2}, \frac{a-b}{2}\right).$$

(xvi) The image of the line $a_1x + b_1y + c_1 = 0$ about the line $ax + by + c = 0$ is

$$2(aa_1 + bb_1)(ax + by + c) = (a^2 + b^2)(a_1x + b_1y + c_1).$$

(xvii) Given two vertices $(x_1, y_1)$ and $(x_2, y_2)$ of an equilateral $\Delta ABC$, then its third vertex is given by.

$$\left[\frac{x_1 + x_2 \pm \sqrt{3}(y_1 - y_2)}{2}, \frac{y_1 + y_2 \mp \sqrt{3}(x_1 - x_2)}{2}\right]$$

(xviii) The equation of the straight line which passes through a given point $(x_1, y_1)$ and makes an angle $\alpha$ with the given straight line $y = mx + c$ are

$$(y - y_1) = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

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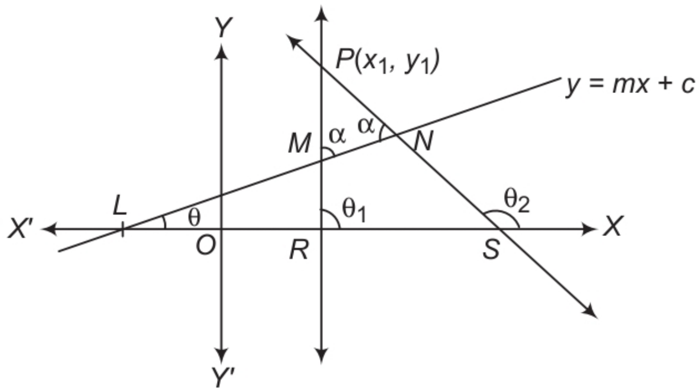


(xix) The equation of the family of lines passing through the intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is

$$(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$$

where, $\lambda$ is any real number.

(xx) Line $ax + by + c = 0$ divides the line joining the points $(x_1, y_1)$ and $(x_2, y_2)$ in the ratio $\lambda : 1$, then $\lambda = -\left(\dfrac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right)$.

If $\lambda$ is positive it divides internally and if $\lambda$ is negative, then it divides externally.

(xxi) Area of a polygon of $n$-sides with vertices $A_1(x_1, y_1), A_2(x_2, y_2), \ldots, A_n(x_n, y_n)$

$$= \frac{1}{2}\left[\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \ldots + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right]$$

(xxii) Equation of the pair of lines through $(\alpha, \beta)$ and perpendicular to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $b(x - \alpha)^2 - 2h(x - \alpha)(y - \beta) + a(y - \beta)^2 = 0$.