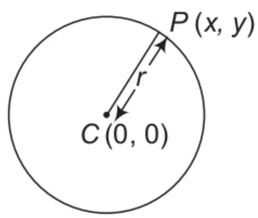
Circles

Circle

Circle is defined as the locus of a point which moves in a plane such that its distance from a fixed point in that plane is constant.



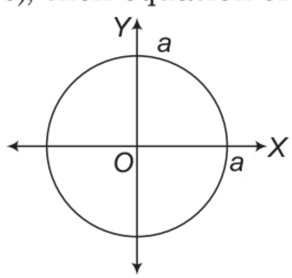
The fixed point is called the centre and the constant distance is called the radius.

Standard Equation of a Circle

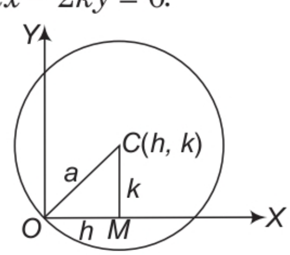
Equation of circle having centre (h, k) and radius a is $(x-h)^2+(y-k)^2=a^2$. This is also known as central form of equation of a circle.

Some Particular Cases of the Central Form

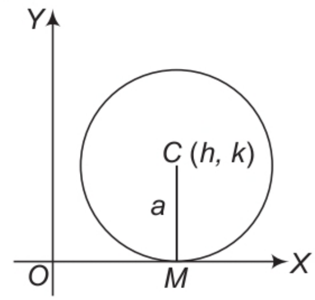
(i) When centre is (0, 0), then equation of circle is $x^2+y^2=a^2$.



(ii) When the circle passes through the origin, then equation of the circle is $x^2+y^2-2hx-2ky=0$.

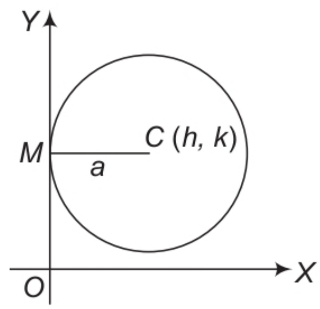


(iii) When the circle touches the X-axis, the equation is $x^2 + y^2 - 2hx - 2ay + h^2 = 0$.



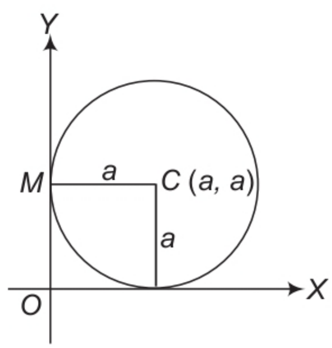
(iv) Equation of the circle, touches the Y-axis is

$$x^2 + y^2 - 2ax - 2ky + k^2 = 0.$$

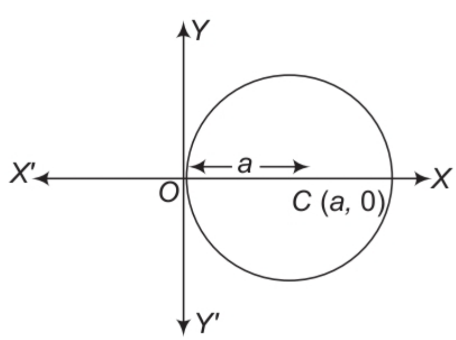


(v) Equation of the circle, touching both axes is

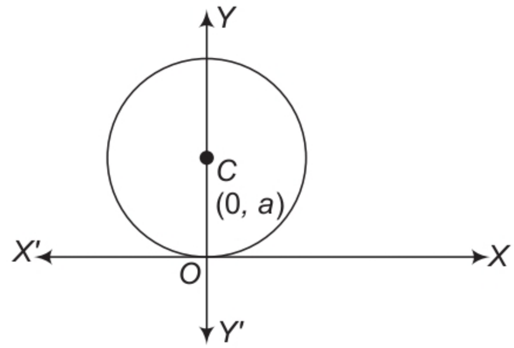
$$x^2 + y^2 - 2ax - 2ay + a^2 = 0.$$



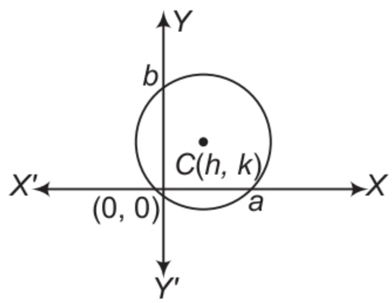
(vi) Equation of the circle passing through the origin and centre lying on the X-axis is $x^2 + y^2 - 2ax = 0$.



(vii) Equation of the circle passing through the origin and centre lying on the Y-axis is $x^2 + y^2 - 2ay = 0$.



(viii) Equation of the circle through the origin and cutting intercepts a and b on the coordinate axes is $x^2 + y^2 - ax - by = 0$.



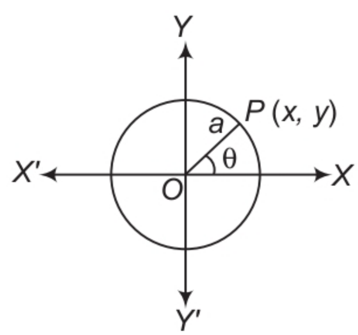
Equation of Circle When Ends Points of Diameter are Given

Equation of the circle, when the coordinates of end points of a diameter are (x_1, y_1) and (x_2, y_2) is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

Equation of Circle Passing Through Three Points

Equation of the circle passes through three non-collinear points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is $\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$.

Parametric Equation of a Circle



The parametric equation of $(x - h)^2 + (y - k)^2 = a^2$ is

$x = h + a\cos\theta,\ y = k + a\sin\theta, 0 \le \theta \le 2\pi$

For circle $x^2 + y^2 = a^2$, parametric equation is

$x = a\cos\theta,\ y = a\sin\theta$

General Equation of a Circle

The general equation of a circle is given by $x^2 + y^2 + 2gx + 2fy + c = 0$, whose centre $= (-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$

(i) If $g^2 + f^2 - c > 0$, then the radius of the circle is real and hence the circle is also real.

(ii) If $g^2 + f^2 - c = 0$, then the radius of the circle is 0 and the circle is known as point circle.

(iii) If $g^2 + f^2 - c < 0$, then the radius of the circle is imaginary. Such a circle is imaginary, which is not possible to draw.

Position of a Point w.r.t. a Circle

A point (x_1, y_1) lies outside, on or inside a circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0, \text{ according as } S_1 > , = \text{ or } < 0$$

where, $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Intercepts on the Axes

The length of the intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with X and Y-axes are $2\sqrt{g^2 - c}$ and $2\sqrt{f^2 - c}$ respectively.

(i) If $g^2 > c$, then the roots of the equation $x^2 + 2gx + c = 0$ are real and distinct, so the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ meets the X-axis in two real and distinct points.

(ii) If $g^2 = c$, then the roots of the equation $x^2 + 2gx + c = 0$ are real and equal, so the circle touches X-axis, then intercept on X-axis is 0.

(iii) If $g^2 < c$, then the roots of the equation $x^2 + 2gx + c = 0$ are imaginary, so the given circle does not meet X-axis in real point. Similarly, the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the Y-axis in real and distinct points, touches or does not meet in real point according to $f^2 >, =$ or $< c$.

Equation of Tangent

A line which touch only one point of a circle.

1. Point Form

(i) The equation of the tangent at the point $P(x_1, y_1)$ to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

(ii) The equation of the tangent at the point $P(x_1, y_1)$ to a circle $x^2 + y^2 = r^2$ is $xx_1 + yy_1 = r^2$.

2. **Slope Form**

(i) The equation of the tangent of slope m to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are

$$y + f = m(x + g) \pm \sqrt{(g^2 + f^2 - c)(1 + m^2)}$$

(ii) The equation of the tangents of slope m to the circle $(x - a)^2 + (y - b)^2 = r^2$ are $y - b = m(x - a) \pm r\sqrt{1 + m^2}$ and the coordinates of the points of contact are

$$\left(a \pm \frac{mr}{\sqrt{1 + m^2}}, b \mp \frac{r}{\sqrt{1 + m^2}}\right).$$

(iii) The equation of tangents of slope m to the circle $x^2 + y^2 = r^2$ are $y = mx \pm r\sqrt{1 + m^2}$ and the coordinates of the point of contact are

$$\left(\pm \frac{rm}{\sqrt{1 + m^2}}, \mp \frac{r}{\sqrt{1 + m^2}}\right).$$

3. **Parametric Form**

The equation of the tangent to the circle $(x - a)^2 + (y - b)^2 = r^2$ at the point $(a + r\cos\theta, b + r\sin\theta)$ is $(x - a)\cos\theta + (y - b)\sin\theta = r$.

Equation of Normal

A line which is perpendicular to the tangent is known as a normal.

1. **Point Form**

(i) The equation of normal at the point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$$

or $$(y_1 + f)x - (x_1 + g)y + (gy_1 - fx_1) = 0.$$

(ii) The equation of normal at the point (x_1, y_1) to the circle $x^2 + y^2 = r^2$ is $\frac{x}{x_1} = \frac{y}{y_1}$.

2. **Slope Form**

The equation of a normal of slope m to the circle $x^2 + y^2 = r^2$ is $my = -x \pm r\sqrt{1 + m^2}$.

3. **Parametric Form**

The equation of normal to the circle $x^2 + y^2 = r^2$ at the point $(r \cos\theta, r \sin\theta)$ is

$$\frac{x}{r \cos\theta} = \frac{y}{r \sin\theta} \quad \text{or} \quad y = x \tan\theta.$$

Important Points to be Remembered

(i) If (x_1, y_1) is one end of a diameter of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then the other end will be $(-2g - x_1, -2f - y_1)$.

(ii) If a line is perpendicular to the radius of a circle at its end points on the circle, then the line is a tangent to the circle and *vice-versa.*

(iii) Normal at any point on the circle is a straight line which is perpendicular to the tangent to the curve at the point and it passes through the centre of circle.

(iv) The line $y = mx + c$ meets the circle in unique real point or touch the circle $x^2 + y^2 = r^2$, if $r = \left|\frac{c}{\sqrt{1+m^2}}\right|$

and the point of contacts are $\left(\frac{\pm mr}{\sqrt{1+m^2}}, \frac{\mp r}{\sqrt{1+m^2}}\right)$.

(v) The line $lx + my + n = 0$ touches the circle $x^2 + y^2 = r^2$, if $r^2(l^2 + m^2) = n^2$.

(vi) Tangent at the point $P(r\cos\theta, r\sin\theta)$ to the circle $x^2 + y^2 = r^2$ is $x \cos\theta + y \sin\theta = r$.

(vii) The point of intersection of the tangent at the points $P(\theta_1)$ and $Q(\theta_2)$ on the circle $x^2 + y^2 = r^2$ is given by

$$x = \frac{r\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)} \text{ and } y = \frac{r\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}.$$

(viii) A line intersect a given circle at two distinct real points, if the length of the perpendicular from the centre is less than the radius of the circle.

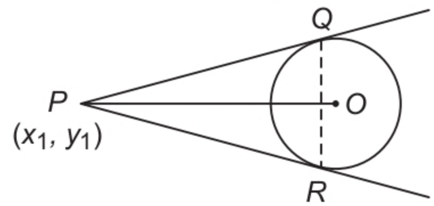
(ix) Length of the intercept cut off from the line $y = mx + c$ by the circle $x^2 + y^2 = a^2$ is $\quad 2\sqrt{\frac{a^2(1+m^2) - c^2}{1+m^2}}$

(x) If P is a point and C is the centre of a circle of radius r, then the maximum and minimum distances of P from the circle are $CP + r$ and $|CP - r|$ respectively.

(xi) Power of a point (x_1, y_1) with respect to the circle
$x^2 + y^2 + 2gx + 2fy + c = 0$ is $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

Pair of Tangents

(i) The combined equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the circle $x^2 + y^2 = r^2$ is



$$(x^2 + y^2 - r^2)(x_1^2 + y_1^2 - r^2) = (xx_1 + yy_1 - r^2)^2$$

or $$SS_1 = T^2$$

where, $$S = x^2 + y^2 - r^2, S_1 = x_1^2 + y_1^2 - r^2$$

and $$T = xx_1 + yy_1 - r^2$$

(ii) The length of the tangents from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to

$$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}$$

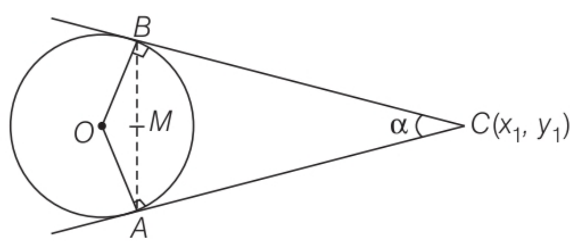
(iii) **Chord of contact** QR of two tangents, drawn from $P(x_1, y_1)$ to the circle $x^2 + y^2 = r^2$ is $xx_1 + yy_1 = r^2$ or $T = 0$.

Similarly, for the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

(iv) Let AB is a chord of contact of tangents from C to the circle $x^2 + y^2 = r^2$. M is the mid-point of AB.



(a) Coordinates of $M \left(\frac{r^2 x_1}{x_1^2 + y_1^2}, \frac{r^2 y_1}{x_1^2 + y_1^2} \right)$

(b) $AB = 2r \frac{\sqrt{x_1^2 + y_1^2 - r^2}}{\sqrt{x_1^2 + y_1^2}}$

(c) $BC = \sqrt{x_1^2 + y_1^2 - r^2}$

(d) Area of quadrilateral $OACB = r\sqrt{x_1^2 + y_1^2 - r^2}$

(e) Area of $\Delta\ ABC = \dfrac{r}{x_1^2 + y_1^2}(x_1^2 + y_1^2 - r^2)^{3/2}$

(f) Area of $\Delta\ OAB = \dfrac{r^3}{x_1^2 + y_1^2}\sqrt{x_1^2 + y_1^2 - r^2}$

(g) Angle between two tangents $\angle\ ACB$ is $2\tan^{-1}\dfrac{r}{\sqrt{S_1}}$.

(v) In general, two tangents can be drawn to a circle from a given point in its plane. If m_1 and m_2 are slope of the tangents drawn from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$, then

$$m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - a^2} \quad \text{and} \quad m_1 \times m_2 = \frac{y_1^2 - a^2}{x_1^2 - a^2}$$

(vi) The pair of tangents from $(0, 0)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are at right angle, if $g^2 + f^2 = 2c$.

Equation of Chord Bisected at a Given Point

The equation of chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ bisected at the point (x_1, y_1) is given by $T = S_1$.

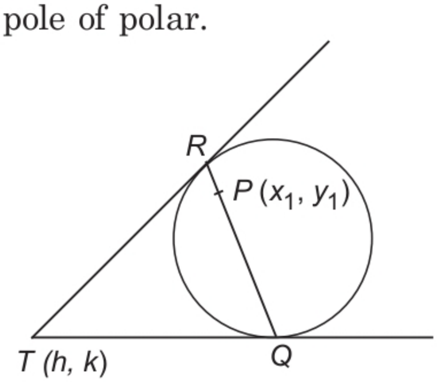
i.e. $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Director Circle

The locus of the point of intersection of two perpendicular tangents to a given circle is called a director circle. For circle $x^2 + y^2 = r^2$, the equation of director circle is $x^2 + y^2 = 2r^2$.

Pole and Polar

If through a point $P(x_1, y_1)$ (within or outside a circle) there be drawn any straight line to meet the given circle at Q and R, the locus of the point of intersection of tangents at Q and R is called the polar of P and point P is called the pole of polar.



(i) Equation of polar to the circle
$x^2 + y^2 = r^2$ is $xx_1 + yy_1 = r^2$.

(ii) Equation of polar to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

(iii) **Conjugate Points** Two points A and B are conjugate points with respect to a given circle, if each lies on the polar of the other with respect to the circle.

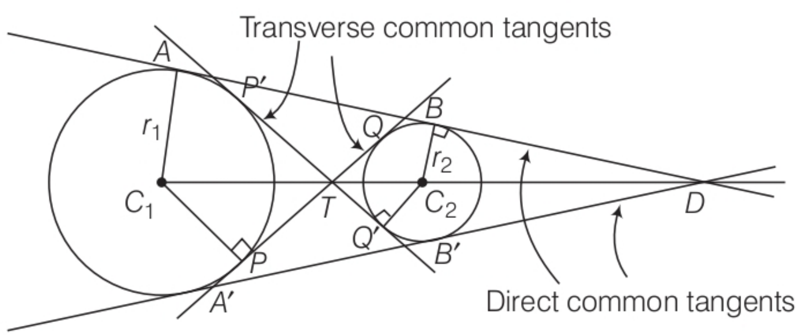
(iv) **Conjugate Lines** If two lines be such that the pole of one lies on the other, then they are called conjugate lines with respect to the given circle.

Common Tangents of Two Circles

Let the centres and radii of two circles are c_1, c_2 and r_1, r_2 respectively. Then, the following cases of intersection of these two circles may arise.

(i) When two circles are separate, four common tangents are possible.

Condition, $$C_1C_2 > r_1 + r_2$$



Clearly, $$\frac{C_1D}{C_2D} = \frac{r_1}{r_2}$$ [externally]

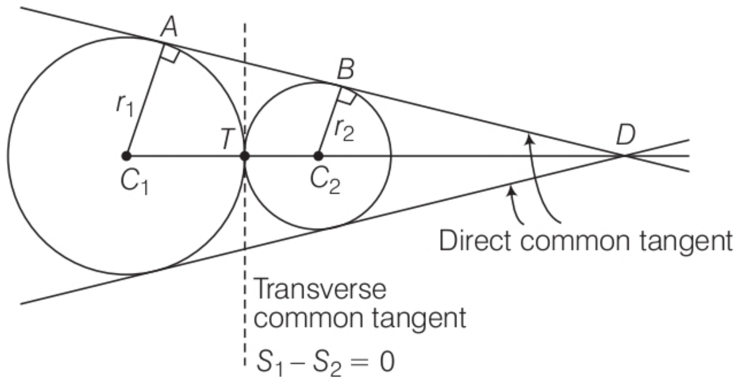
and $$\frac{C_1T}{C_2T} = \frac{r_1}{r_2}$$ [internally]

Length of direct common tangent
$$AB = A'B' = \sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$$

Length of transverse common tangent
$$PQ = P'Q' = \sqrt{(C_1C_2)^2 - (r_1 + r_2)^2}$$

(ii) When two circles touch externally, three common tangents are possible.

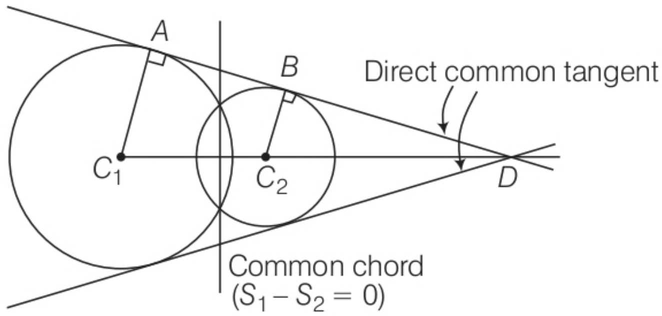
Condition, $$C_1C_2 = r_1 + r_2$$



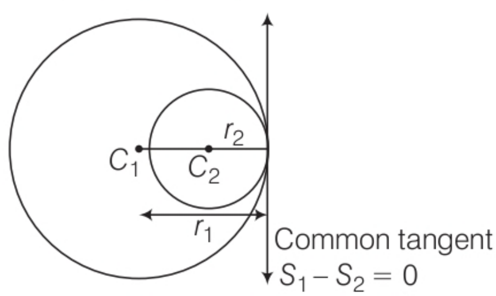
Clearly, $$\frac{C_1D}{C_2D} = \frac{r_1}{r_2} \quad \text{[externally]}$$

and $$\frac{C_1T}{C_2T} = \frac{r_1}{r_2} \quad \text{[internally]}$$

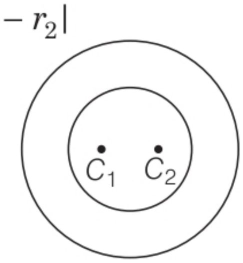
(iii) When two circles intersect, two common tangents are possible.
Condition, $|r_1 - r_2| < C_1C_2 < (r_1 + r_2)$



(iv) When two circles touch internally, one common tangent is possible.
Condition, $C_1C_2 = |r_1 - r_2|$

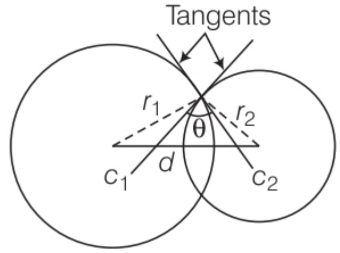


(v) When one circle contains another circle, no common tangent is possible.
Condition, $C_1C_2 < |r_1 - r_2|$



Angle of Intersection of Two Circles

The angle of intersection of two circles is defined as the angle between the tangents to the two circles at their point of intersection is given by



$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$

Orthogonal Circles

Two circles are said to be intersect orthogonally, if their angle of intersection is a right angle.

If two circles

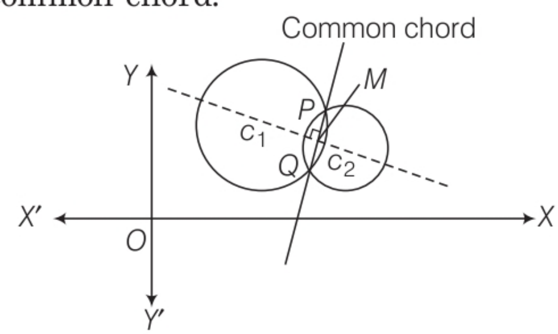
$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ are orthogonal, then

$2g_1g_2 + 2f_1f_2 = c_1 + c_2$

Common Chord

The chord joining the points of intersection of two given intersecting circles is called common chord.



(i) If $S_1 = 0$ and $S_2 = 0$ be two intersecting circles, such that

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$,

then their common chord is given by $S_1 - S_2 = 0$

(ii) If C_1, C_2 denote the centre of the given intersecting circles, then their common chord

$$PQ = 2\,PM = 2\sqrt{(C_1P)^2 - (C_1M)^2}$$

(iii) If r_1 and r_2 be the radii of two orthogonally intersecting circles, then length of common chord is $\dfrac{2r_1r_2}{\sqrt{r_1^2 + r_2^2}}$.

Family of Circles

(i) The equation of a family of circles passing through the intersection of a circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ and line $L = lx + my + n = 0$ is $S + \lambda L = 0$
where, λ is any real number.

(ii) The equation of the family of circles passing through the point $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

(iii) The equation of the family of circles touching the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ at point $P(x_1, y_1)$ is

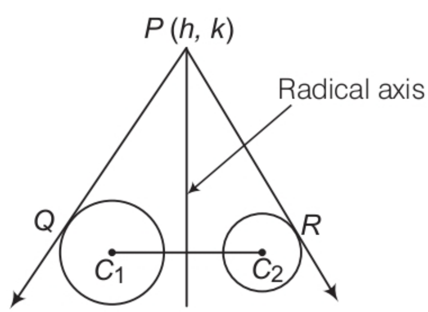
$$x^2 + y^2 + 2gx + 2fy + c + \lambda [xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c] = 0$$

or $S + \lambda L = 0$, where $L = 0$ is the equation of the tangent to $S = 0$ at (x_1, y_1) and $\lambda \in R$.

(iv) Any circle passing through the point of intersection of two circles S_1 and S_2 is $S_1 + \lambda S_2 = 0$, (where $\lambda \neq -1$).

Radical Axis

The radical axis of two circles is the locus of a point which moves in such a way that the length of the tangents drawn from it to the two circles are equal. A system of circles in which every pair has the same radical axis is called a coaxial system of circles. The equation of radical axis of two circles $S_1 = 0$ and $S_2 = 0$ is given by $S_1 - S_2 = 0$.



(i) The radical axis of two circles is always perpendicular to the line joining the centres of the circles.

(ii) The radical axes of three circles, whose centres are non-collinear taken in pairs are concurrent.

(iii) The centre of the circle cutting two given circles orthogonally, lies on their radical axis.

(iv) **Radical Centre** The point of intersection of radical axis of three circles whose centre are non-collinear, taken in pairs, is called their radical centre.

Coaxial System of Circles

A system of circle is said to be coaxial system of circles, if every pair of the circles in the system has same radical axis.

(i) The equation of a system of coaxial circles, when the equation of the radical axis $P \equiv lx + my + n = 0$ and one of the circle of the system $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, is $S + \lambda P = 0$.

where λ is an arbitrary constant.

(ii) Since, the lines joining the centres of two circles is perpendicular to their radical axis. Therefore, the centres of all circles of a coaxial system lie on a straight line, which is perpendicular to the common radical axis.

Limiting Points

Limiting points of a system of coaxial circles are the centres of the point circles belonging to the family.

Let equation of circle be $x^2 + y^2 + 2gx + c = 0$

$\therefore$ Radius of circle $= \sqrt{g^2 - c}$

For limiting point, $r = 0$

$$\therefore \quad \sqrt{g^2 - c} = 0 \Rightarrow g = \pm \sqrt{c}$$

Thus, limiting points of the given coaxial system as $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$.

Important Points to be Remembered

(i) Pole of $lx + my + n = 0$ with respect to $x^2 + y^2 = a^2$ is $\left(-\frac{a^2 l}{n}, -\frac{a^2 m}{n}\right)$.

(ii) Let $S_1 = 0$, $S_2 = 0$ be two circles with radii r_1, r_2, then $\frac{S_1}{r_1} \pm \frac{S_2}{r_2} = 0$ will meet at right angle.

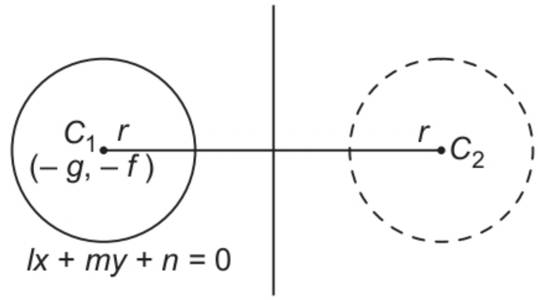
(iii) Family of circles touching a line $L = 0$ at a point (x_1, y_1) on it is $(x - x_1)^2 + (y - y_1)^2 + \lambda L = 0$.

(iv) Circumcircle of a Δ with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is

$$\frac{(x - x_1)(x - x_2) + (y - y_1)(y - y_2)}{(x_3 - x_1)(x_3 - x_2) + (y_3 - y_1)(y_3 - y_2)} = \frac{\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}}{\begin{vmatrix} x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}}$$

Image of the Circle by the Line Minor

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$



and line minor is $lx + my + n = 0$.

Then, the image of the circle is

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

where, (x_1, y_1) is mirror image of centre $(-g, -f)$ with respect to mirror line $lx + my + n = 0$ and $r = \sqrt{g^2 + f^2 - c}$.

Diameter of a Circle

The locus of the middle points of a system of parallel chords of a circle is called a diameter of the circle.

(i) The equation of the diameter bisecting parallel chords $y = mx + c$ of the circle $x^2 + y^2 = a^2$ is $x + my = 0$.

(ii) The diameter corresponding to a system of parallel chords of a circle always passes through the centre of the circle and is perpendicular to the parallel chords.