Sequences and Series

1. **Sequence**
A sequence is a succession of numbers or terms formed according to some rule. e.g. 4, 8, 12, ... A sequence is either finite or infinite according as number of terms in it. A sequence whose range is a subset of the set of real numbers R, is called a real sequence.

2. **Series**
If $a_1, a_2, a_3, \dots, a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + a_4 + \dots + a_n$ is called series.

3. **Progressions**
A sequence whose terms follow a certain pattern or rule, is called progression. e.g. 2, 4, 6, ..., 100

4. **Arithmetic Progression** (AP)
A sequence in which the difference of two consecutive terms is constant, is called arithmetic progression (AP). This difference is called the common difference and denoted by d. Thus, $d = a_{n+1} - a_n$

*n*th Term of an AP

If a is the first term, d is common difference and l is the last term of an AP, then

(i) nth term is given by $a_n = a + (n-1)d$.

(ii) nth term of an AP from the end is

$$a'_n = a_n - (n-1)d \text{ or } l-(n-1)d$$

(iii) $a_n + a'_n$ = constant

Properties of Arithmetic Progression

(i) If a sequence is an AP, then its nth term is a linear expression in n i.e. its nth term is given by $An + B$, where A and B are constants and A is common difference.

(ii) If a constant is added or subtracted from each term of an AP, then the resulting sequence is an AP with same common difference.

(iii) If each term of an AP is multiplied or divided by a non-zero constant, then the resulting sequence is also an AP.

(iv) Any three terms of an AP can be taken as $(a-d)$, a, $(a+d)$ and any four terms of an AP can be taken as $(a-3d)$, $(a-d)$, $(a+d)$, $(a+3d)$.

5. **Sum of *n* Terms of an AP**
If a is the first term, d is the common difference and l is the last term, then

Sum of n terms, $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$

Note *If sum of n terms, S_n of an AP is given, then nth term, $a_n = S_n - S_{n-1}$, where $a_1 = S_1$*

6. **Arithmetic Mean**
(i) If a, A and b are in AP, then $A = \frac{a+b}{2}$ is called the arithmetic mean of a and b.

(ii) If $a, A_1, A_2, A_3, \dots, A_n, b$ are in AP and $A_1, A_2, A_3, \dots, A_n$ are n arithmetic mean between a and b, then

$$d = \frac{b-a}{n+1} \text{ and } A_n = a + \frac{n(b-a)}{n+1}$$

(iii) Sum of n arithmetic mean between a and b is $n\left(\frac{a+b}{2}\right)$.

i.e. $A_1 + A_2 + A_3 + \dots + A_n = n\left(\frac{a+b}{2}\right)$

7. **Geometric Progression** (GP)
A sequence $a_1, a_2, \dots, a_n$ is called geometric progression, if it follows the relation $\frac{a_{k+1}}{a_k} = r$ (constant) for all $k \in N$

the constant ratio is called common ratio (r) of the GP.

*n*th Term of a GP

(i) If a is the first term and r is the common ratio, then the general term or nth term of GP is

$$a_n = ar^{n-1}, \text{ or } l = ar^{n-1}, \text{ where } l \text{ is last term.}$$

(ii) nth term of a GP from the end is

$$a'_n = \frac{l}{r^{n-1}}$$

where, l is last term

Properties of Geometric Progression

(i) If all the terms of GP are multiplied or divided by same non-zero constant, then the resulting sequence is a GP with the same common ratio.

(ii) The reciprocals of the terms of a given GP also form a GP.

(iii) If each term of a GP is raised to some power, the resulting sequence also forms a GP.

(iv) In a finite GP, the product of the terms equidistant from the beginning and from the end is always same and is equal to the product of the first and the last terms.

(v) The resulting sequence formed by taking the product and division of the corresponding terms of two GP's is also a GP.

(vi) Any three terms can be taken in GP as $\frac{a}{r}$, a and ar and any four terms can be taken in GP as $\frac{a}{r^3}, \frac{a}{r}$, ar and ar^3.

8. **Sum of *n* Terms of a GP**
If a and r are the first term and common ratio of a GP respectively, then

(i) Sum of n terms of a GP,

$$S_n = \begin{cases} a\frac{(1-r^n)}{1-r}, & \text{if } |r| < 1 \\ a\frac{(r^n - 1)}{r-1}, & \text{if } |r| > 1 \end{cases}$$

(ii) Sum of an infinite GP,

$$S_\infty = \begin{cases} \frac{a}{1-r}, & |r| < 1 \\ \infty, & |r| \geq 1 \end{cases}$$

9. Geometric Mean (GM)

(i) If a, G and b are in GP, then G is called the geometric mean of a and b and is given by $G = \sqrt{ab}$.

(ii) If $a, G_1, G_2, G_3, \ldots, G_n, b$ are in GP, then $G_1, G_2, G_3, \ldots, G_n$ are n GM's between a and b, then

common ratio, $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$ and $G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$

10. Relation between AM and GM

Let A and G be the AM and GM of two positive real numbers a and b, respectively.

$$\Rightarrow \quad A - G = \frac{(\sqrt{a} - \sqrt{b})^2}{2} \geq 0 \Rightarrow A \geq G$$

11. Sum of *n* Terms of Special Series

(i) Sum of first n natural numbers is

$$\Sigma n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

(ii) Sum of squares of first n natural numbers is

$$\Sigma n^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(iii) Sum of cubes of first n natural numbers is

$$\Sigma n^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$